Math 8583: Theory of Partial Differential Equations: Fall 2007

Homework Assignment 2, corrected (due on Wednesday, October 17, till 10:10 am)

52 points are distributed between 5 problems

1 (10 points). Show that the convolution

$$\Phi_\varepsilon(x) := (\Phi_0 * \eta^\varepsilon)(x - 2\varepsilon), \quad \text{where} \quad \Phi_0(x) := x_+ := \max(x, 0),$$

satisfies all the properties

$$\Phi_\varepsilon,\ \Phi'_\varepsilon,\ \Phi''_\varepsilon \geq 0 \quad \text{on } \mathbb{R}^1, \quad \Phi_\varepsilon \equiv 0 \quad \text{on } (-\infty, \varepsilon], \quad \text{and} \quad \Phi'_\varepsilon \equiv 1 \quad \text{on } [3\varepsilon, \infty),$$

if we take $\eta^\varepsilon(x) := \varepsilon^{-n}\eta(\varepsilon^{-1}x)$, where the function η is defined in Notes (1.12).

2 (10 points). Let f be a functions on a set $K \subset \mathbb{R}^n$, such that

$$\omega_0(\rho) := \sup\{|f(x) - f(y)| :\ x, y \in K,\ |x - y| \leq \rho\} \leq \omega(\rho),$$

where $\omega(\rho)$ is a non-decreasing continuous function on $[0, +\infty)$ such that $\omega(0) = 0$ and $\omega(a + b) \leq \omega(a) + \omega(b)$ for all $a, b \geq 0$. Show that the function

$$F(x) := \inf_{z \in K} \left[f(z) + \omega(|x - z|)\right] \equiv f(x) \quad \text{on} \quad K,$$

and it satisfies $|F(x) - F(y)| \leq \omega(|x - y|)$ for all $x, y \in \mathbb{R}^n$.

Hint. Use the inequality $|\inf f_1(z) - \inf f_2(z)| \leq \sup |f_1(z) - f_2(z)|$.

3 (10 points). Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a harmonic function in an unbounded domain $\Omega \subset \mathbb{R}^n$, such that $u = 0$ on $\partial\Omega$. Suppose that the measure $|\Omega \cap B_1(x)| \leq \mu |B_1|$ for all $x \in \mathbb{R}^n$ with a constant $\mu \in (0, 1)$, and let the origin $0 \in \Omega$, and $u(0) \geq 1$. Show that there is a constant $c_0 > 0$ such that

$$M_R := \sup_{\Omega \cap B_R(0)} u \geq c_0\, e^{c_0 R} \quad \text{for all} \quad R > 0.$$

Hint. Use the Growth lemma to show that $M_R \leq \mu \cdot M_{R+1}$ for all $R > 0$.

4 (10 points). Let $u \in C^2(\mathbb{R}^n)$ be such a function that

$$u \geq 0, \quad \Delta u + u^p \leq 0 \qquad \text{on} \quad \mathbb{R}^n, \quad \text{where} \quad p = \mathit{const} \in (0, 1].$$

Show that $u \equiv 0$ on $\mathbb{R}^n$.

5 (12 points). Let $u = (u(x_1, x_2) \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to the problem

$$\Delta u = -1 \quad \text{in} \quad \Omega := (0, 1) \times (0, 1), \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

Show that:

(a) $u \in C^1(\overline{\Omega})$.

(b) $|D_{11}u|$, $|D_{22}u| \leq 1$ in Ω.

(c) $D_{12}u$ is unbounded in Ω.