Math 8583: Theory of Partial Differential Equations: Fall 2007

Homework 2. Problems and Solutions

1. Show that the convolution

$$\Phi_\varepsilon(x) := (\Phi_0 * \eta^\varepsilon)(x - 2\varepsilon), \quad \text{where} \quad \Phi_0(x) := x_+ := \max(x, 0),$$

satisfies all the properties

$$\Phi_\varepsilon,\ \Phi'_\varepsilon,\ \Phi''_\varepsilon \ge 0 \quad \text{on } \mathbb{R}^1, \quad \Phi_\varepsilon \equiv 0 \quad \text{on } (-\infty, \varepsilon], \quad \text{and} \quad \Phi'_\varepsilon \equiv 1 \quad \text{on } [3\varepsilon, \infty),$$

if we take $\eta^\varepsilon(x) := \varepsilon^{-n}\eta(\varepsilon^{-1}x)$, where the function η is defined in Notes (1.12).

Solution. Taking into account that Φ_0 is monotone and convex, we have

$$\Phi_0(y) \ge 0, \quad 0 \le \Phi_0(y+h) - \Phi_0(y) \le h, \quad 0 \le \Phi_0(y+h) - 2\Phi_0(y) + \Phi_0(y-h)$$

for all $y \in \mathbb{R}^1$ and $h > 0$. Multiplying by $\eta^\varepsilon(x-y)$ and integrating with respect to y yields similar properties of the function $\Phi^{(\varepsilon)} := \Phi_0 * \eta^\varepsilon$:

$$\Phi^{(\varepsilon)}(x) \ge 0, \quad 0 \le \Phi^{(\varepsilon)}(x+h) - \Phi^{(\varepsilon)}(x) \le h, \quad 0 \le \Phi^{(\varepsilon)}(x+h) - 2\Phi^{(\varepsilon)}(x) + \Phi^{(\varepsilon)}(x-h)$$

for all $x \in \mathbb{R}^1$ and $h > 0$. Hence

$$\begin{aligned} 0 &\le \frac{d}{dx}\Phi^{(\varepsilon)}(x) = \lim_{h \to 0^+} \frac{1}{h}\big[\Phi^{(\varepsilon)}(x+h) - \Phi^{(\varepsilon)}(x)\big] \le 1, \\ 0 &\le \frac{d^2}{dx^2}\Phi^{(\varepsilon)}(x) = \lim_{h \to 0^+} \frac{1}{h}\big[\Phi^{(\varepsilon)}(x+h) - 2\Phi^{(\varepsilon)}(x) + \Phi^{(\varepsilon)}(x-h)\big], \end{aligned}$$

so that Φ_ε, Φ'_ε, $\Phi''_\varepsilon \ge 0$ on $\mathbb{R}^1$. Moreover, since $\Phi_0(x) \equiv 0$ for $x \le 0$, $\Phi_0(x+h) - \Phi_0(x) \equiv h$ for $x \ge 0$, $h \ge 0$, and $\eta^\varepsilon(x) \equiv 0$ for $|x| \ge \varepsilon$, we also get

$$\Phi^{(\varepsilon)}(x) \equiv 0 \quad \text{for} \quad x \le -\varepsilon, \qquad \frac{d}{dx}\Phi^{(\varepsilon)}(x) \equiv 1 \quad \text{for} \quad x \ge \varepsilon,$$

which imply $\Phi_\varepsilon \equiv 0$ on $(-\infty, \varepsilon]$, and $\Phi'_\varepsilon \equiv 1$ on $[3\varepsilon, \infty)$.

2. Let f be a functions on a set $K \subset \mathbb{R}^n$, such that

$$\omega_0(\rho) := \sup\{|f(x) - f(y)| : \ x, y \in K,\ |x - y| \le \rho\} \le \omega(\rho),$$

where $\omega(\rho)$ is a non-decreasing continuous function on $[0, +\infty)$ such that $\omega(0) = 0$ and $\omega(a+b) \le \omega(a) + \omega(b)$ for all $a, b \ge 0$. Show that the function

$$F(x) := \inf_{z \in K} \big[f(z) + \omega(|x - z|)\big] \equiv f(x) \quad \text{on} \quad K,$$

and it satisfies $|F(x) - F(y)| \le \omega(|x - y|)$ for all $x, y \in \mathbb{R}^n$.

Solution. For $x, z \in K$, we have $f(x) - f(z) \le \omega_0(|x - z|) \le \omega(|x - z|)$, hence

$$f(x) \le \inf_{z \in K} \big[f(z) + \omega(|x - z|)\big] =: F(x),$$

and the infimum is attained for $z = x$, i.e $f \equiv F$ on K. Further for arbitrary $x, y, z \in \mathbb{R}^n$,

$$\begin{aligned} \omega(|x - z|) &\le \omega(|y - z| + |x - y|) \le \omega(|y - z|) + \omega(|x - y|), \\ f(z) + \omega(|x - z|) &\le f(z) + \omega(|y - z|) + \omega(|x - y|). \end{aligned}$$

By taking the infimum over $z \in K$, we get

$$F(x) \le F(y) + \omega(|x-y|).$$

Obviously, one can interchange x and y in this inequality, so that $|F(x) - F(y)| \le \omega(|x-y|)$.

3. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a harmonic function in an unbounded domain $\Omega \subset \mathbb{R}^n$, such that $u = 0$ on $\partial\Omega$. Suppose that the measure $|\Omega \cap B_1(x)| \le \mu |B_1|$ for all $x \in \mathbb{R}^n$ with a constant $\mu \in (0,1)$, and let the origin $0 \in \Omega$, and $u(0) \ge 1$. Show that there is a constant $c_0 > 0$ such that

$$M_R := \sup_{\Omega \cap B_R(0)} u \ge c_0 \, e^{c_0 R} \quad \text{for all} \quad R > 0.$$

Solution. One can apply the Growth Lemma (Lemma 1.20) to the function u in $\Omega^+ := \Omega \cap \{u > 0\}$. This gives us

$$u(x) \le \mu \cdot \sup_{\Omega \cap B_1(0)} u \quad \text{for each} \quad x \in \Omega^+.$$

In particular, $1 \le u(0) \le \mu \cdot M_1$. Note that

$$\text{if} \quad x \in \Omega \cap B_R(0) \qquad \text{then} \quad (\Omega \cap B_1(x)) \subset (\Omega \cap B_{R+1}(0)).$$

Therefore,

$$M_R := \sup_{\Omega \cap B_R(0)} u \le \mu \cdot M_{R+1} \quad \text{for} \quad R > 0.$$

We now have

$$1 \le M_0 := u(0) \le \mu \cdot M_1 \le \mu^2 \cdot M_2 \le \cdots \le \mu^k \cdot M_k \le \cdots .$$

Set $c := -\ln \mu > 0$. For arbitrary $R > 0$, let $k := [R]$ - the integer part of R, i.e. $k \le R < k+1$. Then

$$M_R \ge M_k \ge \mu^{-k} = e^{-k \ln \mu} = e^{ck} \ge e^{c(R-1)} = \mu \, e^{cR},$$

and the desired estimate follows with $c_0 := \min(\mu, c) = \min(\mu, -\ln \mu) > 0$.

4. Let $u \in C^2(\mathbb{R}^n)$ be such a function that

$$u \ge 0, \quad \Delta u + u^p \le 0 \qquad \text{on} \quad \mathbb{R}^n, \quad \text{where} \quad p = \mathit{const} \in (0, 1].$$

Show that $u \equiv 0$ on $\mathbb{R}^n$.

Solution. In $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^n$, then our statement follows immediately from the mean value theorem: since $\Delta u \le -u^p \le 0$, we have

$$\int\limits_{B_R(x_0)} u \, dx \le |B_R| \cdot u(x_0) = 0 \quad \text{for all} \quad R > 0,$$

which implies $u \equiv 0$. Therefore, we need to show that the case $u > 0$ is impossible. In this case, we must have $0 < u(x_0) < 1$ for some $x_0 \in \mathbb{R}^n$. Indeed, otherwise $u \ge 1$ on $\mathbb{R}^n$. Comparing $u(x)$ with the function

$$v_R(x) := \frac{1}{2n}(R^2 - |x|^2) \quad \text{in} \quad B_R := B_R(0),$$

we then have

$$\Delta u \le -u^p \le -1 = \Delta v_R \quad \text{in} \quad B_R, \qquad u \ge 0 = v_R \quad \text{on} \quad \partial B_R.$$

By the comparison principle, $u(0) \ge v_R(0) \to \infty$ as $R \to \infty$, which is impossible.

Without loss of generality, we can assume $0 < u(0) < 1$ and $u > 0$ on $\mathbb{R}^n$. Consider the function $w(x) := \prod_{k=1}^{n} \cos(n^{-1/2} x_k)$, which satisfies

$$w > 0, \quad \Delta w = -w \quad \text{in} \quad Q := \left\{ x \in \mathbb{R}^n : \max_k |x_k| < \sqrt{n} \cdot \frac{\pi}{2} \right\}; \qquad w = 0 \quad \text{on} \quad \partial Q.$$

Since w/u is continuous on $\overline{Q}$ and vanishes on ∂Q, we also have

$$1 < \frac{w(0)}{u(0)} \le \lambda := \max_Q \frac{w}{u} = \frac{w(z_0)}{u(z_0)}$$

for some point $z_0 \in Q$. Them $w \le \lambda u$,

$$\Delta(w - \lambda u) = \Delta w - \lambda \cdot \Delta u \ge \lambda u^p - w \ge (\lambda u)^p - w^p \ge 0 \quad \text{in} \quad Q.$$

By the maximum principle, we get a contradiction

$$0 = (w - \lambda u)(z_0) \le \sup_{\partial Q} (w - \lambda u) = -\lambda \inf_{\partial Q} u < 0,$$

which proves that $u \equiv 0$ on $\mathbb{R}^n$.

Remark. One can show that there is a function $u \in C^2(\mathbb{R}^n)$ satisfying $u > 0$ and $\Delta u + u^p \le 0$ on $\mathbb{R}^n$ if and only if $n \ge 3$ and $p > n/(n-2)$.

5. Let $u = u(x_1, x_2) \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to the problem

$$\Delta u = -1 \quad \text{in} \quad \Omega := (0,1) \times (0,1), \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

Show that:

(a) $u \in C^1(\overline{\Omega})$.

(b) $|D_{11}u|$, $|D_{22}u| \le 1$ in Ω.

(c) $D_{12}u$ is unbounded in Ω.

Solution. Step 1. Estimate of u. By the comparison principle,

$$0 \le u(x) \le \frac{1}{2} x_1(1 - x_1), \quad 0 \le u(x) \le \frac{1}{2} x_2(1 - x_2) \quad \text{in} \quad \Omega,$$

hence

$$0 \le u(x) \le \frac{1}{2} \min\left\{ x_1(1 - x_1),\ x_2(1 - x_2) \right\} \le \frac{1}{2} d(x), \tag{1}$$

where $d(x) := \operatorname{dist}(x, \partial\Omega) = \min\{x_1, x_2, 1 - x_1, 1 - x_2\}$. Note that the functions

$$v_1(x) := u(x) - \frac{1}{2} x_1(1 - x_1) \quad \text{and} \quad v_2(x) := u(x) - \frac{1}{2} x_2(1 - x_2)$$

are harmonic in Ω and in addition,

$$v_1 = 0 \quad \text{on} \quad \Gamma_1 := \{x_1 = 0 \text{ or } 1,\ 0 < x_2 < 1\}, \quad v_2 = 0 \quad \text{on} \quad \Gamma_2 := \{x_2 = 0 \text{ or } 1,\ 0 < x_1 < 1\}.$$

From properties of harmonic functions it follows $v_1 \in C^\infty(\Omega \cup \Gamma_1)$, $v_2 \in C^\infty(\Omega \cup \Gamma_2)$, and since they are different from u by a polynomial, we actually have

$$u \in C^\infty(\Omega \cup \Gamma_1 \cup \Gamma_2) = C^\infty(\overline{\Omega} \setminus P), \quad \text{where} \quad P := \{(0,0),\ (0,1),\ (1,0),\ (1,1)\}.$$

Step 2. Preliminary estimate of D^2u. We first derive an estimate which will be improved later:

$$\max_{i,j} |D_{ij}u(x)| \le N \cdot d_0^{-1}(x) \quad \text{in} \quad \Omega, \quad \text{where} \quad d_0(x) := \operatorname{dist}(x, P). \tag{2}$$

Here and in the rest of the proof, N denotes absolute constants, which may be different in different expressions. By symmetry, it suffices to prove this estimate in the triangle $\mathbb{T} := \{0 < x_1 < x_2 < 1/2\}$, in which $0 < d(x) = x_1 < x_2 < d_0(x) = |x| < \sqrt{2} \cdot x_2$. For $x = (x_1, x_2) \in \mathbb{T}$, consider separately two possible cases (i) and (ii):

(i) $0 < x_1 < \frac{1}{2}x_2$, i.e. x is close to $\partial\Omega$. In this case,

$$x \in B_r^+ := B_r(y) \cap \{x_1 > 0\}, \quad \text{where} \quad y := (0, x_2), \ \ r := \frac{1}{2}x_2.$$

Since the harmonic function v_1 vanishes on the flat portion of ∂B_{2r}^+, we can use the boundary estimate (1.45):

$$|D_{ij}v_1(x)| \le \sup_{B_r^+} |D_{ij}v_1| \le \frac{N}{r^2} \sup_{B_{2r}^+} |v_1| \le \frac{N}{r} \le \frac{N}{d_0(x)}.$$

(ii) $\frac{1}{2}x_2 < x_1 < x_2$. In this case, we set $r := \frac{1}{4}x_2$. Then $B_r := B_r(x) \subset B_{2r} \subset \Omega$ and we can use the interior estimate (1.16):

$$|D_{ij}v_1(x)| \le \sup_{B_r} |D_{ij}v_1| \le \frac{N}{r^2} \sup_{B_{2r}} |v_1| \le \frac{N}{r} \le \frac{N}{d_0(x)}.$$

Thus in both cases (i) and (ii), we have $|D_{ij}v_1| \le N/d_0$ for all i, j. Since $|D_{ij}u - D_{ij}v_1| \le 1$, the desired estimate (2) follows.

Step 3. Sharp estimates of $D_{11}u$, $D_{22}u$. From the equalities $\Delta u = D_{11}u + D_{22}u = -1$ on $\overline{\Omega} \setminus P$ and $u = 0$ on $\partial\Omega$ it follows that the functions $w_1 := D_{11}u$ and $w_2 := D_{22}u$ satisfy

$$\Delta w_1 = \Delta w_2 = 0 \quad \text{in} \quad \Omega; \quad w_1 = -1, \ w_2 = 0 \quad \text{on} \quad \Gamma_1; \quad w_1 = 0, \ w_2 = -1 \quad \text{on} \quad \Gamma_2.$$

We claim that

$$\sup_Q w = \sup_{(\partial\Omega)\setminus P} w \quad \text{for} \quad w = \pm w_1, \ \pm w_2. \tag{3}$$

This estimate would imply $-1 \le D_{11}u, \ D_{22}u \le 0$. In order to prove this version of the maximum principle, we need to "suppress" the possible growth of w near P, which is permitted by the estimate (2). For this purpose, consider the harmonic function

$$h(x) = h(x_1, x_2) := x_1 x_2 \cdot |x|^{-4} = \text{const} \cdot D_{12} \ln |x|, \quad x \ne 0.$$

For $0 < \delta < 1$, the function $W_1^\delta(x) := h(x_1 + \delta, x_2 + \delta)$ belongs to $C^\infty(\overline{\Omega})$ and satisfies

$$W_1^\delta > 0, \quad \Delta W_1^\delta = 0 \quad \text{in} \quad \Omega; \qquad W_1^\delta \ge c \cdot \delta^{-2} \quad \text{on} \quad \Omega \cap \{|x| = \delta\},$$

where $c = \text{const} > 0$. This function helps to control the growth of w near the origin $(0, 0) \in P$. For the remaining three vertices of Ω, the corresponding functions W_k^δ, $k = 2, 3, 4$, are defined by appropriate translations and rotations in $\mathbb{R}^2$. Then $W^\delta := W_1^\delta + W_2^\delta + W_3^\delta + W_4^\delta \in C^\infty(\overline{\Omega})$, and

$$W^\delta > 0, \quad \Delta W^\delta = 0 \quad \text{in} \quad \Omega; \qquad W^\delta \ge c \cdot \delta^{-2} \quad \text{on} \quad \Omega \cap \{d_0(x) = \delta\}.$$

Next, we compare each of functions $w = \pm w_1, \pm w_2$ with the function

$$w^\delta(x) := M + \beta(\delta) \cdot W^\delta(x) \quad \text{on} \quad \Omega^\delta := \Omega \cap \{d_0(x) > \delta\},$$

where

$$M := \sup_{(\partial\Omega)\setminus P} w, \quad \text{and} \quad \beta(\delta) := \sup_{\Omega\cap\{d_0(x)=\delta\}} \frac{|w(x)|}{W^\delta(x)} \le \frac{N\delta^{-1}}{c\delta^{-2}} = N\delta \to 0^+ \quad \text{as} \quad \delta \to 0^+.$$

Both w and w^δ are harmonic in Ω^δ, and $w \le w^\delta$ on $\partial\Omega^\delta$. By the comparison principle, we have $w \le w^\delta$ in Ω^δ. Then for arbitrary $x \in \Omega$,

$$w(x) \le \lim_{\delta\to 0^+} w^\delta(x) = M = \sup_{(\partial\Omega)\setminus P} w.$$

The estimate (3) is proved.

Step 4. Estimate of D^3u. We claim that

$$\max_{i,j,k} |D_{ijk}u(x)| \le N \cdot d_0^{-1}(x) \quad \text{in} \quad \Omega. \tag{4}$$

As in Step 2, it suffices to prove this estimate in the triangle $\mathbb{T} := \{0 < x_1 < x_2 < 1/2\}$. Note that $w_2 := D_{22}u$ is harmonic in Ω and vanishes for $x_1 = 0$. Following the lines in cases (i) and (ii) of Step 2, we get

$$|D_{22k}u(x)| = |D_k w_2(x)| \le \sup_{(B_r^+ \text{ or } B_r)} |D_k w_2| \le \frac{N}{r} \cdot \sup_{(B_{2r}^+ \text{ or } B_{2r})} |w_2| \le \frac{N}{r} \le \frac{N}{d_0(x)}$$

for $k = 1, 2$. Since $D_{11k}u + D_{22k}u = D_k(\Delta u) = 0$, this estimate is also true for $D_{11k}u$. However, any derivative $D^l u$ of order $|l| = 3$ can be written as $D_{11k}u$ or $D_{22k}u$. Therefore, the estimate (4) is true for all i, j, k.

Step 5. Estimate of $D_{12}u$. We claim that

$$|D_{12}u(x)| \le N \cdot |\ln |d_0(x)|| \quad \text{in} \quad \Omega. \tag{5}$$

Obviously, it suffices to consider the case $|x| \le 1/2$. The estimate (5) follows easily from (4) by integration. If a smooth curve $\gamma := \{x = x(s) : \ s_0 \le s \le s_1\}$ is parameterized by the arc length, then for any function $f \in C^1$, we can write

$$|f(x(s_1))| = \left| f(x(s_0)) + \int_{s_0}^{s_1} \frac{d}{ds} f(x(s))\, ds \right| \le |f(x(s_0))| + \int_{s_0}^{s_1} |Df(x(s))|\, ds.$$

In our case, for $f = D_{12}u$, we can integrate from $x(s_0) := (2|x|)^{-1} \cdot x \in \partial B_{1/2}$ to $x_{s_1} = x \in B_{1/2}$ in radial direction. This yields

$$|D_{12}u(x)| \le N + \int_{|x|}^{1/2} \frac{N}{s}\, ds \le N + N \cdot |\ln |x|| = N + N \cdot |\ln d_0(x)| \le N \cdot |\ln d_0(x)|.$$

Step 6. Estimate of Du. From the previous Steps 3 and 5, we have $D_{ij}u \le N \cdot |\ln d_0|$. Integrating along the arc $|x| = r < 1/2$ and taking into account that $D_1u(r, 0) = 0$, we obtain $D_1u(x) \le Nr|\ln r|$ on this arc. A similar estimates is also true for $D_2u(x)$. Hence we have

$$\max_k |D_k u(x)| \le N\, d_0(x) \cdot |\ln |d_0(x)|| \quad \text{in} \quad \Omega. \tag{6}$$

In particular, it follows $u \in C^1(\overline{\Omega})$, and $Du = 0$ on P.

Step 7. Unboundedness of $D_{12}u$. Compare $u(x)$ with the function

$$W = W_1 W_2, \quad \text{where} \quad W_1 := \frac{1}{2}x_1 x_2, \quad W_2 := -\ln|x|, \quad \text{in the domain} \quad S := \{x_1 > 0,\, x_2 > 0,\, |x| < 1\} \subset \Omega.$$

We have

$$\Delta W = W_1 \Delta W_2 + 2\, DW_1 \cdot DW_2 + W_2 \Delta W_1 = 2\, DW_1 \cdot DW_2 = -\frac{2x_1 x_2}{|x|^2} \geq -1 = \Delta u \quad \text{in} \quad S,$$

and $W = 0 \leq u$ on ∂S. By the comparison principle, $W \leq u$ in S, which implies

$$f(t) := u(t,t) \geq -\frac{\sqrt{2}}{2} \cdot t^2 \ln t, \qquad 0 < t \leq \frac{\sqrt{2}}{2}.$$

By Taylor's formula,

$$f(t) = f(0) + f'(0) \cdot t + \frac{1}{2} f''(\bar{t}) \cdot t^2 = \frac{1}{2} f''(\bar{t}) \cdot t^2, \quad \text{where} \quad 0 < \bar{t} < t.$$

Comparing these two expressions, we see that

$$f''(\bar{t}) \geq -\sqrt{2} \cdot \ln t \to \infty \quad \text{as} \quad t \to 0^+.$$

On the other hand, f'' is a linear combination of $D_{11}u$, $D_{22}u$, and $D_{12}u$ with bounded coefficients. Since $D_{11}u$ and $D_{22}u$ are bounded, the mixed derivative $D_{12}u$ must be unbounded.