Math 8583: Theory of Partial Differential Equations: Fall 2007
Homework Assignment 3. Problems and Solutions

1. For $n = 1, 2, \ldots$, denote $Q_n := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_i < 1 \text{ for all } i\}$.
Let $u_n \in C^2(Q_n) \cap C(\overline{Q_n})$ be a solution to the problem

$$\Delta u_n = -1 \quad \text{in} \quad Q_n, \qquad u_n = 0 \quad \text{on} \quad \partial Q_n.$$

Show that

$$u_n(x_1, \ldots, x_n) \le u_m(x_1, \ldots, x_m) \quad \text{for all} \quad 1 \le m \le n, \quad x = (x_1, \ldots, x_n) \in Q_n.$$

Solution. We have $Q_n \subseteq Q_m$ for $n \ge m$. Therefore, $u_m \ge 0$ in Q_m implies $u_n = 0 \le u_m$ on $(\partial Q_n) \subset \overline{Q_m}$. We also have $\Delta u_m = \Delta u_n = 0$ in Q_n. By the comparison principle, $u_n \le u_m$ on $\overline{Q_n}$.

2. Evaluate the integral

$$\int_{-\infty}^{+\infty} e^{-x^2} \cos x \, dx.$$

Solution. The Cauchy problem

$$u_t = u_{xx} \quad \text{for} \quad t > 0, \ -\infty < x < +\infty, \qquad u(0, x) \equiv u_0(x) := \cos x$$

has a bounded solution $u(t, x) := e^{-t} \cos x$. On the other hand, this solution can se represented in the form

$$u(t, y) = \int_{-\infty}^{+\infty} \Gamma(t, y - x)\, u_0(x)\, dx, \quad \text{where} \quad \Gamma(t, x) := \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{x^2}{4t}\Big).$$

In particular,

$$e^{-t} = u(t, 0) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{x^2}{4t}\Big) \cos x \, dx, \qquad t > 0.$$

Taking $t := 1/4$, we get

$$\int_{-\infty}^{+\infty} e^{-x^2} \cos x \, dx = e^{-1/4}\sqrt{\pi}.$$

3. Let $\Omega_1 \subseteq \Omega_2$ be bounded smooth domains in $\mathbb{R}^n$. For $k = 1, 2$, let functions $u_k \in C^2(\overline{\Omega}_k)$ be such that

$$u_k > 0, \ \Delta u_k + \lambda_k u_k = 0 \quad \text{in} \quad \Omega_k, \qquad u_k = 0 \quad \text{on} \quad \partial\Omega_k, \quad \text{where} \quad \lambda_k = const.$$

Show that $0 < \lambda_2 \le \lambda_1$.

Solution. Using Green's formula (Lemma 1.16), we obtain

$$(\lambda_1 - \lambda_2) \int_{\Omega_1} u_1 u_2 \, dx = \int_{\Omega_1} \Big[u_1 \cdot (-\lambda_2 u_2) - u_2 \cdot (-\lambda_1 u_1)\Big]\, dx$$

$$= \int_{\Omega_1} (u_1 \Delta u_2 - u_2 \Delta u_1)\, dx = \int_{\partial\Omega} \Big(u_1 \frac{\partial u_2}{\partial \boldsymbol{\nu}} - u_2 \frac{\partial u_1}{\partial \boldsymbol{\nu}}\Big) dS \ge 0,$$

because $u_1 = 0$ on $\partial\Omega_1$, $u_2 \ge 0$ on $\overline{\Omega}_1 \supseteq \partial\Omega_1$, and since $u_1 > 0$ in Ω_1, the derivative in the direction of the unit outward normal $\partial u_1/\partial\boldsymbol{\nu} \le 0$ on $\partial\Omega_1$. Since $u_1 u_2 > 0$ in Ω_1, we must have $\lambda_1 \ge \lambda_2$.

Remark. Let Ω be a bounded smooth domain in $\mathbb{R}^n$, and let $u \in C^2(\overline{\Omega})$ be a function satisfying

$$u > 0, \quad \Delta u + \lambda u = 0 \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

One can show that

$$\lambda = \lambda(\Omega) := \inf_{v \in \mathcal{A}(\Omega)} \frac{||Dv||_2^2}{||v||_2^2} > 0, \quad \text{where} \quad \mathcal{A}(\Omega) := \{v \in C(\overline{\Omega}) :\ v = 0 \ \text{ on } \ \partial\Omega\}. \tag{2}$$

By definition of $\lambda(\Omega)$, there exists a sequence $u_k \in \mathcal{A}(\Omega)$, such that

$$||u_k||_2 = 1 \quad \text{for all } k, \qquad \text{and} \quad \lambda \le \int\limits_\Omega |Du_k|^2 dx \to \lambda \quad \text{as} \quad k \to \infty.$$

Rewriting this property in the form

$$0 \le m_k := \int\limits_\Omega \Big(|Du_k|^2 - \lambda u_k^2\Big) dx \to 0 \quad \text{as} \quad k \to \infty,$$

one can show that u_k converges in $L^2(\Omega)$ to a weak solution u of the problem (1) in the sense

$$\int\limits_\Omega u\,(\Delta\phi + \lambda\phi)\,dx \qquad \forall \phi \in C_0^\infty(\Omega).$$

This weak solution (possibly after modifying on a set of zero measure) is in fact a classical solution of (1), normalized by the condition $||u||_2 = 1$.

The equality (2) means that λ is the maximal constant such that

$$\lambda \cdot ||v||_2^2 := \lambda \cdot \int\limits_\Omega v^2\,dx \le ||Dv||_2^2 := \int\limits_\Omega |Dv|^2\,dx,$$

or equivalently, $N := \lambda^{-1/2}$ is the best (smallest) constant in the Poincaré inequality

$$||v||_2 \le N \cdot ||Dv||_2, \qquad v \in \mathcal{A}(\Omega).$$

Vote that each function $v \in \mathcal{A}(\Omega)$ can be approximated, with respect to the norm $||v||_2 + ||Dv||_2$, by functions $v^\varepsilon \in C_0^1(\Omega)$. For example, one can take

$$v^\varepsilon := \Phi_\varepsilon(v_+) - \Phi_\varepsilon(v_-), \qquad \text{where} \quad v_\pm := \max(0, \pm v),$$

and Φ_ε are functions used in the proof of Growth Lemma 1.20. Therefore,

$$\lambda(\Omega) = \inf_{v \in C_0^1(\Omega)} \frac{||Dv||_2^2}{||v||_2^2} > 0. \tag{3}$$

We assume that functions $v \in C_0^1(\Omega)$ are extended as $v \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. Then for $\Omega_1 \subseteq \Omega_2$, we have $C_0^1(\Omega_1) \subseteq C_0^1(\Omega_2)$, and from (3) it follows $\lambda(\Omega_1) \ge \lambda(\Omega_2) > 0$.

4. Show that arbitrary vector field $\boldsymbol{F} = (F_1, F_2, F_3) \in C_0^\infty(\mathbb{R}^{\nvDash})$ can be decomposed into two: $\boldsymbol{F} = \boldsymbol{G} + \boldsymbol{H}$, such that

$$\operatorname{curl} \boldsymbol{G} := \boldsymbol{\nabla} \times \boldsymbol{G} \equiv 0, \qquad \operatorname{div} \boldsymbol{H} := \boldsymbol{\nabla} \cdot \boldsymbol{H} \equiv 0.$$

Moreover, $\operatorname{curl} \boldsymbol{F} \equiv 0$ if and only if $\boldsymbol{F} = \operatorname{grad} U$ for some scalar function (potential) U, and $\operatorname{div} \boldsymbol{F} \equiv 0$ if and only if $\boldsymbol{F} = \operatorname{curl} \boldsymbol{V}$ for some vector field $\boldsymbol{V}$.

Solution. Using formula

$$\operatorname{curl}(\operatorname{curl} \boldsymbol{A}) = \operatorname{grad}(\operatorname{div} \boldsymbol{A}) - \Delta \boldsymbol{A} \quad \text{with} \quad \boldsymbol{A} := \Gamma * \boldsymbol{F},$$

where Γ is the fundamental solution for the Laplacian, we can write

$$\boldsymbol{F} = \Delta(\Gamma * \boldsymbol{F}) = \Delta \boldsymbol{A} = \boldsymbol{G} + \boldsymbol{H},$$

where

$$\begin{aligned} \boldsymbol{G} &:= \operatorname{grad}(\operatorname{div} \boldsymbol{A}) \quad \text{satisfies} \quad \operatorname{curl} \boldsymbol{G} = 0, \\ \boldsymbol{H} &:= -\operatorname{curl}(\operatorname{curl} \boldsymbol{A}) \quad \text{satisfies} \quad \operatorname{div} \boldsymbol{H} = 0, \end{aligned}$$

because $\operatorname{curl}(\operatorname{grad}) = 0$ and $\operatorname{div}(\operatorname{curl}) = 0$.

If $\operatorname{curl} \boldsymbol{F} := \boldsymbol{\nabla} \times \boldsymbol{F} = 0$, then

$$\operatorname{curl} \boldsymbol{A} := \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times (\Gamma * \boldsymbol{F}) = \Gamma * (\boldsymbol{\nabla} \times \boldsymbol{F}) = \boldsymbol{0},$$

which implies $\boldsymbol{H} = 0$. Then $\boldsymbol{F} = \boldsymbol{G} = \operatorname{grad} U$, where $U := \operatorname{div} \boldsymbol{A}$.

If $\operatorname{div} \boldsymbol{F} := \boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$, then

$$\operatorname{div} \boldsymbol{A} := \boldsymbol{\nabla} \cdot \boldsymbol{A} = \boldsymbol{\nabla} \cdot (\Gamma * \boldsymbol{F}) = \Gamma * (\boldsymbol{\nabla} \cdot \boldsymbol{F}) = 0,$$

which implies $\boldsymbol{G} = 0$. Then $\boldsymbol{F} = \boldsymbol{H} = \operatorname{curl} \boldsymbol{V}$, where $\boldsymbol{V} := -\operatorname{curl} \boldsymbol{A}$.

5. Let $u \in C^\infty(\overline{\mathbb{R}^n_+})$ satisfy

$$u \ge 0, \quad \Delta u \equiv 0 \quad \text{in} \quad \mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : \ x_n > 0\}, \qquad u(x', 0) \equiv 0.$$

Show that $u(x', x_n) \equiv \mathit{const} \cdot x_n$ in $\mathbb{R}^n_+$.

Solution. We will use the estimate

$$\sup_{B_r^+} \frac{u(x)}{x_n} \le N \cdot \inf_{B_r^+} \frac{u(x)}{x_n}, \qquad \text{where} \quad B_r^+ := \{x \in \mathbb{R}^n : \ |x| < r, \ x_n > 0\},$$

with a constant N depending only on n. From this estimate it follows

$$M := \sup_{\mathbb{R}^n_+} \frac{u(x)}{x_n} = \lim_{r \to +\infty} \sup_{B_r^+} \frac{u(x)}{x_n} \le \lim_{r \to +\infty} \inf_{B_r^+} \frac{u(x)}{x_n} = N \cdot m, \qquad \text{where} \quad m := \inf_{\mathbb{R}^n_+} \frac{u(x)}{x_n}.$$

Note that the function $v(x) := u(x) - m \cdot x_n$ also satisfies all the assumptions $v \ge 0$, $\Delta v = 0$ in $\mathbb{R}^n_+$, and $v(x', 0) \equiv 0$. Applying the above estimate to this function, we get

$$0 \le M - m = \sup_{\mathbb{R}^n_+} \frac{v(x)}{x_n} \le N \cdot \inf_{\mathbb{R}^n_+} \frac{v(x)}{x_n} = 0,$$

i.e. $M = m$, and $u(x) \equiv m \cdot x_n$ in $\mathbb{R}^n_+$.