Math 8584: Theory of Partial Differential Equations: Spring 2004

Problems for take-home Final Exam
The solutions are due on Friday, May 14, till 3:30 pm.

The exam consists of 5 problems, 10 points each. The adjusted score for the course is

$$S = (1 \text{ best HW, out of } 2) * 5 + (\text{In-Class Exam}) * 6 + (\text{Final}) * 9, \quad S_{\max} = 1000.$$

1. Let $u \in C^2(\overline{B_R})$, where $B_R := \{x \in \mathbb{R}^n : |x| < R\}$, and let

$$0 \le u \le 1, \quad \Delta u = 0 \quad \text{in} \quad B_R; \quad \text{and} \quad u \equiv 0 \quad \text{on} \quad (\partial B_R) \cap \{|x - x_0| < r\}$$

for some $x_0 \in B_R$ and $r > 0$. Show that $u(x_0) \le \frac{N}{r}(R - |x_0|)$ with a constant N depending only on the dimension n.

Hint: Since $u(x_0) \le 1$, and we can choose N large enough, it suffices to consider x_0 close to ∂B_R, i.e. $R - |x_0|$ is much smaller than r. The point $y_0 := \frac{R}{|x_0|} x_0 \in \partial B_R$. Then we can take a ball $B_{r/4}(z_0)$ which touches ∂B_R from outside at the point y_0, and $B_{r/2}(z_0)$ contains x_0. Compare u with harmonic (or superharmonic) function which vanishes on $\partial B_{r/4}(z_0)$ and equals to 1 on $\partial B_{r/2}(z_0)$.

2. Let u be a smooth function on $\mathbb{R}^n$ satisfying

$$u > 0, \qquad \Delta u = u^p \qquad \text{on} \quad \mathbb{R}^n, \qquad \text{where} \quad p = \mathit{const} \in (0, 1).$$

Show that

$$\sup_{|x|<r} u(x) \ge c \cdot r^{\frac{2}{1-p}} \qquad \text{for all} \quad r > 0 \qquad \text{with a constant} \quad c = c(n, p) > 0.$$

Hint: Use the comparison function $v(x) = c\,|x|^{\frac{2}{1-p}} \in C^2$. If the desired estimate fails, compare u and v on the set $\{u > v\}$ (which contains 0). This should give a contradiction.

3. Let u be a solution (of arbitrary sign) in $C^2(B_2)$ to the equation

$$Lu := (aDu, Du) = 0 \quad \text{in} \quad B_2 := \{x \in \mathbb{R}^n : |x| < 2\},$$

where the matrix function $a(x) = [a_{ij}(x)]$ satisfies the *uniform ellipticity condition*

$$a_{ij} = a_{ij}, \quad \nu|\xi|^2 \le (a\xi, \xi) \le \nu^{-1}|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n, \quad \nu = \mathit{const} \in (0, 1]. \tag{UE}$$

Show that

$$\sup_{B_1} u^{2n} \le N \cdot \int_{B_2} u^{2n} dx,$$

with a constant $N = N(n, \nu) > 0$.

Hint: Take a smooth cut-off function $\zeta \in C_0^\infty(B_2)$ such that $0 \le \zeta \le 1$ in B_2, $\zeta \equiv 1$ on B_1. Show that $L(\zeta^2 u^2) \ge -Nu^2$ and use this estimate together with the Aleksandrov-Bakel'man-Pucci inequality.

Remark. Actually in this estimate, one can replace u^{2n} by $|u|^p$ for arbitrary $p > 0$, then N depends also on p.

4. Let $u \in C^2(\mathbb{R}^n)$ be a **bounded** solution to the equation $Lu := (aDu, Du) + (b, Du) = 0 \quad \text{on} \quad \mathbb{R}^n$, where the matrix function $a(x) = [a_{ij}(x)]$ satisfies (UE) and the vector function $b(x) = [b_i(x)]$ is bounded: $|b(x)| \le K = const$ on $\mathbb{R}^n$. In addition, suppose that a and b are 1-periodic, i.e. $a(x+z) \equiv a(x)$, $b(x+z) \equiv b(x)$ for any $z \in Z^n$ (vectors in $\mathbb{R}^n$ with integer components). Show that $u \equiv const$ on $\mathbb{R}^n$.

Hint: One can use the following

Harnack Inequality (HE). *Let $u \in C^2(B_{2r})$, $r > 0$, be such that*

$$u \ge 0, \quad Lu := (aDu, Du) + (b, Du) = 0 \quad \text{in} \quad B_{2r} = B_{2r}(x_0) \subset \mathbb{R}^n,$$

where $a = [a_{ij}(x)]$ satisfies (UE) and $|b| = |b(x)| \le K = const$. Then

$$\sup_{B_r} u \le N \cdot \inf_{B_r} u \quad \text{with} \quad N = N(n, \nu, Kr).$$

Applying (HE) to the function $M - u$, we derive

Corollary. *If $u \in C^2(B_{2r})$ (not necessary positive) satisfies $Lu = 0$ in B_{2r}, then*

$$\sup_{B_r}(M - u) \le N \cdot (M - u(x_0)), \quad \text{where} \quad M = const \ge \sup_{B_{2r}} u.$$

Step 1. Apply Corollary with large r and $u(x_0)$ close to $M = \sup v$, where $v(x) = u(x+z) - u(x)$, in order to show that $u(x+z) \equiv u(x)$ for all $z \in Z^n$, i.e u is periodic.

Step 2. Apply Corollary again to the function u in order to show that it is constant.

5. Let $\Omega := R^{n-1} \times (0, 1) \subset \mathbb{R}^n$, and let the function $u \in C^\infty(\bar{\Omega})$ be such that

$$U_{2,\alpha} := \max_{i,j}[D_{ij}u]_{0,\alpha;\Omega} < \infty, \quad \text{where} \quad [f]_{0,\alpha;\Omega} := \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, \quad \alpha = const \in (0, 1),$$

and $u \equiv 0$ on $\partial\Omega$. Show that

$$U_{2,\alpha} \le N \cdot [\Delta u]_{0,\alpha;\Omega} \quad \text{with} \quad N = N(n, \alpha). \tag{1}$$

Hint: One can use the boundary estimate

$$\max_{i,j}[D_{ij}u]_{0,\alpha;B_r^+} \le N(n, \alpha) \cdot \left\{ [\Delta u]_{0,\alpha;B_{2r}^+} + r^{-2-\alpha} \sup_{B_{2r}^+} |u| \right\}$$

which is true for arbitrary smooth function on $\overline{B_{2r}^+} := \{x = (x', x_n) \in \mathbb{R}^n : |x| < 2r, x_n > 0\}$, which vanishes for $x_n = 0$, and also the interior estimate of same kind. However, since we do not have the estimate for $\sup_{B_{2r}^+} |u|$, one needs some preparations before using this estimate. First of all, replacing u by $u + const \cdot x_n(1 - x_n)$, we reduce the proof to the case $f(0) = 0$, where $f := \Delta u$. Then

$$|f(x)| \le F_\alpha |x|^\alpha \quad \text{for all} \quad x \in \Omega, \quad \text{where} \quad F_\alpha := [f]_{0,\alpha;\Omega}.$$

In order to evaluate $|u|$, the comparison function $NF_\alpha \cdot w$ is helpful, where

$$w(x) := \prod_{k=1}^{n-1} \cosh(\lambda x_k) \cdot \cos x_n, \quad \text{and} \quad (n-1)\lambda^2 < 1.$$

This function satisfies $\Delta w = -cw$ in Ω with $c := 1 - (n-1)\lambda^2 > 0$.

Alternative way. One can also use Problem 5 from in-class Exam in the following way. Extend the function $f = \Delta u$ to the whole space $\mathbb{R}^n$ with preserving its seminorm $[f]_{0,\alpha}$. Then by subtracting the function in Problem 5, the problem $\Delta u = f$ in Ω with $u = 0$ on $\partial\Omega$ is reduced to $\Delta u = 0$ in Ω with $u = g$ on $\partial\Omega$, where $[g]_{2,\alpha} \le NF_\alpha$. This way is not very short. As an intermediate step, the estimate for third derivatives may be used: $|D^3u(x)| \le NF_\alpha d^{\alpha-1}(x)$, where $d(x) = dist\,(x, \partial\Omega)$.