

Math 8584: Theory of Partial Differential Equations: Spring 2008

Midterm exam is scheduled on Wednesday, April 2, 2008.

Books, notes, and calculators are permitted. The exam consists of 3 take home problems and 2–3 in-class problems. Each of these two part contributes 50% to the total score for this exam.

1. Let $u \in C^2(Q_K) \cap C(\overline{Q_K})$ be a function satisfying the properties

$$0 < u \leq 1, \quad Lu = \sum_{i,j=1}^n a_{ij} D_{ij} u = 0 \quad \text{in } Q_K := \{x = (x', x_n) \in \mathbb{R}^n : K \cdot |x'| < x_n < 1\},$$

and

$$u = 0 \quad \text{on } \Gamma_K := \{x = (x', x_n) \in \mathbb{R}^n : K \cdot |x'| = x_n \leq 1\},$$

where $K = \text{const} > 0$, and the coefficients $a_{ij} = a_{ij}(x)$ satisfy the uniform ellipticity condition

$$a_{ij} = a_{ji}, \quad \nu |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \nu^{-1} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Show that there is a constants $\alpha = \alpha(n, \nu, K) > 1$ such that

$$u(x) = u(x', x_n) \leq (2x_n)^\alpha \quad \text{in } Q_K,$$

and $\alpha \rightarrow +\infty$ as $K \rightarrow +\infty$.

2. Let u_1 and u_2 be functions in $C^2(Z_R) \cap C(\overline{Z_R})$ satisfying the properties

$$u_1, u_2 > 0; \quad Lu_1 = 0, \quad Lu_2 + \lambda^2 u_2 = 0$$

in the cylinder $Z_R := \{x = (x', x_n) \in \mathbb{R}^n : |x'| < R, 0 < x_n < R\}$, where L is an elliptic operator of same kind as in the previous problem, and $\lambda \in \mathbb{R}^1$. Let

$$u_1 = u_2 = 0 \quad \text{on } \Gamma_R := \{x = (x', x_n) \in \mathbb{R}^n : |x'| < R, x_n = 0\}.$$

Show that there exists a constant $N = N(n, \nu, \lambda)$ such that

$$\sup_{Z_{R/2}} \frac{u_1}{u_2} \leq N \cdot \frac{u_1}{u_2}(0, R/2), \quad \sup_{Z_{R/2}} \frac{u_2}{u_1} \leq N \cdot \frac{u_2}{u_1}(0, R/2).$$

Hint. Note that both functions

$$\tilde{u}_1(x_0, x) = u_1(x) \quad \text{and} \quad \tilde{u}_2(x_0, x) = e^{\lambda x_0} \cdot u_2(x)$$

satisfy the equation

$$\frac{\partial^2 \tilde{u}}{\partial x_0^2} + L\tilde{u} = 0 \quad \text{in } \mathbb{R}^1 \times Z_R.$$

3. Show that there are no functions $u = u(t, x) \in C^2(H_T) \cap C(\overline{H_T})$ satisfying the properties

$$u > 0, \quad u_t = \Delta_x u \quad \text{in } H_T := \{(t, x) : 0 < t < T, x \in \mathbb{R}^n\},$$

and $u(0, x) \equiv \exp(x^4)$.

Hint. Compare $u(t, x)$ with functions $u_R(t, x) \in C^2(H_T) \cap C(\overline{H_T}) \cap L^\infty(H_T)$ satisfying the heat equation in H_T with the initial data $u_R(0, x) \equiv \exp(x^4) \cdot \zeta(x)$, where

$$\zeta \in C^\infty(\mathbb{R}^n), \quad 0 \leq \zeta \leq 1, \quad \zeta(x) \equiv 1 \quad \text{for } |x| < R, \quad \text{and} \quad \zeta(x) \equiv 0 \quad \text{for } |x| > 2R.$$