(Updated on March 24, 2008)

Math 8584: Theory of Partial Differential Equations: Spring 2008

Homework Assignment 1 (due on Wednesday, March 26)
50 points are distributed between 4 problems.

1. (10 points) Let $u \in C^2(B_2)$ be a bounded solution of the equation

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u = 0 \quad \text{in} \quad B_2 := \{x \in \mathbb{R}^n : \, |x| < 2\}, \tag{1}$$

where the coefficients $a_{ij} = a_{ij}(x)$ satisfy the uniform ellipticity condition

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}\xi_l\xi_j \le \nu^{-1}|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{2}$$

Show that there are constants $\alpha = \alpha(n, \nu) > 0$ and $N = N(n, \nu) > 0$ such that

$$\sup_{x,y \in B_1} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \le N \cdot \sup_{B_2} |u|.$$

Hint. Applying the Harnack inequality to $u + \mathit{const}$, derive the estimate in Statement 1 for

$$\omega(\rho) := \operatorname*{osc}_{B_\rho} u := \sup_{B_\rho} u - \inf_{B_\rho} u.$$

2. (15 points + extra 5 point) For $j = 1, 2$, let

$$Q_j := \{x = (x', x_n) \in \mathbb{R}^n : \, K_j \cdot |x'| < x_n < 1\}, \quad \text{where} \quad K_1 > K_2 > 0,$$

and let $u_j \in C^2(\overline{Q_j})$ be functions satisfying

$$u_j > 0, \quad Lu_j = 0 \quad \text{in} \quad Q_j, \qquad u_j = 0 \quad \text{on} \quad \partial Q_j \cap \{x_n = K_j \cdot |x'|\}.$$

Show that

$$\omega(\rho) := \sup_{Q_1 \cap \{0 < x_n < \rho\}} \frac{u_1}{u_2} \to 0 \quad \text{as} \quad \rho \to 0^+.$$

Hint. Use estimates in Sec. 3.3 and 3.4.

3. (10 points) Let $u \in C^2(\overline{Q^+})$ satisfy

$$u > 0, \quad \Delta u = 0 \quad \text{in} \quad Q^+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : \, x_1 > 0, \, 0 < x_2 < \pi\}, \qquad u = 0 \quad \text{on} \quad \partial Q^+.$$

Show that $u = \mathit{const} \cdot \sinh x_1 \cdot \sin x_2$ in Q^+.

(over)

Hints. Denote $v := \sinh x_1 \cdot \sin x_2$. From Statement 2 it follows that for $R \geq 1$,

$$\begin{aligned} m(R) &:= \inf_{Q^+ \cap \{x_1 < R\}} \frac{u}{v} = \inf_{Q^+ \cap \{x_1 = R\}} \frac{u}{v}, \\ M(R) &:= \sup_{Q^+ \cap \{x_1 < R\}} \frac{u}{v} = \sup_{Q^+ \cap \{x_1 = R\}} \frac{u}{v}. \end{aligned}$$

Show that $M(R) \leq N \cdot m(R)$. Replacing u by $u - \mathrm{const} \cdot v$, reduce the proof to the case when $m(R) \to 0$ as $R \to \infty$, and then do the rest.

4. (15 points) Let $u \in C^2(\overline{Q})$ satisfy

$$u > 0, \quad \Delta u = 0 \quad \text{in} \quad Q := \{x = (x_1, x_2) \in \mathbb{R}^2 : -\infty < x_1 < \infty,\, 0 < x_2 < \pi\}, \qquad u = 0 \quad \text{on} \quad \partial Q.$$

Show that $u = (c_1 e^{x_1} + c_2 e^{-x_1}) \sin x_2$ in Q, where c_1 and c_2 are non-negative constants.

You can use (without proof) the following facts:

Statement 1. Let $\omega(\rho)$ be a non-negative, non-decreasing function on an interval $(0, \rho_0]$, such that

$$\omega(q^{-1}\rho) \leq q^{-\alpha}\omega(\rho) \quad \text{for all} \quad \rho \in (0, \rho_0], \quad \text{where} \quad q = \mathrm{const} > 1.$$

Then

$$\omega(\rho) \leq \left(\frac{q\rho}{\rho_0}\right)^{\alpha} \omega(\rho_0) \quad \text{for all} \quad \rho \in (0, \rho_0].$$

Proof of this Statement is similar to deriving of (3.15) from (3.17) in Lecture Notes.

Statement 2. Let Ω be a bounded open set in $\mathbb{R}^n$, and let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$v > 0. \quad Lu = Lv = 0 \quad \text{in} \quad \Omega; \qquad u = 0 \quad \text{on} \quad \Gamma \subset \partial\Omega.$$

Then

$$M := \sup_{\Omega} \frac{u}{v} \leq M_0 := \sup_{(\partial\Omega) \setminus \Gamma} \frac{u}{v}.$$

Proof. By definition of M_0,

$$u \leq M_0 v \quad \text{on} \quad (\partial\Omega) \setminus \Gamma.$$

Since $u = 0$ on Γ, this inequality holds on the whole boundary $\partial\Omega$, and by the comparison principle, in Ω.