Math 8584: Theory of Partial Differential Equations: Spring 2008

Homework 1. Problems and solutions

1. Let $u \in C^2(B_2)$ be a bounded solution of the equation

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u = 0 \quad \text{in} \quad B_2 := \{x \in \mathbb{R}^n : \, |x| < 2\}, \tag{1}$$

where the coefficients $a_{ij} = a_{ij}(x)$ satisfy the uniform ellipticity condition

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}\xi_l\xi_j \le \nu^{-1}|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{2}$$

Show that there are constants $\alpha = \alpha(n,\nu) > 0$ and $N = N(n,\nu) > 0$ such that

$$\sup_{x,y\in B_1} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le N \cdot \sup_{B_2} |u|.$$

Solution. Fix $x \in B_1$ and $\rho \in (0, 1/2]$. We claim that

$$\omega\rho := \operatorname*{osc}_{B_\rho} u \le 2^{-\alpha}\omega(2\rho) \quad \text{for some} \quad \alpha = \alpha(n,\nu). \tag{3}$$

Replacing u by $u -$ const, we may assume $\inf\limits_{B_{2\rho}(x)} u = 0$, so that $u \ge 0$ in $B_{2\rho}(x)$. By Harnack inequality,

$$\sup_{B_\rho(x)} u \le N_0(n,\nu) \cdot \inf_{B_\rho(x)} u.$$

Therefore,

$$\omega(\rho) = \sup_{B_\rho(x)} u - \inf_{B_\rho(x)} u \le (1 - N_0^{-1}) \sup_{B_\rho(x)} u \le (1 - N_0^{-1}) \sup_{B_{2\rho}(x)} u = (1 - N_0^{-1}) \cdot \omega(2\rho),$$

and (3) follows with $2^{-\alpha} = 1 - N_0^{-1}$.

Applying Statement 1 with $\rho_0 = 1$, we get

$$\omega(\rho) \le (2\rho)^\alpha \operatorname*{osc}_{B_1(x)} \le 2^{1+\alpha}\rho^\alpha \sup_{B_2} |u|,$$

and the desired estimate holds true with $N = 2^{1+\alpha}$.

2. For $j = 1, 2$, let

$$Q_j := \{x = (x', x_n) \in \mathbb{R}^n : \, K_j \cdot |x'| < x_n < 1\}, \quad \text{where} \quad K_1 > K_2 > 0,$$

and let $u_j \in C^2(\overline{Q_j})$ be functions satisfying

$$u_j > 0, \quad Lu_j = 0 \quad \text{in} \quad Q_j, \qquad u_j = 0 \quad \text{on} \quad \partial Q_j \cap \{x_n = K_j \cdot |x'|\}.$$

Show that

$$\omega(\rho) := \sup_{Q_1\cap\{0<x_n<\rho\}} \frac{u_1}{u_2} \to 0 \quad \text{as} \quad \rho \to 0^+.$$

Solution. Denote $V_\rho := Q_1 \cap \{\rho/4 < x_n < \rho\}$. By Harnack inequality,

$$M(\rho) := \sup_{V_\rho} u_2 \le N_1 \cdot \inf_{V_\rho} u_2 \quad \text{for} \quad 0 < \rho \le 1/2,$$

where $N_1 = N_1(n, \nu, K_1, K_2)$. For fixed $\rho \in (0, 1/2]$, the function $v := \omega(\rho) \cdot u_2 - u_1$ satisfies

$$v \ge 0, \quad Lv = 0 \quad \text{in} \quad V_\rho,$$

and

$$v = \omega(\rho) \cdot u_2 \ge N_1^{-1}\omega(\rho)M(\rho) \quad \text{on} \quad (\partial Q_1) \cap \{\rho/4 \le x_n \le \rho\} \subset \partial V_\rho.$$

Combining Lemma 3.8 with the interior Harnack inequality, we obtain

$$v \ge c \cdot \omega(\rho)M(\rho) \ge c \cdot \omega(\rho)u_2 \quad \text{on} \quad Q_1 \cap \{x_n = \rho/2\},$$

where $c = c(n, \nu, K_1, K_2) \in (0, 1)$. Therefore,

$$(1 - c)\omega(\rho) \cdot u_2 - u_1 \ge 0 \quad \text{on} \quad \partial(Q_1 \cap \{x_n < \rho/2\}).$$

By the maximum principle, this inequality holds true on $Q_1 \cap \{x_n < \rho/2\}$, which means

$$\omega(\rho/2) = \sup_{Q_1\cap\{0<x_n<\rho/2\}} \frac{u_1}{u_2} \le (1 - c)\,\omega(\rho) =: 2^{-\alpha}\omega(\rho).$$

Now the desired property follows from Statement 1.

3. Let $u \in C^2(\overline{Q^+})$ satisfy

$$u > 0, \quad \Delta u = 0 \quad \text{in} \quad Q^+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\, 0 < x_2 < \pi\}, \qquad u = 0 \quad \text{on} \quad \partial Q^+.$$

Show that $u = const \cdot \sinh x_1 \cdot \sin x_2$ in Q^+.

Solution. Denote $v := \sinh x_1 \cdot \sin x_2$. From Statement 2 it follows that as $R \nearrow \infty$,

$$m(R) := \inf_{Q^+\cap\{x_1<R\}} \frac{u}{v} = \inf_{Q^+\cap\{x_1=R\}} \frac{u}{v} \searrow m(\infty) = \text{const} \ge 0,$$
$$M(R) := \sup_{Q^+\cap\{x_1<R\}} \frac{u}{v} = \sup_{Q^+\cap\{x_1=R\}} \frac{u}{v} \nearrow M(\infty) \le \infty.$$

Applying Corollary 3.14, we obtain $M(R) \le N \cdot m(R)$. Replacing u by $u - m(\infty) \cdot v$, we reduce the proof to the case when $m(\infty) = 0$. Finally,

$$0 \le \sup_{Q^+} \frac{u}{v} = \lim_{R\to\infty} M(R) \le N \cdot \lim_{R\to\infty} m(R) = 0,$$

i.e. $u \equiv m(\infty) \cdot v$.

4. Let $u \in C^2(\overline{Q})$ satisfy

$$u > 0, \quad \Delta u = 0 \quad \text{in} \quad Q := \{x = (x_1, x_2) \in \mathbb{R}^2 : -\infty < x_1 < \infty,\, 0 < x_2 < \pi\}, \qquad u = 0 \quad \text{on} \quad \partial Q.$$

Show that $u = (c_1 e^{x_1} + c_2 e^{-x_1}) \sin x_2$ in Q, where c_1 and c_2 are non-negative constants.

Solution Denote $v_1 := e^{x_1} \sin x_2,\ v_2 := e^{-x_1} \sin x_2$. As in the previous proof, we have

$$M_1(R) := \sup_{Q \cap \{x_1 = R\}} \frac{u}{v_1} \le N \cdot m_1(R) := N \cdot \inf_{Q \cap \{x_1 = R\}} \frac{u}{v_1}.$$

Replacing u by $u + v_2$, we can assume that $M_1(R) \to +\infty$ as $R \to -\infty$. Therefore,

$$\inf_{Q \cap \{x_1 \le R\}} \frac{u}{v_1} = m_1(R) \searrow m(\infty) = \text{const} \ge 0 \quad \text{as} \quad R \nearrow \infty.$$

Replacing u by $u - m(\infty) \cdot v_1$, we reduce the proof to the case $m(\infty) = 0$. The rest of the proof is left to the reader. It is very similar to the proof of the previous statement.

You used the following facts:

Statement 1. Let $\omega(\rho)$ be a non-negative, non-decreasing function on an interval $(0, \rho_0]$, such that

$$\omega(q^{-1}\rho) \le q^{-\alpha} \omega(\rho) \quad \text{for all} \quad \rho \in (0, \rho_0], \quad \text{where} \quad q = \text{const} > 1.$$

Then

$$\omega(\rho) \le \left(\frac{q\rho}{\rho_0}\right)^{\alpha} \omega(\rho_0) \quad \text{for all} \quad \rho \in (0, \rho_0].$$

Proof of this Statement is similar to deriving of (3.15) from (3.17) in Lecture Notes.

Statement 2. Let Ω be a bounded open set in $\mathbb{R}^n$, and let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$v > 0. \quad Lu = Lv = 0 \quad \text{in} \quad \Omega; \qquad u = 0 \quad \text{on} \quad \Gamma \subset \partial\Omega.$$

Then

$$M := \sup_{\Omega} \frac{u}{v} \le M_0 := \sup_{(\partial\Omega) \setminus \Gamma} \frac{u}{v}.$$

Proof. By definition of M_0,

$$u \le M_0 v \quad \text{on} \quad (\partial\Omega) \setminus \Gamma.$$

Since $u = 0$ on Γ, this inequality holds on the whole boundary $\partial\Omega$, and by the comparison principle, in Ω.