

## Math 8651: Theory of Probability Including Measure Theory: Fall 2004

### Drill Problems for Midterm Exam

Midterm Exam is scheduled on *Wednesday, October 27*, during the class time: 10:10 am – 11:00 am.

No books. Calculators and notes are permitted. The exam will contain 5 problems, two or three of which will be chosen out of the following list.

1. Fix  $p = \text{const} \in (0, 1)$ , and set

$$P_n(k) := \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for integers } 0 \leq k \leq n.$$

(i) Show that

$$\sum_{k=0}^n k P_n(k) = np, \quad \sum_{k=0}^n k(k-1) P_n(k) = n(n-1)p^2.$$

(ii) For each  $\varepsilon > 0$ , show that

$$\sum_{k: |k-pn| > \varepsilon n} P_n(k) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

*Hint:* The last sum does not exceed  $\sum (\varepsilon n)^{-2} (k-pn)^2 P_n(k)$ .

2. Let  $f$  be a function on  $\Omega$  with values in  $\mathbb{R}^1$ . For each subset  $A \subseteq \Omega$ , define

$$f(A) := \{y \in \mathbb{R}^1 : y = f(x) \text{ for some } x \in A\}.$$

Verify which of the following relations are true, and for each of them, either prove it or give a counter-example: for  $A, B \subseteq \Omega$ ,

$$f(A \cup B) = f(A) \cup f(B), \quad f(AB) = f(A) \cdot f(B), \quad f(A^c) = (f(A))^c.$$

3. (**This problem is excluded from the Midterm Exam**). Let  $A$  be a Borel measurable set in  $\mathbb{R}^1$ . Show that the set

$$F := \{x \in \mathbb{R}^1 : \lim_{\varepsilon \rightarrow 0^+} \frac{\lambda(A \cap (x - \varepsilon, x + \varepsilon))}{2\varepsilon} = I_A(x)\}$$

is such that its complement  $E := F^c$  has Lebesgue measure  $\lambda(E) = 0$ . In other words, the limit  $= I_A$  almost everywhere (a.e.) in  $\mathbb{R}^1$ . Here  $I_A$  is the indicator function of  $A$ :

$$I_A(x) = 1 \quad \text{if } x \in A, \quad \text{and} \quad I_A(x) = 0 \quad \text{if } x \in A^c.$$

(over)

4. Let  $f(x)$  be a non-decreasing, continuous function on  $\mathbb{R}^1$  such that  $f(0) = 0$ ,

$$f(x + j) \equiv f(x) + j \quad \text{for all } x \in \mathbb{R}^1 \quad \text{and} \quad j \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\},$$

and there exists a Borel set  $E_1 \subseteq [0, 1]$  of Lebesgue measure  $\lambda(E_1) = 0$ , such that the set

$$f(E_1) := \{y \in \mathbb{R}^1 : y = f(x) \text{ for some } x \in E_1\}$$

has Lebesgue measure  $\lambda(f(E_1)) = 1$ . Show that the function

$$g(x) := \sum_{k=1}^{\infty} 4^{-k} f(2^k x)$$

is strictly increasing on  $\mathbb{R}^1$  (i.e.  $g(x) < g(y)$  for  $x < y$ ),

$$g(x + j) \equiv g(x) + j \quad \text{for all } x \in \mathbb{R}^1 \quad \text{and} \quad j \in \mathbb{Z},$$

and there exists a Borel set  $E \subseteq [0, 1]$  of Lebesgue measure  $\lambda(E) = 0$ , such that the set  $\lambda(g(E)) = 1$ .

*Hint.* You can use (without proof) the regularity property, which implies that for any Borel (or even Lebesgue) set  $S$  of measure  $\lambda(S) = 0$ , and any  $\varepsilon > 0$ , there exists an open set  $O_\varepsilon \supseteq S$  of measure  $\lambda(O_\varepsilon) < \varepsilon$ . In turn, any open set in  $\mathbb{R}^1$  is represented as a countable union of disjoint intervals.