Math 8651: Theory of Probability Including Measure Theory: Fall 2004

Homework Assignment 1 (due on Wednesday, September 29, till 11:00 am)

50 points are distributed between 5 problems, 10 points each.

1. Using the *binomial formula* $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, show that for integers $n \geq 2$,

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \ldots \pm n\binom{n}{n} = 0.$$

2**(i).** Show that for arbitrary sets $A, B, C \subseteq \Omega$, the *symmetric difference*

$$A \triangle B := (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

satisfies the property $A \triangle B \subseteq (A \triangle C) \cup (B \triangle C)$.

(ii). Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $A, B, C \in \mathcal{F}$. Show that

$$|P(A) - P(B)| \leq P(A \triangle B) \leq P(A \triangle C) + P(B \triangle C).$$

3. Let $\mathcal{E}$ be a family of subsets of a set Ω, i.e. $\mathcal{E} \subseteq 2^{\Omega}$. Show that for arbitrary set $B \in \sigma(\mathcal{E})$, there exists a countable subfamily $\mathcal{E}_B := \{B_1, B_2, \ldots, B_n, \ldots\} \subseteq \mathcal{E}$ (depending on B), such that $B \in \sigma(\mathcal{E}_B)$.

4. Let A be a Borel subset of the interval $[-1, 1]$, and its Lebesgue measure $\lambda(A) > 1$. Show that for some $x \in A$, the point $x + 1$ also belongs to A.

5. Let $f(x)$ be an arbitrary bounded function on the interval $(0, 1)$. Show that the set

$$A := \{x \in (0,1) : \exists \lim_{y \to x} f(y) = f(x)\}$$

is a Borel set.

Hint. For each $n = 1, 2, \cdots$, the interval $[0, 1)$ is represented as the union of disjoint intervals:

$$[0,1) = \bigcup_{k=1}^{2^n} I_{n,k}, \quad \text{where } I_{n,k} := [(k-1)2^{-n}, k2^{-n}).$$

Consider piecewise constant functions $\overline{f}_n$ and $\underline{f}_n$, which are defined by the equalities

$$\overline{f}_n(x) = \sup_{I_{n,k}} f, \quad \underline{f}_n(x) = \inf_{I_{n,k}} f, \quad \text{for} \quad x \in I_{n,k}.$$

The set $Z = \{k2^{-n} : n = 1, 2, \cdots; k = 1, 2, \ldots, 2^n\}$ is countable, and for each $x \in (0,1) \setminus Z$, we have

$$x \in A \iff \overline{f}_n(x) - \underline{f}_n(x) \to 0 \text{ as } n \to \infty.$$