Math 8651: Theory of Probability Including Measure Theory: Fall 2004

Homework 1. Problems and Solutions

1. Using the *binomial formula* $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, show that for integers $n \geq 2$,

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \ldots \pm n\binom{n}{n} = 0.$$

Solution. Note that

$$k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = n \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = n \cdot \binom{n-1}{k-1} \quad \text{for } k = 1, 2, \ldots, n.$$

Hence

$$\begin{aligned} \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \ldots \pm n\binom{n}{n} &= n \cdot \left[\binom{n-1}{0} - \binom{n-1}{1} + \binom{n-1}{2} - \ldots \pm \binom{n-1}{n}\right] \\ &= n \cdot [1 + (-1)]^{n-1} = 0. \end{aligned}$$

Alternatively, one can get same result by differentiating both parts of the equality

$$(1-t)^n = 1 - \binom{n}{1}t + \binom{n}{2}t^2 + \ldots \pm \binom{n}{n}t^n$$

at the point $t = 1$.

2(i). Show that for arbitrary sets $A, B, C \subseteq \Omega$, the *symmetric difference*

$$A \triangle B := (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

satisfies the property $A \triangle B \subseteq (A \triangle C) \cup (B \triangle C)$.

(ii). Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $A, B, C \in \mathcal{F}$. Show that

$$|P(A) - P(B)| \leq P(A \triangle B) \leq P(A \triangle C) + P(B \triangle C).$$

Solution. **(i)** We have

$$\begin{aligned} A \setminus B &= AB^c = A(C^c \cup C)B^c = (AC^cB^c) \cup (ACB^c) \\ &\subseteq (AC^c) \cup (CB^c) = (A \setminus C) \cup (C \setminus B). \end{aligned}$$

Interchanging A and B, we also get $B \setminus A \subseteq (B \setminus C) \cup (C \setminus A)$, and

$$\begin{aligned} A \triangle B &= (A \setminus B) \cup (B \setminus A) \subseteq (A \setminus C) \cup (C \setminus B) \cup (B \setminus C) \cup (C \setminus A) \\ &= [(A \setminus C) \cup (C \setminus A)] \cup [(B \setminus C) \cup (C \setminus B)] = (A \triangle C) \cup (B \triangle C). \end{aligned}$$

(ii) Compare with the proof of Corollary 11, Ch. 1, in Lecture Notes.

3. Let $\mathcal{E}$ be a family of subsets of a set Ω, i.e. $\mathcal{E} \subseteq 2^{\Omega}$. Show that for arbitrary set $B \in \sigma(\mathcal{E})$, there exists a countable subfamily $\mathcal{E}_B := \{B_1, B_2, \ldots, B_n, \ldots\} \subseteq \mathcal{E}$ (depending on B), such that $B \in \sigma(\mathcal{E}_B)$.

Solution. Introduce the family

$$\mathcal{P} := \{B : \ B \in \sigma(\mathcal{E}), \text{ and } \exists \text{ a countable subfamily } \mathcal{E}_B \subseteq \mathcal{E}, \text{ such that } B \in \sigma(\mathcal{E}_B)\}.$$

Each set $E \in \mathcal{E}$ also belongs to $\mathcal{P}$, because we can take $\mathcal{E}_E := \{E\}$ (the family $\mathcal{E}_E$ contains just one member E). In this case, $\sigma(\mathcal{E}_E) = \sigma(E) = \{\emptyset, \Omega, E, E^c\}$, and $E \in \sigma(\mathcal{E}_E)$. Therefore, $\mathcal{P} \supseteq \mathcal{E}$.

If we show that $\mathcal{P}$ is a σ-field, then also $\mathcal{P} \supseteq \sigma(\mathcal{E})$ – the minimal σ-field containing $\mathcal{E}$. Since $\mathcal{P} \subseteq \sigma(\mathcal{E})$, we must have $\mathcal{P} \subseteq \sigma(\mathcal{E})$, i.e. the desired property holds for all $B \in \sigma(\mathcal{E})$.

In order to show that $\mathcal{P}$ is a σ-field, we need to check the three properties:

(i) $\emptyset \in \mathcal{P}$. This property is obvious, because $\emptyset$ belongs to $\sigma(E)$ generated by an arbitrary $E \in \mathcal{E}$.

(ii) $A \in \mathcal{P} \Longrightarrow A^c \in \mathcal{P}$. If $A \in \mathcal{P}$, then $\exists$ a countable $\mathcal{E}_A \subseteq \mathcal{E}$ such that $A \in \sigma(\mathcal{E}_A)$. Since $\sigma(\mathcal{E}_A)$ is a σ-field, we also have $A^c \in \sigma(\mathcal{E}_A)$, i.e. $A^c \in \mathcal{P}$.

(iii') A countable sequence $\{A_n\} \subseteq \mathcal{P} \Longrightarrow \cup A_n \in \mathcal{P}$. For each $n = 1, 2, \ldots$, $\exists$ a countable $\mathcal{E}_n$ such that $A_n \in \sigma(\mathcal{E}_n)$. For $A := \cup A_n$, define $\mathcal{E}_A := \cup \mathcal{E}_n$. Note that $\mathcal{E}_A$ is countable, because it is a countable union of countable sets. For each n, from $\mathcal{E}_n \subseteq \mathcal{E}_A$ it follows $A_n \in \sigma(\mathcal{E}_n) \subseteq \sigma(\mathcal{E}_A)$. Since $\sigma(\mathcal{E}_A)$ is a σ-field, we also have $A = \cup A_n \in \sigma(\mathcal{E}_A)$, and $A \in \mathcal{P}$.

4. Let A be a Borel subset of the interval $[-1, 1]$, and its Lebesgue measure $\lambda(A) > 1$. Show that for some $x \in A$, the point $x + 1$ also belongs to A.

Solution. By additivity of λ, we have

$$\lambda(A) = \lambda(A_0) + \lambda(A_1), \quad \text{where} \quad A_0 := A \cap [-1, 0], \quad A_1 := A \cap (0, 1].$$

Since λ is translation invariant, the set $A_2 := 1 + A_0 := \{1 + x : x \in A_0\} \subseteq [0, 1]$ has measure $\lambda(A_2) = \lambda(A_0)$. From $A_1 \cup A_2 \subseteq [0, 1]$ it follows $\lambda(A_1 \cup A_2) \le \lambda([0, 1]) = 1$. The sets A_1 and A_2 cannot be disjoint, because this would imply

$$1 \ge \lambda(A_1 \cup A_2) = \lambda(A_1) + \lambda(A_2) = \lambda(A_1) + \lambda(A_0) = \lambda(A),$$

in contradiction to our assumption $\lambda(A) > 1$. Hence $\exists$ a point $a \in A_1 A_2$. Since $a \in A_2 = 1 + A_0$, we can write $a = 1 + x$ with $x \in A_0 \subseteq A$, and $a = 1 + x \in A_1 \subseteq A$.

5. Let $f(x)$ be an arbitrary bounded function on the interval $(0, 1)$. Show that the set

$$A := \{x \in (0, 1) : \exists \lim_{y \to x} f(y) = f(x)\}$$

is a Borel set.

Solution. For each $n = 1, 2, \cdots$, the interval $[0, 1)$ is represented as the union of disjoint intervals:

$$[0, 1) = \bigcup_{k=1}^{2^n} I_{n,k}, \quad \text{where} \quad I_{n,k} := [(k-1)2^{-n}, k2^{-n}).$$

Consider piecewise constant functions $\overline{f}_n$ and $\underline{f}_n$, which are defined by the equalities

$$\overline{f}_n(x) = \sup_{I_{n,k}} f, \quad \underline{f}_n(x) = \inf_{I_{n,k}} f, \quad \text{for} \quad x \in I_{n,k}.$$

The set $Z = \{k2^{-n} : n = 1, 2, \cdots; k = 1, 2, \ldots, 2^n\}$ is countable, and for each $x \in (0, 1) \setminus Z$, we have

$$x \in A \iff f_n(x) := \overline{f}_n(x) - \underline{f}_n(x) \to 0 \quad \text{as} \quad n \to \infty.$$

By Example 1 in Lecture Notes, Sec. 1.1, the set

$$A_0 := \{x \in (0, 1) : \exists \lim_{n \to \infty} f_n(x) = 0\} = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_{n,k},$$

where $A_{n,k} := \{x \in (0, 1) : |f_n(x)| < k^{-1}\}$. Since f_n is a piecewise constant functions, each set $A_{n,k}$ is a finite union of intervals, and therefore it is a Borel set ($A_{n,k} \in \mathcal{B}$). The set $A_0 \in \mathcal{B}$, because it is a result of countably many operations on sets in $\mathcal{B}$. At the points $z \in Z$, the convergence $f_n(z) \to 0$ implies only the existence of the one-sided limits $f(z + 0) := \lim_{y \to z^+} f(y) = f(z)$, so that we must exclude a subset $Z_0 \subseteq Z$ of points z, for which $f(z - 0) := \lim_{y \to z^-} f(y)$ either does not exist, or $f(z - 0) \ne f(z)$. The set $Z_0 \in \mathcal{B}$ because it is countable. Finally, $A = A_0 \setminus Z_0 \in \mathcal{B}$.