SIAM J. Appl. Math.
Vol. 66, No. 1, pp. 187–205

AN APPROXIMATE METHOD FOR SCATTERING BY THIN STRUCTURES*

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Abstract. Scattering of waves by a thin structure is considered in this work. The Helmholtz equation with variable coefficient models the wave phenomena. The scatterer is assumed to have a high index of refraction while at the same time it is very small in one of the dimensions. We show that if the index scales as $O(1/h)$, where h is the thickness of the scatterer, then an approximate solution, based on perturbation analysis, can be obtained. The approximate solution consists of a leading order term plus a corrector, each of which solves an integral equation in two dimensions for a three-dimensional problem. We provide error analysis on the approximation. The approximate method can be viewed as an efficient computational approach since it can potentially greatly simplify scattering calculations. Numerical results provide an assessment of the accuracy of the approximate solution.

Key words. scattering, Helmholtz equation, approximate solution, asymptotics, error estimates

AMS subject classifications. 65R20, 78A45, 45E99, 34E10, 78M99

DOI. 10.1137/040617388

1. Introduction.

The problem under investigation arises in the study of photonic band gap (PBG) structures. Optical devices that exploit photonic band gap phenomena to guide and manipulate light are expected to play an important role in optical communication networks and optical computing. Thin-film or membrane devices are particularly attractive because of the relative ease with which they can be made.

A typical thin-film device is made of a material with a high index of refraction. The high index is needed to confine light within the structure. To manipulate light within the structure, holes are drilled into film. Typical structures under study can be found in several recent papers [3, 5, 6].

In order to simulate how light behaves in such a structure, it is necessary to solve the wave equation. In most studies, the structure is surrounded by air. Thus, the domain in which the wave equation must be solved will be all of $\mathbb{R}^3$. The thin film structure can be modeled by prescribing index of refraction to a subdomain of $\mathbb{R}^3$.

The classical approach to performing the required simulation of wave propagation in such a complicated structure is the finite-difference time-domain (FDTD) method [2], with absorbing boundary conditions. While the computation proceeds in a straightforward manner, it is very time consuming.

In this paper, we propose an approximate method to solve the scattering problem. The method starts with the time-harmonic wave equations and applies a perturbation approach based on an identified small parameter. The advantage of our method is that it reduces the complexity of the computation by one dimension. The Lippmann–Schwinger formulation of the scattering problem will involve a three-dimensional

*Received by the editors October 20, 2004; accepted for publication (in revised form) May 16, 2005; published electronically October 17, 2005.

http://www.siam.org/journals/siap/66-1/61738.html

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(3-D) (volume) integral equation. Our method simplifies the calculation to solving a sequence of two-dimensional (2-D) integral equations.

The present work addresses only the case of the scalar wave equation. Maxwell's equation, which is the correct model for the propagation phenomena, will be treated in a separate, future work.

This paper is organized as follows. We give a description of the problem we wish to solve in the next section. The perturbation approach is presented in section 3. Justification of the approximate method follows in sections 4 and 5. Section 6 contains numerical examples in two dimensions. The paper closes with a discussion.

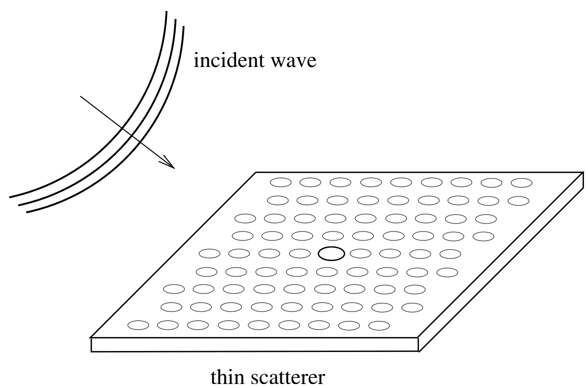

Fig. 2.1. *Scattering by a thin structure.*

2. Problem statement. The situation we are attempting to model is depicted in Figure 2.1. The propagation of waves is modeled by the Helmholtz equation

$$\Delta u + k^2 \epsilon(x,z) u = 0, \tag{2.1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $z \in \mathbb{R}$. In (2.1) we call $\epsilon(x,z)$ the dielectric constant (an abuse in terminology), which will be set to unity in air and to some value in the structure. The field $u(x,z)$ is of normalized real frequency k and comprises two components:

$$u(x,z) = u_{\mathrm{i}}(x,z) + u_{\mathrm{s}}(x,z),$$

where the incident wave u_{i} is given, and the scattered field u_{s} satisfies the Sommerfeld radiation conditions

$$\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}} = O\left(\frac{1}{r^2}\right) \quad \text{for } r = \sqrt{|x|^2 + z^2} \to \infty.$$

The thin structure is incorporated into the definition of $\epsilon(x,z)$. Let $\Omega \times (-h/2, h/2)$ be the region occupied by the thin structure, where Ω is a bounded domain in $\mathbb{R}^2$. Then the dielectric constant is defined by

$$\epsilon(x,z) = \begin{cases} 1 & \text{for} \quad |z| > h/2, \\ \epsilon_0(x)/h & \text{for} \quad |z| < h/2, x \in \Omega, \\ 1 & \text{for} \quad |z| < h/2, x \notin \Omega. \end{cases} \tag{2.2}$$

Thus, we have assumed that $\epsilon_0(x)$ is supported in Ω.

The problem we wish to solve is to find the scattered field $u_{\rm s}(x,z)$ given a thin structure (2.2) and an incident field $u_{\rm i}(x,z)$. For the purpose of simulating waves in PBG structures, the total field $u(x,z)$ for (x,z) in the thin structure, $\Omega \times (-h/2, h/2)$, is of great interest.

3. Perturbation approach. One can rewrite the problem for $u(x,z)$ in (2.1) as an integral equation [1] by observing that since

$$\Delta u_{\rm i} + k^2 u_{\rm i} = 0,$$

we have that

$$\Delta u_{\rm s} + k^2 u_{\rm s} = k^2(1-\epsilon)u.$$

If $G(x,z,x',z')$ is the free space Green's function for the Helmholtz equation in $\mathbb{R}^3$, i.e., G satisfies

$$(\Delta_{(x',z')} + k^2)G = \delta_{(x,z)}$$

in $\mathbb{R}^3$ with the Summerfeld radiation condition, then by using integration by parts and the decay at infinity, we get

$$u_{\rm s}(x,z) = k^2 \int_\Omega \int_{-h/2}^{h/2} \left(1 - \frac{\epsilon_0(x')}{h}\right) G(x,z,x',z')u(x',z')dz'dx'.$$

Hence we have that the field satisfies the well-known Lippmann–Schwinger integral equation

$$u(x,z) = u_{\rm i}(x,z) + k^2 \int_\Omega \int_{-h/2}^{h/2} \left(1 - \frac{\epsilon_0(x')}{h}\right) G(x,z,x',z')u(x',z')dz'dx'. \tag{3.1}$$

We note that $G(x,z,x',z')$ is given by

$$G(x,z,x',z') = \frac{1}{4\pi} \frac{e^{ik\sqrt{|x-x'|^2+|z-z'|^2}}}{\sqrt{|x-x'|^2+|z-z'|^2}}.$$

See [1] for the full proof of equivalence. To solve for the field, we need to view (3.1) as an integral equation satisfied by $u(x,z)$ for $(x,z) \in \Omega \times (-h/2, h/2)$. Once we have found $u(x,z)$ for (x,z) in the thin domain, we can then use (3.1) as a way to compute the field outside the thin domain. Therefore, our first step will be to find an asymptotic approximation for u inside the thin region.

To find a first order approximation, we will scale the variable in the z direction, $z = h\zeta$, so that the integral equation is now

$$u(x,\zeta) = u_{\rm i}(x,h\zeta) + k^2 \int_\Omega \int_{-1/2}^{1/2} \left(1 - \frac{\epsilon_0(x')}{h}\right) G(x,h\zeta,x',h\zeta')u(x',\zeta')hd\zeta'dx', \quad (x,\zeta) \in \Omega \times (-1/2, 1/2). \tag{3.2}$$

Formally, we assume a perturbation series ansatz

$$u(x,\zeta) = u_0(x) + hu_1(x,\zeta) + \cdots, \quad (x,\zeta) \in \Omega \times [-1/2, 1/2]. \tag{3.3}$$

The goal is now to obtain equations by which $u_0(x,z)$ and $u_1(x,z)$ can be found.

Substituting (3.3) into (3.2), we see that

$$u_0(x) + hu_1(x,\zeta) + \cdots = u_{\mathrm{i}}(x,0) + h\frac{\partial u_{\mathrm{i}}}{\partial z}(x,0) + O(h^2) \tag{3.4}$$
$$+ k^2 \int_\Omega \int_{-1/2}^{1/2} (h - \epsilon_0(x'))G(x,h\zeta,x',h\zeta')\left[u_0(x') + hu_1(x',\zeta') + \cdots\right] d\zeta' dx'.$$

The classical way to find u_0 and u_1 is to equate like powers of h on both sides of (3.4). However, the situation is complicated by the fact that $G(x,z,x',z')$ is singular.

We make the observation that if the integral

$$\int_{-1/2}^{1/2} G(x,h\zeta,x',h\zeta')d\zeta'$$

converges as $h \to 0$ to

$$G(x,0,x',0),$$

then, setting equal like powers of h in (3.4), we see that $u_0(x)$ satisfies the integral equation

$$u_0(x) = u_{\mathrm{i}}(x,0) - k^2 \int_\Omega \epsilon_0(x')G(x,0,x',0)u_0(x')dx'. \tag{3.5}$$

We will justify this step in the next section and show that $u_1(x,z)$ can be calculated, as well as justified, in section 5.

Once we have solved for $u_0(x)$ and $u_1(x,z)$, we can insert them in the right-hand side of (3.1) to obtain an approximation of the field for all points outside the thin domain. Note that the equation for u_0 and, as we shall see, that for u_1 are 2-D integral equations. Therefore, in terms of computational cost we have reduced the dimension of the problem by one.

4. Justification for the first term. Now we provide a rigorous error estimate for the approximation (3.5) derived above. The solution $u(x,z)$ satisfies

$$u(x,z) = u_{\mathrm{i}}(x,z) + k^2 \int_\Omega \int_{-h/2}^{h/2} \left(1 - \frac{\epsilon_0(x')}{h}\right) G(x,z,x',z')u(x',z')dz'dx'. \tag{4.1}$$

The candidate for an approximation to u on the thin strip to substitute into (3.1) is $u_0(x)$, the solution to the lower dimensional problem:

$$u_0(x) = u_{\mathrm{i}}(x,0) - k^2 \int_\Omega \epsilon_0(x')G(x,0,x',0)u_0(x')dx'. \tag{4.2}$$

To show this is a good approximation, let

$$\zeta = z/h \quad \text{and} \quad \tilde{u}(x,\zeta) = u(x,z),$$

so that

$$\tilde{u}(x,\zeta) = u_{\mathrm{i}}(x,h\zeta) + k^2 \int_\Omega \int_{-1/2}^{1/2} \left(h - \epsilon_0(x')\right) G(x,h\zeta,x',h\zeta')\tilde{u}(x',\zeta')d\zeta'dx'. \tag{4.3}$$

For convenience define S to be the scaled strip

$$S = \Omega \times (-1/2, 1/2).$$

We will show the following uniform norm estimate.

PROPOSITION 1. *There exists a constant C independent of h (but depending on k) such that*

$$\|u_0(x) - \tilde{u}(x,\zeta)\|_{L^\infty(S)} \le Ch.$$

Using (4.2) and (4.3) and interchanging the order of integration, we can write

$$\tilde{u}(x,\zeta) - u_0(x) = u_{\mathrm{i}}(x, h\zeta) - u_{\mathrm{i}}(x,0) + hk^2 \int_\Omega \int_{-1/2}^{1/2} G(x, h\zeta, x', h\zeta')\tilde{u}(x',\zeta')d\zeta' dx'$$
$$+ k^2 \int_{-1/2}^{1/2} \int_\Omega \epsilon_0(x') \left[G(x,0,x',0)u_0(x') - G(x,h\zeta,x',h\zeta')\tilde{u}(x',\zeta')\right] dx' d\zeta'.$$

We add and subtract appropriate terms to obtain

$$\tilde{u}(x,\zeta) - u_0(x) = u_{\mathrm{i}}(x, h\zeta) - u_{\mathrm{i}}(x,0) + hk^2 \int_\Omega \int_{-1/2}^{1/2} G(x, h\zeta, x', h\zeta')\tilde{u}(x',\zeta')d\zeta' dx'$$
$$+ k^2 \int_{-1/2}^{1/2} \int_\Omega \epsilon_0(x')\tilde{u}(x',\zeta') \left[G(x,0,x',0) - G(x,h\zeta,x',h\zeta')\right] dx' d\zeta'$$
$$+ k^2 \int_{-1/2}^{1/2} \int_\Omega \epsilon_0(x') G(x,0,x',0) \left[u_0(x') - \tilde{u}(x',\zeta')\right] dx' d\zeta'.$$

For a given $\epsilon_0 \in L^\infty(\Omega)$ which is also piecewise continuous, define the integral operator

$$T : L^2(S) \to L^2(S)$$

by

$$T(f) = \int_{-1/2}^{1/2} \int_\Omega \epsilon_0(x') G(x,0,x',0) f(x',\zeta') dx' d\zeta'.$$

By an abuse of notation, we will also use T to denote the same integral operator on the space of continuous functions, $C^0(\overline{S})$, equipped with the L^∞ norm:

$$T : C^0(\overline{S}) \to C^0(\overline{S}).$$

(Note that $T(f)$ will always be independent of ζ, so the range of T is really only functions on Ω.) Then $\tilde{u} - u_0$ satisfies

$$(I + k^2 T)(\tilde{u} - u_0) = u_{\mathrm{i}}(x, h\zeta) - u_{\mathrm{i}}(x,0) + hk^2 \int_\Omega \int_{-1/2}^{1/2} G(x, h\zeta, x', h\zeta')\tilde{u}(x',\zeta')d\zeta' dx'$$
$$+ k^2 \int_{-1/2}^{1/2} \int_\Omega \epsilon_0(x')\tilde{u}(x',\zeta') \left[G(x,0,x',0) - G(x,h\zeta,x',h\zeta')\right] dx' d\zeta'. \tag{4.4}$$

The lemmas that follow are used to bound the right-hand side of (4.4) and to show that we can invert $(I + k^2 T)$.

LEMMA 1. *There exists a constant C independent of h and ζ' but depending on k such that*

$$\sup_{(x,\zeta)\in S}\int_\Omega |G(x,0,x',0)-G(x,h\zeta,x',h\zeta')|dx' \le Ch.$$

Proof. The difference of these Green's functions can be written as

$$\begin{aligned}
&G(x,0,x',0)-G(x,h\zeta,x',h\zeta')\\
&=\frac{1}{4\pi}\frac{e^{ik|x-x'|}}{|x-x'|}-\frac{1}{4\pi}\frac{e^{ik\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}}{\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}\\
&=\frac{1}{4\pi}e^{ik|x-x'|}\left[\frac{1}{|x-x'|}-\frac{1}{\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}\right]\\
&\quad+\frac{1}{4\pi}\frac{1}{\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}\left[e^{ik|x-x'|}-e^{ik\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}\right]. \qquad (4.5)
\end{aligned}$$

We first work on the second term on the right-hand side of (4.5). Since we know that for $(x,\zeta)\in S$,

$$\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}-|x-x'|\le h,$$

there exists a constant C independent of h and ζ' but depending on (real) k such that

$$|e^{ik|x-x'|}-e^{ik\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}|\le Ch. \qquad (4.6)$$

Since

$$\frac{1}{\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}\le\frac{1}{|x-x'|},$$

which is integrable with respect to x' on Ω, we have that

$$\int_\Omega\frac{dx'}{\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}$$

is bounded independently of h, ζ', and $(x,z)\in S$. This along with (4.6) gives that we can choose C independent of h, ζ', and $(x,\zeta)\in S$ such that

$$\int_\Omega\frac{1}{4\pi}\frac{1}{\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}|e^{ik|x-x'|}-e^{ik\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}|dx'\le Ch.$$

The integral of the first term on the right-hand side of (4.5) can be bounded as follows:

$$\begin{aligned}
&\int_\Omega\left|\frac{1}{4\pi}e^{ik|x-x'|}\left[\frac{1}{|x-x'|}-\frac{1}{\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}\right]\right|dx'\\
&\le\int_\Omega\left|\frac{1}{|x-x'|}-\frac{1}{\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}\right|dx'\\
&=\int_\Omega\left(\frac{1}{|x-x'|}-\frac{1}{\sqrt{|x-x'|^2+h^2|\zeta-\zeta'|^2}}\right)dx'
\end{aligned}$$

since the integrand is nonnegative. Now choose R large enough so that if $B_R(x)$ is the ball of radius R centered at x in $\mathbb{R}^2$,

$$\Omega \subset B_R(x)$$

for all $x \in \Omega$. Then the above is bounded by

$$\int_{B_R(x)} \left(\frac{1}{|x-x'|} - \frac{1}{\sqrt{|x-x'|^2 + h^2|\zeta - \zeta'|^2}} \right) dx'.$$

Change to polar coordinates centered at x with

$$r = |x - x'|.$$

The integral transforms to

$$\begin{aligned} &2\pi \int_0^R \left(\frac{1}{r} - \frac{1}{\sqrt{r^2 + h^2|\zeta-\zeta'|^2}} \right) r dr \\ &= 2\pi \left[R - \sqrt{R^2 + h^2|\zeta - \zeta'|^2} + h|\zeta - \zeta'| \right] \end{aligned}$$

by direct calculation. This is then $O(h)$, where the constant is independent of $(x,\zeta) \in S$ and $\zeta' \in (-1/2, 1/2)$. This, combined with the bounds on the first integral, proves the lemma. □

Recall that our scaled domain S is given as

$$S = \Omega \times (-1/2, 1/2).$$

LEMMA 2. *Let $\epsilon_0(x)$ be piecewise continuous on Ω. Then the operator $T : L^2(S) \to L^2(S)$ given by*

$$(Tf)(x) = \int_S \epsilon_0(x') G(x, 0, x', 0) f(x', \zeta') dx' d\zeta',$$

where

$$G(x, 0, x', 0) = \frac{1}{4\pi} \frac{e^{ik|x-x'|}}{|x-x'|},$$

is compact. Moreover, if we view T on the Banach space of continuous functions,

$$T : C^0(\overline{S}) \to C^0(\overline{S}),$$

it is also a compact operator. Furthermore, $(I + k^2 T)$ is continuously invertible on both $L^2(S)$ and $C^0(\overline{S})$.

Proof. Since ϵ_0 is piecewise continuous, the kernel $\epsilon_0(x')G(x,0,x',0)$ is a finite sum of weakly singular kernels. Hence the fact that T is a compact operator on $C^0(\overline{S})$ follows from Theorem 1.11 of [1]. To show T is compact on $L^2(S)$, we will show that for any sequence $\{f_n\}$ such that $\|f_n\|_{L^2(S)} < M$ and $f_n \rightharpoonup 0$, the sequence $Tf_n \to 0$ in $L^2(S)$. Let

$$D := \{(x - x', \zeta - \zeta') \; ; \; (x, \zeta), (x', \zeta') \in S\}$$

and define

$$g(y,\eta) := e^{ik|y|}/|y| \;\text{ for }\; (y,\eta) \in D.$$

Then since $g \in L^1(D)$, there exists

$$g_m \in C^\infty(D) \quad \text{such that} \quad \|g - g_m\|_{L^1(D)} \to 0 \quad \text{as} \quad m \to \infty.$$

In the estimates that follow we use a standard result (see, for example, [7]). If $g \in L^1(D)$ and $f \in L^p(S)$ $(1 < p < \infty)$, then $(g * f) \in L^p(S)$ and

$$\|g * f\|_{L^p(S)} \le \|g\|_{L^1(D)} \|f\|_{L^p(S)}. \tag{4.7}$$

Now consider

$$\begin{aligned}
\|Tf_n\|^2_{L^2(S)} &= \int_S \left[\int_S \epsilon_0(x') G(x,0,x',0) f_n(x',\zeta') dx' d\zeta' \right]^2 dx d\zeta \\
&= \frac{1}{4\pi} \int_S \left[\int_S \epsilon_0(x') g(x-x', \zeta-\zeta') f_n(x',\zeta') dx' d\zeta' \right]^2 dx d\zeta.
\end{aligned}$$

Let $M_\epsilon = \|\epsilon_0\|_\infty$. By adding and subtracting g_m, we can bound the above by

$$\begin{aligned}
&\frac{M_\epsilon^2}{4\pi} \int_S \left[|g_m * f_n| + |(g - g_m) * f_n| \right]^2 dx d\zeta \\
&\le \frac{M_\epsilon^2}{2\pi} (\|g_m * f_n\|^2_{L^2(S)} + \|(g - g_m) * f_n\|^2_{L^2(S)}) \\
&\le \frac{M_\epsilon^2}{2\pi} (\|g_m * f_n\|^2_{L^2(S)} + \|g - g_m\|^2_{L^1(D)} M^2),
\end{aligned} \tag{4.8}$$

with the last inequality obtained by using (4.7) with $p = 2$. Also, since

$$|g_m * f_n| \le \|g_m\|_{L^\infty(D)} \|f_n\|_{L^1(S)}$$

and $\|f_n\|_{L^1(S)}$ is bounded, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \|g_m * f_n\|_{L^2(S)} = \| \lim_{n\to\infty} (g_m * f_n)\|_{L^2(S)} = 0$$

since $\{f_n\}$ goes to zero weakly. For any given $\varepsilon > 0$, choose m large enough so that

$$\|g - g_m\|^2_{L^1(D)} M^2 < \varepsilon.$$

Then for this m we can choose n large enough that

$$\|g_m * f_n\|^2_{L^2(S)} < \varepsilon$$

also. Hence from (4.8)

$$\|Tf_n\|^2_{L^2(S)} \le C\varepsilon.$$

Therefore, $\lim_{n\to\infty} \|Tf_n\|^2_{L^2(S)} = 0$, and hence T is compact on $L^2(S)$.

Now, by the Fredholm theory (Corollary 1.17 of [1]), $I + k^2T$ is invertible if $(I + k^2T)f = 0$ has only the zero solution. The following argument holds on the Banach space X for both $X = L^2(S)$ and $X = C^0(\overline{S})$. Let $f \in X$ be a solution to

$(I + k^2 T)f = 0$. Since for any $f \in X$, Tf depends only on $x \in \Omega$, the solution f satisfies

$$f = -k^2 T f$$

and hence also depends only on x. Let

$$w(x, z) = \int_\Omega \epsilon_0(x')G(x, z, x', 0)f(x')dx'.$$

By a slight abuse of notation, in what follows we will use Ω to denote $\Omega \times \{0\}$. Using the equation for G, its conditions at infinity, and standard arguments of single layer potential theory [4],

(a) $(\triangle + k^2)w(x, z) = 0$ in $\mathbb{R}^3 \setminus \Omega$,
(b) w satisfies the radiation condition,
(c) $[w]_\Omega = 0$,
(d) $\left[\frac{\partial w}{\partial z}\right]_\Omega = \epsilon_0 f$,

where $[\cdot]$ denotes the jump across Ω, i.e.,

$$[g] = \lim_{z\to 0^+} g(x, z) - \lim_{z\to 0^-} g(x, z).$$

Let B_R be any ball in $\mathbb{R}^3$ containing Ω, multiply through by $\overline{w}$, and integrate by parts to get

$$\int_{B_R\setminus\Omega} |\nabla w|^2 - k^2 \int_{B_R\setminus\Omega} |w|^2 - \int_{\partial B_R} \overline{w}\frac{\partial w}{\partial \nu} - \int_\Omega \overline{w}\left[\frac{\partial w}{\partial z}\right]_\Omega = 0.$$

Since $(I + k^2 T)f = 0$, we conclude that $f = -k^2 w$ on Ω. Substituting this and property (d) in the identity above, we get

$$\int_{B_R\setminus\Omega} |\nabla w|^2 - k^2 \int_{B_R\setminus\Omega} |w|^2 - \int_{\partial B_R} \overline{w}\frac{\partial w}{\partial \nu} + \frac{1}{k^2}\int_\Omega \epsilon_0|f|^2 = 0.$$

Hence

$$\operatorname{Im}\int_{\partial B_R} \overline{w}\frac{\partial w}{\partial \nu} = 0.$$

We can now use the Rellich lemma (see Theorem 3.12 of [1]) on any domain $U \in \mathbb{R}^3$ arbitrarily close to Ω to obtain $w = 0$ in $B_R \setminus U$. Hence $w = 0$ on $\mathbb{R}^3 \setminus \Omega$. Thus the jump $\left[\frac{\partial w}{\partial z}\right]_\Omega = 0$. By property (d), we can conclude that $f = 0$. □

Note an immediate corollary of the above lemma is that u_0 exists, is unique, and is in $C^0(\overline{S})$.

Proof of Proposition 1. Note that for each fixed h, the right-hand side of (4.4) is a continuous function. We take a Taylor expansion of the smooth incident wave $u_\mathbf{i}$ for the first term. We then invoke Lemma 1 for the third term to obtain the bound

$$\|(I + k^2T)(\tilde{u} - u_0)\|_{L^\infty(S)} \le h\|u_\mathbf{i}\|_{C^1(S)} + C_1 h\|\tilde{u}\|_{L^\infty(S)} + C_2 h\|\tilde{u}\|_{L^\infty(S)}.$$

By the boundedness of $(I + k^2T)^{-1}$, we have that

$$\begin{aligned}\|\tilde{u} - u_0\|_{L^\infty(S)} &\le C_3 h\|u_\mathbf{i}\|_{C^1(S)} + C_4 h\|\tilde{u}\|_{L^\infty(S)}\\ &\le C_3 h\|u_\mathbf{i}\|_{C^1(S)} + C_4 h\|\tilde{u} - u_0\|_{L^\infty(S)} + C_4 h\|u_0\|_{L^\infty(S)}.\end{aligned}$$

Since we know u_0 is continuous and $u_{\rm i}$ is bounded in C^1, we have that there exists a C_5 independent of h such that

$$(1 - C_4 h)\|\tilde{u} - u_0\|_{L^\infty(S)} \le C_5 h,$$

from which the result follows for h small enough. □

5. The next term and its justification. First we find formally an equation for u_1, the next term in the expansion. We begin with the ansatz and Taylor expansion

$$\tilde{u}(x,\zeta) = u_0(x) + h u_1(x,\zeta) + O(h^2),$$

$$u_{\rm i}(x, h\zeta) = u_{\rm i}(x,0) + h\zeta \frac{\partial u_{\rm i}}{\partial z}(x,0) + O(h^2),$$

and, in order to obtain some sort of expansion for G, we consider the function v_h,

$$v_h(x,z) = \int_S \epsilon_0(x') G(x,z,x',h\zeta') u_0(x') d\zeta' dx'.$$

From Lemma 1, it seems that as $h \to 0$, $v_h(x, h\zeta)$ should converge $O(h)$ to $v_0(x)$, where

$$v_0(x) = \int_\Omega \epsilon_0(x') G(x,0,x',0) u_0(x') dx'.$$

So it seems reasonable to define

$$v_1(x,\zeta) = \lim_{h\to 0} \frac{v_h(x,h\zeta) - v_0(x)}{h},$$

so that if the limit exists, we have

$$v_h(x,h\zeta) = v_0(x) + h v_1(x,\zeta) + o(h).$$

Insert these expansions into (4.3) and match like powers of h. The $O(1)$ terms give the equation for u_0. The terms of $O(h)$ yield

$$u_1(x,\zeta) = \zeta \frac{\partial u_{\rm i}}{\partial z}(x,0) - k^2 \int_\Omega \int_{-1/2}^{1/2} \epsilon_0(x') G(x,0,x',0) u_1(x,\zeta') d\zeta' dx'$$
$$+ k^2 \int_\Omega \int_{-1/2}^{1/2} G(x,0,x',0) u_0(x') d\zeta' dx' - k^2 v_1(x,\zeta).$$

We use the definition of the operator T defined in the last section to obtain

$$u_1(x,\zeta) = -k^2 T u_1 + \zeta \frac{\partial u_{\rm i}}{\partial z}(x,0) - k^2 v_1(x,\zeta) + k^2 \int_\Omega G(x,0,x',0) u_0(x') dx'. \tag{5.1}$$

To complete our definition of u_1 we need to find an expression for $v_1(x,\zeta)$.

LEMMA 3. *Given any fixed $(x,\zeta) \in S$ and any ρ small enough such that the 2-D ball around x of radius ρ, $B_{x,\rho}$, is contained in Ω, then*

$$\lim_{h\to 0} \frac{1}{h} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{B_{x,\rho}} [G(x,h\zeta,x',h\zeta') - G(x,0,x',0)] dx' d\zeta' = -\frac{1}{2}\left(\zeta^2 + \frac{1}{4}\right).$$

Proof. Before dividing by h, the integral above can be written as

$$
\begin{aligned}
I &= \frac{2\pi}{4\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^\rho \left(\frac{e^{ik\sqrt{r^2+h^2(\zeta-\zeta')^2}}}{\sqrt{r^2+h^2(\zeta-\zeta')^2}} - \frac{e^{ikr}}{r}\right) r dr d\zeta' \\
&= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{ik}\left(e^{ik\sqrt{\rho^2+h^2(\zeta-\zeta')^2}} - e^{ik\rho}\right) d\zeta' - \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{ik}\left(e^{ik|h(\zeta-\zeta')|} - 1\right) d\zeta' \\
&= I_1 + I_2.
\end{aligned}
$$

Since

$$
\frac{\partial}{\partial h}\left(e^{ik\sqrt{\rho^2+h^2(\zeta-\zeta')^2}}\right)
$$

is integrable, one can compute

$$
\lim_{h\to 0^+}\frac{1}{h}I_1 = \lim_{h\to 0^+}\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{ik}\frac{\partial}{\partial h}\left(e^{ik\sqrt{\rho^2+h^2(\zeta-\zeta')^2}}\right) d\zeta' = 0.
$$

The second term,

$$
\lim_{h\to 0^+}\frac{1}{h}I_2 = -\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} |(\zeta-\zeta')| d\zeta' = -\frac{1}{2}\left(\zeta^2 + \frac{1}{4}\right). \qquad \square
$$

Note that in the previous lemma, although the limit holds pointwise, it is not uniform as x approaches the boundary of Ω. It is this observation that will lead to a boundary correction which we will examine in a forthcoming paper. For shorthand, in what follows we use the notation

$$
G = G(x, h\zeta, x', h\zeta') \quad \text{and} \quad G_0 = G(x, 0, x', 0).
$$

PROPOSITION 2. *For any $g \in L^\infty(\Omega)$ such that $g \in C^0(B_{x,\rho})$ for some ρ small enough,*

$$
\lim_{h\to 0}\frac{1}{h}\int_S g(x')[G - G_0] dx' d\zeta' = -\frac{1}{2}g(x)\left(\zeta^2 + \frac{1}{4}\right).
$$

Proof. For any small $\varepsilon > 0$, choose ρ small enough so that

$$
|g(x') - g(x)| < \varepsilon
$$

for any $x', x \in B_{x,\rho}$. Consider

$$\begin{aligned}\int_S g(x')[G-G_0]dx'd\zeta' &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\Omega\setminus B_{x,\rho}}[G-G_0]g(x')dx'd\zeta' \\ &\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{B_{x,\rho}}[G-G_0]g(x')dx'd\zeta' \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\Omega\setminus B_{x,\rho}}[G-G_0]g(x')dx'd\zeta' \\ &\quad + g(x)\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{B_{x,\rho}}[G-G_0]dx'd\zeta' \\ &\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{B_{x,\rho}}[G-G_0][g(x')-g(x)]dx'd\zeta' \\ &= I_1+I_2+I_3.\end{aligned}$$

We first examine I_1. For fixed ρ, we are away from the singularity and can expand the integrand. Define

$$h(y)=\frac{e^{\sqrt{y}}}{\sqrt{y}}.$$

By the mean value theorem, for $y>0$,

$$|h(y+\delta)-h(y)|\le \delta \sup_{[y,y+\delta]}|h'|. \tag{5.2}$$

Note that here,

$$h'(y)=\frac{e^{\sqrt{y}}}{2}\left[\frac{1}{y}-\frac{1}{y^{3/2}}\right].$$

Apply (5.2) with

$$y=|x-x'|^2$$

and

$$\delta=h^2|\zeta-\zeta'|^2.$$

This yields

$$|G-G_0|\le Ch^2\left[\frac{1}{\rho^2}-\frac{1}{\rho^3}\right],$$

where C is independent of h, x', and ζ'. Since $g\in L^\infty$ and ρ is fixed,

$$\lim_{h\to 0}\frac{1}{h}I_1=0.$$

By Lemma 3,

$$\lim_{h\to 0}\frac{1}{h}I_2=-\frac{1}{2}g(x)\left(\zeta^2+\frac{1}{4}\right).$$

For the last term,

$$\begin{aligned}\frac{1}{h}|I_3| &\leq \left\|\frac{G-G_0}{h}\right\|_{L^1(S)} \|g(x')-g(x)\|_{L^\infty(B_{x,\rho})}\\ &\leq C\varepsilon\end{aligned}$$

by Lemma 1 and our choice of ρ. From the limits above, we can choose h small enough that

$$\left|\frac{1}{h}\int_S g(x')[G-G_0]dx'd\zeta' + \frac{1}{2}g(x)\left(\zeta^2+\frac{1}{4}\right)\right| \leq C\varepsilon+\varepsilon$$

from which the result follows. □

Note that u_0 is continuous from Lemma 2. By setting $g(x) = \epsilon_0(x)u_0(x)$ in Proposition 2, we have that

$$v_1(x) = -\frac{1}{2}\epsilon_0(x)u_0(x)\left(\zeta^2+\frac{1}{4}\right)$$

pointwise almost everywhere in Ω, assuming $\epsilon_0(x)$ is piecewise continuous. Using (5.1), this means that u_1 satisfies

$$\begin{aligned}u_1(x,\zeta) &= \zeta\frac{\partial u_{\mathrm{i}}}{\partial z}(x,0) - k^2\int_S \epsilon_0(x')u_1(x',\zeta')G_0 d\zeta' dx'\\ &\quad + k^2\int_\Omega u_0(x')G_0 dx' + \frac{1}{2}k^2\left(\zeta^2+\frac{1}{4}\right)u_0(x)\epsilon_0(x).\end{aligned} \tag{5.3}$$

To compute a simpler expression for u_1, we first note that from the symmetry of the integral with respect to ζ' we have

$$T\left(\zeta\frac{\partial u_{\mathrm{i}}}{\partial z}(x,0)\right) \equiv 0.$$

Hence $u_1(x,\zeta)$ has the form

$$u_1(x,\zeta) = \hat{u}_1(x) + \zeta\frac{\partial u_{\mathrm{i}}}{\partial z}(x,0) + \frac{1}{2}\zeta^2k^2u_0(x)\epsilon_0(x), \tag{5.4}$$

where $\hat{u}_1(x)$ is the solution to the lower dimensional integral equation

$$\begin{aligned}(I+k^2T)\hat{u}_1(x) &= k^2\int_\Omega G(x,0,x',0)u_0(x')dx'\\ &\quad -\frac{k^4}{24}\int_\Omega G(x,0,x',0)\epsilon_0^2(x')u_0(x')dx' + \frac{1}{8}k^2u_0(x)\epsilon_0(x).\end{aligned} \tag{5.5}$$

One can verify this by taking $(I+k^2T)$ of both sides of (5.4) and using (5.3) to eliminate u_1. We show the following convergence estimate.

Proposition 3. *Suppose that $\epsilon_0(x)$ is piecewise continuous. Let $\tilde{u}(x,\zeta), u_0(x)$, and $u_1(x,\zeta)$ be given by* (4.3), (4.2), (5.4), *and* (5.5), *respectively. Then as $h\to 0$,*

$$\|\tilde{u}-(u_0+hu_1)\|_{L^2(S)} = o(h).$$

Proof. Define the error by

$$e = \tilde{u} - (u_0 + hu_1).$$

Then, by using the integral equations for each term we obtain

$$\begin{aligned}
e = {} & u_{\mathrm{i}}(x, h\zeta) + k^2 h \int_S G(x, h\zeta, x', h\zeta')\tilde{u}(x', \zeta')d\zeta' dx' \\
& - k^2 \int_S \epsilon_0(x') G(x, h\zeta, x', h\zeta')\tilde{u}(x', \zeta')d\zeta' dx' - u_{\mathrm{i}}(x, 0) + k^2 T u_0 \\
& - h\zeta \frac{\partial u_{\mathrm{i}}}{\partial z}(x, 0) + hk^2 T u_1 - hk^2 \int_\Omega u_0(x') G(x, 0, x', 0)dx' - \frac{h}{2}k^2 \left(\zeta^2 + \frac{1}{4}\right) u_0(x)\epsilon_0(x).
\end{aligned}$$

By adding $k^2 Te$ to both sides and rearranging terms,

$$\begin{aligned}
(I + k^2 T)e = {} & u_{\mathrm{i}} - \left[u_{\mathrm{i}}(x, 0) + h\zeta \frac{\partial u_{\mathrm{i}}}{\partial z}(x, 0) \right] \\
& + k^2 h \int_S G(x, h\zeta, x', h\zeta')\tilde{u}(x', \zeta')d\zeta' dx' - hk^2 \int_\Omega u_0(x') G(x, 0, x', 0)dx' \\
& + k^2 \int_S \epsilon_0(x') u_0(x') [G_0 - G] dx' d\zeta' - h\frac{k^2}{2} \left(\zeta^2 + \frac{1}{4}\right) u_0(x)\epsilon_0(x) \\
& + k^2 \int_S \epsilon_0(x') [G_0 - G] (\tilde{u}(x', \zeta') - u_0(x')) d\zeta' dx'.
\end{aligned}$$

We will refer to each set of expressions on each line on the right-hand side of the above as $term_1$, $term_2$, $term_3$, and $term_4$. Now, $term_1$ is clearly $O(h^2)$ in L^2 by a Taylor expansion of u_{i}. In $term_2$, we can use Lemma 1 to approximate G by G_0 and commit an error of $O(h)$. Hence $term_2$ becomes

$$hk^2 T \left(\frac{\tilde{u} - u_0}{\epsilon_0} \right) + o(h),$$

which from Proposition 1 and the boundedness of T is $o(h)$ in $L^2(S)$. Consider $term_3/h$. By Proposition 2 and the fact that $g = \epsilon_0 u_0$ is piecewise continuous, this ratio approaches zero pointwise almost everywhere. So,

$$\left\{ \left(\frac{term_3}{h} \right)^2 \right\}$$

is a sequence of functions converging pointwise almost everywhere to zero, and by Lemma 1, they are uniformly bounded on a bounded domain (and hence in L^1). The Lebesgue dominated convergence theorem therefore yields $term_3/h \to 0$ in $L^2(S)$. Finally, for $term_4$, we can again use Proposition 2 with $g = \epsilon_0(\tilde{u} - u_0)$ to obtain

$$term_4 = h\frac{k^2}{2} \left(\zeta^2 + \frac{1}{4}\right) \epsilon_0(x)(\tilde{u}(x) - u_0(x)) + o(h),$$

which is $o(h)$ in $L^2(S)$ by Proposition 1. We have now shown that

$$\|(I + k^2 T)e\|_{L^2(S)} = o(h).$$

The result follows from Lemma 2. $\square$

6. Numerical results. In this section we will show some numerical results. Our goal is to demonstrate the properties of the approximation method, and in order to reduce the computational complexity, we consider 2-D scattering. We will compare results obtained using the approximate method with those obtained by solving the full Lippmann–Schwinger equation numerically.

In two dimensions, we reduce the region Ω to a line segment $\Omega = [-L, L]$. Of course, more general regions consisting of multiple line segments can be considered. The fundamental solution in two dimensions is

$$G(x, \zeta, x', \zeta') = \frac{i}{4} H_0^{(1)} \left(k|(x, \zeta) - (x', \zeta')| \right),$$

where $H_0^{(1)}$ is a Hankel function of the first kind. One can justify that the formula for u_0 is the same as in the 3-D case. The equation satisfied by u_1 hinges on Lemma 3. The result of Lemma 3 applies to the 2-D case without modification. This can be shown by direct calculation. Therefore, the equations for u_0 and u_1 are again (3.5) and (5.4)–(5.5), with the Green's function replaced by the above 2-D version.

In order to obtain solutions to which we compare our approximate solutions, we will solve a 2-D scattering problem. The equation we need to solve is the 2-D version of (3.1). We use piecewise bilinear functions to approximate the exact solution u and discretize the integral equation (3.1) to solve for u. Of particular interest is the solution $u(x, z)$ in the scatterer $S = \Omega \times [-h/2, h/2]$.

We will solve for the approximate solutions $u_0(x)$ using (3.5), and $u_1(x, z)$ using (5.4)–(5.5). To accomplish this, we discretize the integral equations using piecewise linear representations of $u_0(x)$ and $\hat{u}_1$.

The length of the scatterer is $L = 5$, and the thickness is $h = 0.1$. In solving the 2-D problem, we choose a mesh size of 0.02 in the direction of the membrane and 0.025 across its thickness. When solving for u_0 and $\hat{u}_1$, we discretize the interval with mesh size 0.02.

We will solve the scattering problem for three wave numbers, $k = 4$, 8, and 12. We choose the incident wave to be a plane wave of the form

$$u_{\mathrm{i}} = \exp ik(x \cos \theta + z \sin \theta).$$

The wavelengths in the scatterer under these conditions are computed and summarized in Table 6.1. Therefore, one can compare the wavelength with the scatterer thickness $h = 0.1$. For example, at $k = 8$ the thickness is approximately 1/4 the wavelength when $\epsilon = 3$ and approximately 1/3 the wavelength when $\epsilon = 9$.

TABLE 6.1
Wavelength in the scatterer as a function of wave number k and dielectric constant ϵ.

k	$\epsilon = 3$	$\epsilon = 9$
4	0.91	0.52
8	0.45	0.26
12	0.30	0.17

We are particularly interested in the accuracy of the solution on the scatterer itself. In the first experiment, we solve the scattering problem with $k = 8$ with incident wave hitting a uniform scatterer at -45^{o} degrees. For dielectric constant $\epsilon = 3$, the results are shown in Figure 6.1. This is a good situation as the wavelength in the scatterer is more than 4 times the thickness. The error, as can be seen in the figure, is quite small, with the largest disagreement occurring at the bottom of the

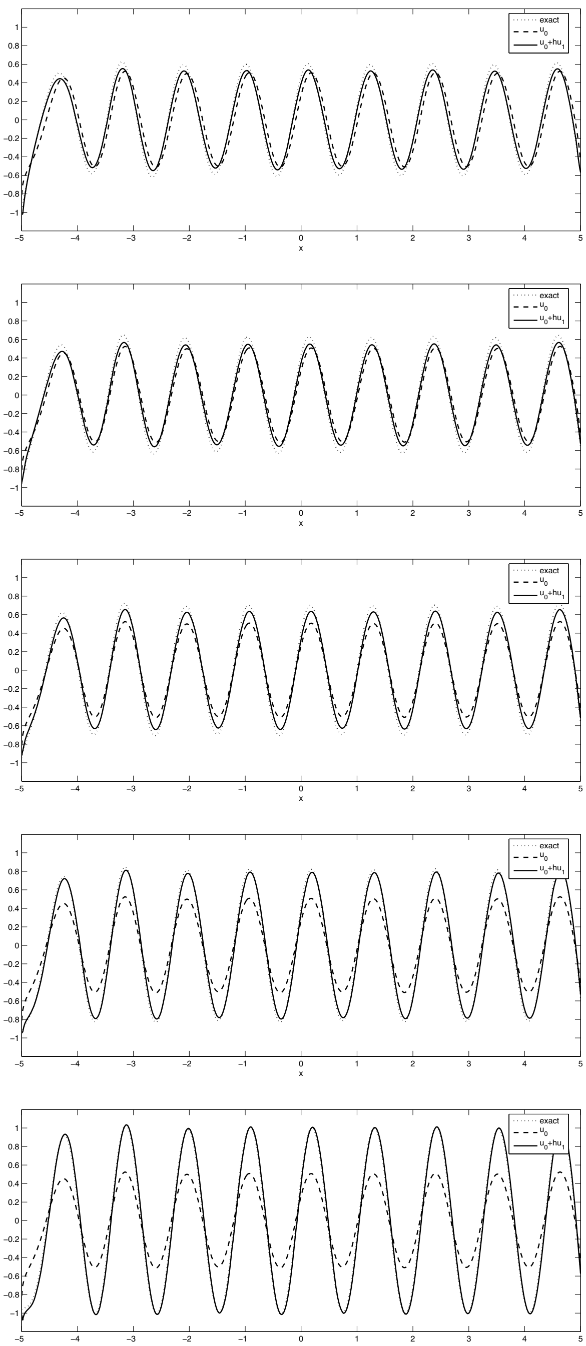

FIG. 6.1. *Top to bottom: real part of* $u(x,z)$ *for* $\epsilon = 3$ *at* $z = -0.05, -0.025, 0, 0.025, 0.05$. *Shown in dots are the exact solutions, and in solid, the approximation* $u_0 + hu_1$. *Also, shown in dashes is the leading order approximation* u_0.

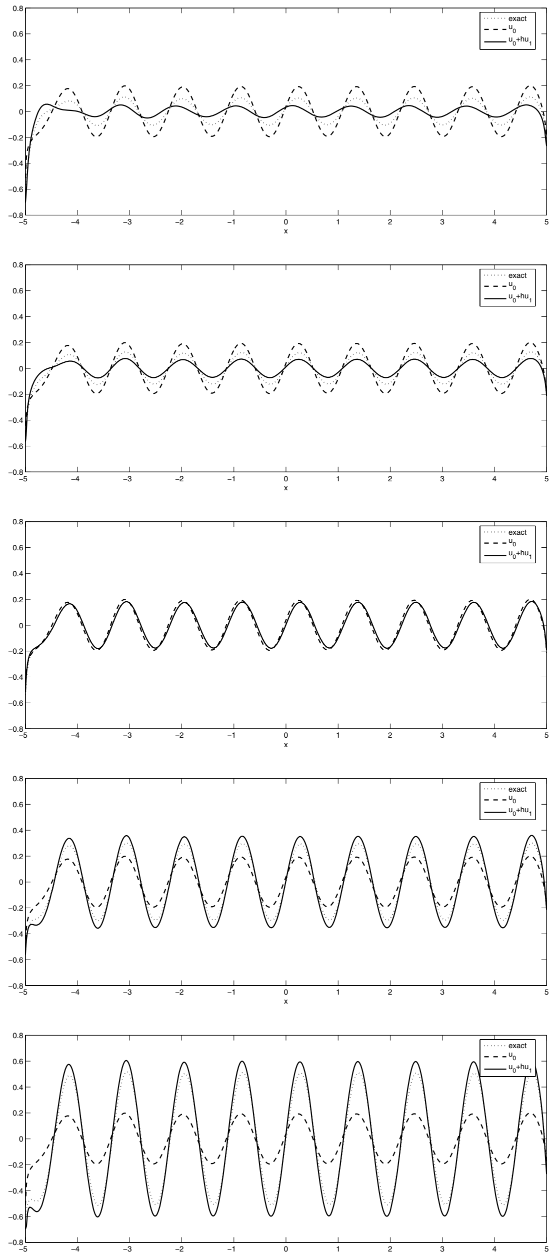

FIG. 6.2. *Top to bottom: real part of* $u(x,z)$ *when* $\epsilon = 9$ *at* $z = -0.05, -0.025, 0, 0.025, 0.05$.

scatterer. There is also some discrepancy at the leading and trailing edges of the scatterer since we have not accounted for the boundary layer.

When the dielectric constant ϵ is 9, the approximation deteriorates. Under this condition, the wavelength is only 2.6 times bigger than the thickness of the scatterer. While the corrector u_1 does improve over the leading order approximation u_0, the error is still quite noticeable. The results are shown in Figure 6.2.

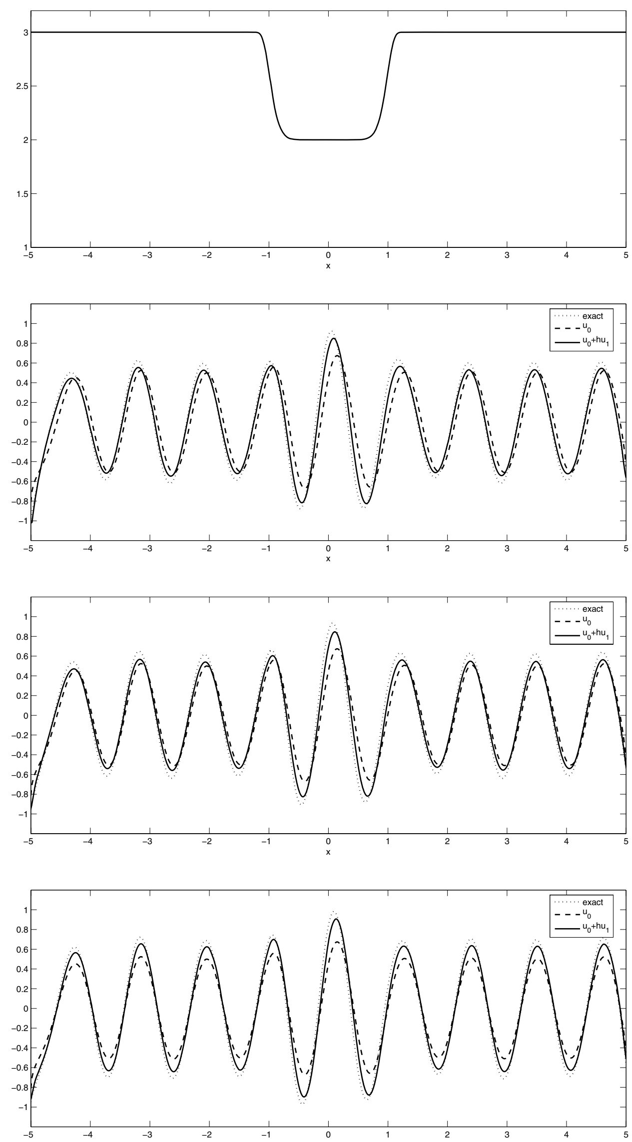

FIG. 6.3. *Top to bottom: the function $\epsilon(x)$, followed by the real part of $u(x, z)$ at $z = -0.05, -0.025, 0$.*

A final example is given when ϵ is x dependent. The distribution of $\epsilon(x)$ is shown in Figure 6.3, together with the solution at various z-values.

Next we calculate the relative L^2 error of the approximation. The L^2 norm is calculated by taking the values of a function at the node points of the regular mesh and interpolating with bilinear splines to obtain an estimate of the function. The interpolated function is then squared and integrated over the domain. The results of the calculations are displayed in Table 6.2. When $\epsilon = 3$, the error increases as a function of k. This is to be expected as the wavelength in the scatterer becomes smaller in comparison to the thickness. When $\epsilon = 9$, the errors follow the same trend

TABLE 6.2
Relative L^2 error of the approximation $u_0 + hu_1$.

	k	Angle of incidence 45°	Normal incidence
$\epsilon = 3$	4	0.0614	0.0529
	8	0.1065	0.1129
	12	0.1413	0.1614
$\epsilon = 9$	4	0.0226	0.0248
	8	0.2058	0.2146
	12	0.6332	0.6567

as k is increased. However, notice that the error is actually smaller for $k = 4$ when $\epsilon = 9$ than when $\epsilon = 3$.

According to our estimates, for a fixed k, as we decrease h and scale ϵ as $1/h$, the error should decrease according to $o(h)$. The numerical examples presented here are meant to give an indication of the accuracy of our approximation.

7. Discussion. In this work, we developed an approximate method for solving a scattering problem where the scatterer is thin. We assume that the dielectric constant of the scatterer scales as $1/h$, where h is the thickness of the scatterer. We formulate the scattering problem using the Lippmann–Schwinger equation. Solution to this equation is approximated by a series in h. Both the leading order solution and the first order corrector can be found by solving an integral equation involving one fewer spatial variable than the original problem. This could lead to substantial savings in realistic computations.

We show that the leading order approximation is $O(h)$ accurate, while the approximation including the first order corrector is $o(h)$ accurate. Boundary layer correctors will be needed to improve the approximation. Finally, we present numerical examples that provide some quantitative assessment of the accuracy of the approximation.

Acknowledgments. The authors are grateful to Habib Ammari, Jay Gopalakrishnan, and Murali Rao for helpful discussions on this work. In particular, we thank Habib Ammari for his clever suggestion which led to the simpler uniqueness proof in Lemma 2.

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