MATH 4242: Applied Linear Algebra Section 010

Fall'08 – Quiz #1 (09/17/08)

These are sketches for the solution of the quiz

1. [3 points] Using Gauss–Jordan elimination (show all the intermediate steps and write down what you are doing in each of them), compute the inverse of

$$A = \begin{pmatrix} 2 & 2 \\ 6 & 5 \end{pmatrix}.$$

Is A regular? Is A non–singular? What is the inverse of $\begin{pmatrix} 2 & 6 \\ 2 & 5 \end{pmatrix}$?

We apply the Gauss–Jordan elimination process

$$\left(\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{array}\right) \overset{r_2 \to r_2 - 3r_1}{\rightsquigarrow} \left(\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right) \overset{r_1 \to r_1 + 2r_2}{\rightsquigarrow} \left(\begin{array}{cc|cc} 2 & 0 & -5 & 2 \\ 0 & -1 & -3 & 1 \end{array}\right)$$

$$\overset{r_1 \to (1/2) r_1}{\rightsquigarrow} \left(\begin{array}{cc|cc} 1 & 0 & -5/2 & 1 \\ 0 & -1 & -3 & 1 \end{array}\right) \overset{r_2 \to -r_2}{\rightsquigarrow} \left(\begin{array}{cc|cc} 1 & 0 & -5/2 & 1 \\ 0 & 1 & 3 & -1 \end{array}\right)$$

The inverse is $A^{-1} = \begin{pmatrix} -5/2 & 1 \\ 3 & -1 \end{pmatrix}$. A is regular because we didn't need to change rows to arrive at upper triangular form. A is non–singular because it has an inverse (or because every regular matrix is non–singular). Finally

$$\begin{pmatrix} 2 & 6 \\ 2 & 5 \end{pmatrix}^{-1} = (A^{\top})^{-1} = (A^{-1})^{\top} = \begin{pmatrix} -5/2 & 3 \\ 1 & -1 \end{pmatrix}$$

2. [3 points] Let B be a square matrix such that $B^2 - 3B + I = 0$, where $B^2 = B\,B$ and I is the identity matrix. Prove that the inverse of B is $3I - B$.

There are two properties to check:

$$B(3I - B) = 3B - B^2 = I, \qquad (3I - B)\,B = 3B - B^2 = I.$$

With these two, we have proved that B has an inverse and that $B^{-1} = 3I - B$.

3. [4 points] Consider the following linear system

$$\begin{cases} x + y + z + t = 0, \\ y + 3z + t = 1, \\ 2y + 6z + 2t = 2. \end{cases}$$

Write the augmented matrix of the system and apply Gauss elimination so that the coefficient matrix is in reduced row echelon form. Identify: the rank of the matrix, basic variables and free variables. Give all the solutions to the system. Finally, write a new right–hand side that makes the system incompatible.

We first apply Gauss elimination to transform the augmented matrix to upper row echelon form:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 2 & 6 & 2 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Rank = 2 = # pivots. Basic variables: x and y (those corresponding to columns where there are pivots). Free variables: z and t (all others). Solution of the system:

$$\left\{\begin{array}{rl} x+y & = -z-t \\ y & = 1-3z-t \end{array}\right. \iff \left\{\begin{array}{rl} x & = -z-t-(1-3z-t) = 2z-1 \\ y & = 1-3z-t \end{array}\right. \qquad z \text{ and } t \text{ free}$$

A possible right–hand side that makes the system incompatible is $(0,1,0)$ (this has to be checked).

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Fall'08 – Quiz #2 (10/01/08)

These are sketches for the solution of the quiz

1. [2 points] Find a basis for the space $\{p(t) = a + b\,t + c\,t^2 \mid p(1) = 0\}$.

   The set can also be written as $\{p(t) = a + b\,t + c\,t^2 \mid a+b+c = 0\} = \operatorname{span}\{1-t, 1-t^2\}$. Since the vectors $1-t$ and $1-t^2$ are linearly independent (it has to be proved), they form a basis for the space.

2. [2 points] What is the dimension of $\operatorname{span}\{(1,1,1,1),(1,-1,1,-1),(1,0,1,0)\}$? (Give a convincing argument, not only a number)

   Simple computations show that the first two vectors are linearly independent, whereas the three of them are dependent. Therefore the dimension of the span is two.

3. [3 points] Prove that $\ker A \subseteq \ker A^2$ for every square matrix A. Prove that if A is non–singular, then $\ker A = \ker A^2$. Give an example of A singular where this equality holds.

   If $A\mathbf{x} = \mathbf{0}$, then $A^2\mathbf{x} = A(A\mathbf{x}) = A\mathbf{0} = \mathbf{0}$. This proves the first result.

   When A is non–singular, so is A^2 (this can be proved with the determinant for instance) and hence both kernels reduce to the zero vector. Otherwise, if A is non–singular and $A^2\mathbf{x} = \mathbf{0}$, then $A\mathbf{x} = A^{-1}A^2\mathbf{x} = \mathbf{0}$ and we have proved that $A\mathbf{x} = \mathbf{0}$ if and only if $A^2\mathbf{x} = \mathbf{0}$, which is the desired result.

   Finally, examples of the last property are the zero matrix for all sizes. Also $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

4. [3 points] Consider the matrix

$$B = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 6 & 2 & 0 \\ 1 & 3 & 3 & -4 \end{pmatrix}.$$

Its row echelon form is

$$\begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Give a basis for the range of B. Give a basis for the kernel of B. What are the dimensions of the corange and the cokernel of B? The corange of B is a subspace of $\mathbb{R}^k$, for which k?

The columns of B corresponding to positions of pivots in the row echelon form give a basis for the range of B. Note that the rank of B is two and therefore any two linearly independent columns of B form a basis of the range of B.

The kernel of B is the same as that of its row echelon form. Continuing with the solution of the associated homogeneous system, we obtain that its solutions are

$$\begin{cases} x_1 = -3x_2 - 2x_4 \\ x_3 = 2x_4 \end{cases} \qquad x_2 \text{ and } x_4 \text{ are free}$$

Therefore, a basis for $\ker B$ is $\{(-3,1,0,0),(-2,0,2,1)\}$.

The dimension of the corange is the same as the dimension of the range, that is, it is the rank of B: 2. The dimension of the cokernel is the number of rows minus the dimension of the corange: 1. The corange is the subspace spanned by the rows of B: it is therefore a subspace of $\mathbb{R}^4$.

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Fall'08 – Quiz #3 (10/29/08)

These are sketches for the solution of the quiz

1. (3 points) Consider the line $\{(x,y) \in \mathbb{R}^2 \mid x - y = 0\}$ and the point $(2,3)$. Find the distance of the point to the line. Find the point of the line that is closest to $(2,3)$.

   The line can also be written as $\operatorname{span}\{(1,1)\}$. We have to find the solution to

   $$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} x = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

   that is $x = 5/2$. The closest point is $\frac{5}{2}(1,1) = (\frac{5}{2},\frac{5}{2})$ and the distance is

   $$\|(\tfrac{5}{2},\tfrac{5}{2}) - (2,3)\| = \tfrac{1}{\sqrt{2}}.$$

2. (3 points) Find the minimum of

   $$x^2 + x\,y + y^2 + x - y + 1.$$

   Prove that it is unique.

   The quadratic function can be written as

   $$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 2\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + 1$$

It is located at the solution of the system

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

which is $(x^*, y^*) = (-1, 1)$. The minimum is the value of the function at this point, that is, 0. The minimum is unique because the matrix is positive definite (this can be seen when solving the system, since all the pivots are positive).

3. (4 points) We have measured a magnitude y for four different times t and obtained the following table:

t_i	1	2	3	4
y_i	2	0	6	1

We want to approximate this group of points by the best linear fit $\alpha+\beta\, t$ with respect to the Euclidean norm. Write down the equations satisfied by the coefficients α and β (do not solve the equations).

We want to find the least squares solution to the system

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \\ 1 \end{pmatrix},$$

which is the solution to the system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 6 \\ 1 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 9 \\ 24 \end{pmatrix}$$

Show that changing the norm to the following weighted norm in the least squares method

$$\|(e_1, e_2, e_3, e_4)\| = \sqrt{2e_1^2 + e_2^2 + e_3^2 + e_4^2}$$

gives the same coefficients for the line as if we take this new table (we have taken the first point twice):

t_i	1	1	2	3	4
y_i	2	2	0	6	1

(you don't have to solve the systems to know that the solutions are the same).

The weighted option gives the system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 6 \\ 1 \end{pmatrix},$$

and repeating the first point gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 0 \\ 6 \\ 1 \end{pmatrix},$$

Both are the system

$$\begin{pmatrix} 5 & 11 \\ 11 & 31 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 11 \\ 26 \end{pmatrix}$$

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Fall'08 – Quiz #4 (11/26/08)

These are sketches for the solution of the quiz

1. (5 points) In the space of 2×2 real matrices, $\mathcal{M}_{2\times 2}$, consider the function $L : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ given by
$$L[A] = A + \tfrac{1}{2}A^\top.$$
Show that L is linear. Find its matrix representative with respect to the standard basis
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
(Hint: $\mathcal{M}_{2\times 2}$ is a four dimensional space so the matrix should be 4×4).

The first part of the exercise is done by showing that for all $A, B \in \mathcal{M}_{2\times 2}$ and $c \in \mathbb{R}$

$$L[A+B) = (A+B) + \tfrac{1}{2}(A+B)^\top = A + \tfrac{1}{2}A^\top + B + \tfrac{1}{2}B^\top = L[A] + L[B],$$

$$L[c\,A] = c\,A + \tfrac{1}{2}(c\,A)^\top = c\Big(A + \tfrac{1}{2}A^\top\Big) = c\,L[A].$$

We now compute the images of the elements of the basis and decompose the result in the basis again:

$$\begin{aligned}
L\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] &= \begin{pmatrix} 3/2 & 0 \\ 0 & 0 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
L\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] &= \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
L\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] &= \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix} = \text{etc} \\
L\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] &= \begin{pmatrix} 0 & 0 \\ 0 & 3/2 \end{pmatrix} = \text{etc}
\end{aligned}$$

The results for each of the elements of the basis are written as the columns of a 4 matrix

$$\begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 3/2 \end{pmatrix}.$$

Note that

$$L\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} \frac{3}{2}\,a & b+\frac{1}{2}\,c \\ c+\frac{1}{2}\,b & \frac{3}{2}\,d \end{pmatrix}, \qquad \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 3/2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\,a \\ b+\frac{1}{2}\,c \\ c+\frac{1}{2}\,b \\ \frac{3}{2}\,d \end{pmatrix}$$

2. (5 points) Consider the following affine transformation in the plane

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Describe the image of the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ under this transformation. Show that this transformation has an inverse by finding it.

We first compute the image of the three vertices

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \qquad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The image of a triangle by an affine transformation is again a triangle, so the answer is the triangle with vertices $(1,1)$, $(2,0)$ and $(1,3)$. If

$$\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^{-1}\left(\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which gives the formula for the inverse of T.

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Fall'08 – Quiz #5 (12/10/08)

These are sketches for the solution of the quiz

1. (5 points) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\,\Lambda\,Q^\top$ (spectral decomposition), where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

First of all we need to compute the **eigenvalues** of A, that is the roots of $\det(A-\lambda I)$. With some computations we can obtain

$$\det(A - \lambda\,I) = -(\lambda-2)^2(\lambda+1),$$

so the eigenvalues are $\lambda = 2$ (double) and $\lambda = -1$. Second, we compute the **eigenvectors**. For $\lambda = 2$ we solve the homogeneous system $(A - 2I)\mathbf{x} = \mathbf{0}$

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad x = y - z.$$

Therefore $V_2 = \ker(A-2I) = \operatorname{span}\{(1,1,0),(1,0,-1)\}$. With $\lambda = -1$, we have to solve $(A+I)\mathbf{x} = \mathbf{0}$,

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ -1 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore $V_{-1} = \operatorname{span}\{(1,-1,1)\}$. Thirdly, we have to find **orthonormal bases of eigenvectors**. To do that we apply the Gram–Schmidt method separately on each of the subspaces (details not given) to obtain

$$V_2 = \operatorname{span}\{(1/\sqrt{2}, 1/\sqrt{2}, 0), (1/\sqrt{6}, -1/\sqrt{6}, -2/\sqrt{6})\}, \qquad V_{-1} = \operatorname{span}\{(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})\}.$$

Now we put these three vectors as columns of Q and the eigenvalues (in the same order) as the elements of the diagonal of Λ

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Remark: the result is not unique. Depending on how we choose the bases for V_2 and V_{-1} the corresponding orthonormal bases will vary. Also, taking the eigenvalues in a different order, will create a permutation of elements in Λ. The important thing is that eigenvalues are given in the same order as eigenvectors. Otherwise the decomposition does not hold.

Using the spectral decomposition of A, find that of A^{-1}.

We don't need any computation. Note that $Q^{-1} = Q^\top$ and therefore $(Q^\top)^{-1} = Q$. We apply the formula for the inverse of the product

$$A^{-1} = (Q\Lambda Q^\top)^{-1} = (Q^\top)^{-1}\Lambda^{-1}Q^{-1} = Q\Lambda^{-1}Q^\top.$$

Remark: The matrix Q is the same and the diagonal matrix has to be inverted: the eigenvalues of A^{-1} are the inverses of the eigenvalues of A but their corresponding eigenvectors are the same.

2. (5 points) In this exercise P is $m \times r$ with orthonormal columns, Q is $n \times r$ with orthonormal columns and Σ is a diagonal $r \times r$ matrix with positive elements on its diagonal. Finally $A = P\Sigma Q^\top$ (this is the SVD of A).

   (a) Prove that $P\mathbf{y} = \mathbf{0}$ if and only if $\mathbf{y} = \mathbf{0}$. Prove that $P\Sigma\mathbf{y} = \mathbf{0}$ if and only if $\mathbf{y} = \mathbf{0}$.

   The columns of P are orthonormal and therefore independent. This means that $\ker P = \{\mathbf{0}\}$ which is the first assertion. (Note that the fact that the columns of P are orthonormal can also be written as $P^\top P = I$, which can be used as an alternative proof of the first statement). For the second assertion, note the simple chain of implications

$$P\Sigma\mathbf{y} = \mathbf{0} \iff P(\Sigma\mathbf{y}) = \mathbf{0} \iff \Sigma\mathbf{y} = \mathbf{0} \iff \mathbf{y} = \mathbf{0}.$$

   We have used the first statement and the fact that Σ is non–singular (it is a diagonal matrix with positive elements on the diagonal).

(b) Using the previous result, prove that $\ker A = \ker Q^\top$.

The argument is very similar to the previous one: we use (a) to see that

$$A\mathbf{x} = \mathbf{0} \quad \Longleftrightarrow \quad (P\Sigma)(Q^\top \mathbf{x}) = \mathbf{0} \quad \Longleftrightarrow \quad Q^\top \mathbf{x} = \mathbf{0},$$

which is the same thing as saying that $\ker A = \ker Q^\top$.

(c) Show that the columns of Q form an orthonormal basis for the corange of A (hint: relate coranges and kernels).

We first use that for any matrix B, $\mathrm{corng} B = (\ker B)^\perp$. From point (b) we obtain that

$$\mathrm{corng} A = (\ker A)^\perp = (\ker Q^\top)^\perp = \mathrm{corng} Q^\top.$$

The rows of $Q^\top$ (the columns of Q) are orthonormal, so they are an orthonormal basis of their span (the span of the rows of $Q^\top$ is the corange of $Q^\top$, which is the same as the range of Q.)

(d) Where in the SVD of A can you find an orthonormal basis for the range of A? (hint: transpose and apply (c)).

If we transpose the SVD of A we obtain

$$A^\top = Q \Sigma P^\top,$$

so the roles of P and Q are reversed. Point (c) applied to this new SVD asserts that the columns of P form an orthonormal basis of the corange of $A^\top$, which is the same set as the range of A.

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Fall'08 – Midterm # 1 (10/10/08)

These are sketches for the solution of the exam

1. (25 points) Consider the following 3×3 matrix

$$A = \begin{pmatrix} 1 & -1 & -4 \\ -1 & 2 & 6 \\ 2 & -3 & -12 \end{pmatrix}.$$

(a) (10 pts) Compute A^{-1} using Gauss–Jordan elimination. Check that your result is correct by doing the product of A with it.

The result is

$$\begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ 1/2 & -1/2 & -1/2 \end{pmatrix}.$$

Some intermediate steps in the Gauss–Jordan elimination process (you have to detail what the elementary operations are

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -4 & 1 & 0 & 0 \\ -1 & 2 & 6 & 0 & 1 & 0 \\ 2 & -3 & -12 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -4 & -2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 3 & -2 & -2 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

(b) (5 pts) Is A regular? (Justify your answer)

In the Gauss elimination process we did not need to change rows to get to upper triangular form, so A is regular.

(c) (5 pts) Using A^{-1} solve the system

$$\begin{cases} x - y - 4z &= 1, \\ -x + 2y + 6z &= 0, \\ 2x - 3y - 12z &= 2. \end{cases}$$

If A is non–singular, $A^{-1}\mathbf{b}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$. Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ 1/2 & -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -\frac{1}{2} \end{pmatrix}.$$

(d) (5 pts) Write the matrix

$$B = \begin{pmatrix} 1 & -1 & -4 \\ 2 & -3 & -12 \\ -1 & 2 & 6 \end{pmatrix}$$

as the product of a permutation matrix by A. Use this result to compute the inverse of B without having to use Gauss-Jordan elimination again.

B is the result of interchanging the second and third rows of A. Therefore $B = PA$, where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Using the formula for the inverse of a product of matrices, $B^{-1} = A^{-1}P^{-1}$. But $P^{-1} = P$ and we have

$$B^{-1} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ 1/2 & -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 1/2 & -1/2 & -1/2 \end{pmatrix}.$$

2. (20 points) Consider the matrix

$$F = \begin{pmatrix} 3 & 1 & -1 \\ -3 & 1 & 3 \\ 6 & 4 & 1 \end{pmatrix}.$$

(a) (10 pts) Compute the LU decomposition of F.

We apply Gauss elimination keeping record of the elementary operators:

$$\begin{pmatrix} 3 & 1 & -1 \\ -3 & 1 & 3 \\ 6 & 4 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 2 \\ 6 & 4 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} & & \\ -1 & & \\ & & \end{pmatrix} \quad \begin{pmatrix} & & \\ -1 & & \\ 2 & & \end{pmatrix} \quad \begin{pmatrix} & & \\ -1 & & \\ 2 & 1 & \end{pmatrix}$$

(The second line is the record of operations). Then

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) (5 pts) Compute $\det F$.

By the definition of determinant, we have that $\det F = \det U = 3 \cdot 2 \cdot 1 = 6$. Also we can use that $\det F = \det L \det U = 1 \cdot 6 = 6$.

(c) (5 pts) Use the LU decomposition of F to solve the linear system

$$F\mathbf{x} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}.$$

Let $\mathbf{a} = U\mathbf{x}$. We solve

$$L\mathbf{a} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \quad \text{and} \quad U\mathbf{x} = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The first system is solved using forwards substitution. Its solution is $a = 2$, $b = 0$, $c = -4$. The second system is solve by backwards substitution. The solution is then $x = -2$, $y = 4$, $z = -4$.

3. (10 points) Give a basis for the space

$$\{p(t) = a\,t^2 + b\,t + c \,|\, p'(1) = 0\},$$

where $p'(t)$ denotes the derivative of p.

$p'(t) = 2a\,t + b$ and then $p'(1) = 2a + b$. Therefore we can write the set as $\{p(t) = a\,t^2 + b\,t + c \,|\, a = (-1/2)\,b\} = \{b\,(-\frac{1}{2}t^2 + t) + c \,|\, b, c \in \mathbb{R}\} = \mathrm{span}\{-\frac{1}{2}t^2 + t, 1\}$. The polynomials $\{-\frac{1}{2}t^2 + t, 1\}$ are linearly independent. Therefore $\{-\frac{1}{2}t^2 + t, 1\}$ is a basis for the given space.

4. (20 points) In this problem we will be working in the vector space of 2×2 real matrices.

(a) (10 pts) Are the matrices

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

linearly independent?

Note that

$$x\,A_1 + y\,A_2 + z\,A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

if and only if

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The rank of the matrix is three (you have to go to upper echelon form to prove that). Therefore the unique solution of this system is the trivial one $x = y = z = 0$, which means that A_1, A_2, A_3 are linearly independent.

(b) (10 pts) Give a necessary and sufficient condition for a matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

to be in span $\{A_1, A_2, A_3\}$.

Proceeding as in (a) we have to find a necessary and sufficient condition for the system

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}$$

to be compatible. We apply Gauss elimination to the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & a_{11} \\ 1 & 1 & 0 & a_{12} \\ 1 & 1 & 0 & a_{21} \\ 1 & 2 & 1 & a_{22} \end{array}\right) \rightsquigarrow \cdots \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & a_{11} \\ 0 & \frac{1}{2} & -\frac{1}{2} & a_{12} - \frac{1}{2}a_{11} \\ 0 & 0 & 2 & a_{22} + a_{11} - 3a_{12} \\ 0 & 0 & 0 & a_{12} - a_{21} \end{array}\right)$$

and see that compatibility of the system is equivalent to $a_{12} = a_{21}$. Therefore a matrix is in the span of A_1, A_2 and A_3 if and only if it is symmetric ($a_{12} = a_{21}$).

5. (25 pts) Consider the matrix

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

(a) (5 pts) What is the rank of C? Give a basis for the range of C. What is the dimension of the kernel of C?

If we transform C to upper row echelon form we obtain

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \end{pmatrix}.$$

From here it is clear that $\operatorname{rank} C = 2$ and therefore the range of C is a two–dimensional subspace of $\mathbb{R}^2$, that is, the range of C is $\mathbb{R}^2$ and any basis for this space is a basis for the range. The dimension of the kernel of C is the number of columns minus the rank: $3 - 2 = 1$.

That the rank of C is two can be proved by 'visual' inspection. The first and second columns of C are not proportional to each other and therefore they are independent. This can be used to prove that the range of C is $\mathbb{R}^2$ and everything else.

(b) (5 pts) Prove that the linear system $C\mathbf{x} = \mathbf{b}$ is compatible for every $\mathbf{b} \in \mathbb{R}^2$. Give an interpretation of this fact in terms of the range of C.

Because the rank of C is two and there are only two equations, the upper echelon form for the augmented system does not allow for incompatibilities. The system $C\mathbf{x} = \mathbf{b}$ is solvable if and only if

$$\mathbf{b} \in \operatorname{span}\{(1,3),(2,2),(3,1)\} = \operatorname{range} C = \mathbb{R}^2,$$

so there is no condition and the system is always solvable.

(c) (10 pts) Find a matrix D such that

$$C\,D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Hint: you have to solve two linear systems. For each of the systems you only need one solution, not all of them).

Note that D has to be a 3×2 matrix. If $\mathbf{d}_1$ and $\mathbf{d}_2$ are the respective columns of D, then

$$C\mathbf{d}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad C\mathbf{d}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These systems are solvable (by (b)) but not uniquely, since we will always have a free variable. We can try to solve both systems at the same time

$$\left(\begin{array}{ccc|c|c} 1 & 2 & 3 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c|c} 1 & 2 & 3 & 1 & 0 \\ 0 & -4 & -8 & -3 & 1 \end{array}\right).$$

Either we continue solving by backwards substitution from here, or we can further simplify the system in the style of Gauss–Jordan elimination (only on columns with pivots). In this case we get

$$\left(\begin{array}{ccc|c|c} 1 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 2 & \frac{3}{4} & -\frac{1}{4} \end{array}\right).$$

The columns of D are:

$$\mathbf{d}_1 = \begin{pmatrix} -\frac{1}{2}+z \\ \frac{3}{4}-2z \\ z \end{pmatrix}, \qquad \mathbf{d}_2 = \begin{pmatrix} \frac{1}{2}+z \\ -\frac{1}{4}-2z \\ z \end{pmatrix}.$$

The value of z is arbitrary and can be taken different in $\mathbf{d}_1$ and $\mathbf{d}_2$. Taking $z = 0$ in both sides, we have

$$D = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 \end{pmatrix}.$$

(d) (5 pts) Use the matrix D of (c) to obtain one solution of

$$C\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

without solving the corresponding linear system. (Hint. What is the value of $C\,D\,\mathbf{b}$?)

Because $CD\mathbf{b} = I\mathbf{b} = \mathbf{b}$ for all $\mathbf{b}$, we have that $D\mathbf{b}$ is a solution of $C\mathbf{x} = \mathbf{b}$. In our case, we only have to multiply

$$D\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

6. (Optional: 20 additional points) The ideas of the previous problem can be generalized in the form of the following theorem: *Let C be a $n \times m$ matrix. Then the rank of C is n (the number of rows) if and only if there exists a matrix D such that $C\,D = I_n$.* Prove it.

   (Hint. There are many ways of proving this result. You can try to write down several characterizations of the fact that the rank of a $n \times m$ matrix C is n, in terms of its range and of the solvability of systems $C\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$. Then you can try to prove that compatibility of systems $C\mathbf{x} = \mathbf{b}$ for arbitrary right–hand sides is equivalent to existence of a right–inverse, that is, a matrix D such that $C\,D = I_n$.)

The argument can be done as follows. Let C be an $n \times m$ matrix. The rank of C is equal to n (the number of rows) if and only if the range of C is $\mathbb{R}^n$, if and only if $C\mathbf{x} = \mathbf{b}$ is solvable for any $\mathbf{b} \in \mathbb{R}^n$.

Then, we can solve the systems

$$C\mathbf{d}_i = \mathbf{e}_i, \qquad i = 1, \dots, n$$

where $\mathbf{e}_i$ are the vectors of the canonical basis for $\mathbb{R}^n$, that is, the columns of I_n. Putting these solutions as the columns of a matrix D we have that $CD = I_n$.

Reciprocally, if $CD = I_n$, then $C(D\mathbf{b}) = (CD)\mathbf{b} = I\mathbf{b} = \mathbf{b}$, which means that $\mathbf{x} = D\mathbf{b}$ is a solution of the system $C\mathbf{x} = \mathbf{b}$. Therefore $C\mathbf{x} = \mathbf{b}$ is compatible for all $\mathbf{b}$ and this is equivalent to the rank of C being n.

MATH 4242: Applied Linear Algebra Section 010

Fall'08 – Midterm # 2 (11/14/08)

These are sketches for the solution of the exam

1. (10 points) Find a minimizer and the minimum value for the quadratic function

$$2x^2 - 2x\,y + 3y^2 - 6x + 8y + 3.$$

In the process, show that you are actually obtaining a minimum.

We can write the quadratic form in the following form

$$q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 2 \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} + 3.$$

The minimum of this quadratic form (if it exists) is in the solution

$$\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

The solution of this system is $(x^*, y^*) = (1, -1)$. When solving, we realize that the matrix is regular and we obtain positive pivots, so it is positive definite and we have computed the minimizer. The minimum value is $q(1,-1) = -4$.

2. (20 points) In $\mathcal{P}^{(2)} = \{p(t) = a + b\,t + c\,t^2 \,|\, a, b, c \in \mathbb{R}\}$ we consider the following inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(t)\,q(t)\,\mathrm{d}t.$$

(a) Find an orthonormal basis of $W = \{p(t) = a + b\,t + c\,t^2 \,|\, p(0) = 0\}$.

We can write $W = \{a + b\,t + c\,t^2 \,|\, a = 0\} = \operatorname{span}\{t, t^2\}$. Now we apply Gram–Schmidt's method (I'm including the needed computations at the end)

$$p_1 = t, \qquad p_2 = t^2 - \frac{\langle t^2, p_1 \rangle}{\langle p_1, p_1 \rangle}\, p_1 = t^2.$$

$$\langle p_1, p_1 \rangle = \langle t, t \rangle = \int_{-1}^{1} t^2 \mathrm{d}t = \frac{2}{3}, \qquad \langle t^2, p_1 \rangle = \int_{-1}^{1} t^3 \mathrm{d}t = 0, \qquad \langle p_2, p_2 \rangle = \int_{-1}^{1} t^4 \mathrm{d}t = \frac{2}{5}.$$

Normalizing p_1 and p_2 we obtain an orthonormal basis of W

$$q_1 = \sqrt{\frac{3}{2}}t, \qquad q_2 = \sqrt{\frac{5}{2}}t^2.$$

(b) Find the orthogonal projection of $p(t) = 1 + t$ onto W.

The formula for the projection, given an orthonormal basis of W is

$$\langle 1+t, q_1\rangle\, q_1 + \langle 1+t, q_2\rangle q_2 = \frac{3}{2}\,\langle 1+t, t\rangle\, t + \frac{5}{2}\langle 1+t, t^2\rangle\, t^2 = t + \frac{5}{3}t^2,$$

where we have computed

$$\langle 1+t, t\rangle = \int_{-1}^{2} (t+t^2)\mathrm{d}t = \frac{2}{3}, \qquad \langle 1+t, t^2\rangle = \int_{-1}^{1} (t^2+t^3)\mathrm{d}t = \frac{2}{3}.$$

3. (15 points) We have measured some data (t_i, y_i) and we are looking for the best weighted linear fit $y = \alpha + \beta t$ using the weights c_i given in the table:

t_i	0	1	2	−1
y_i	10	−4	−2	0
c_i	2	1	1	3

Find the equation of this line.

We use Weighted Least Squares for the incompatible system

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ -2 \\ 0 \end{pmatrix}$$

that we obtain when we try to impose the line $y = \alpha + \beta t$ to go through all the points. The LS system is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 3 \end{pmatrix} \begin{pmatrix} 10 \\ -4 \\ -2 \\ 0 \end{pmatrix}$$

that is

$$\begin{pmatrix} 7 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} 14 \\ -8 \end{pmatrix} \qquad \Longrightarrow \qquad \alpha^* = 2, \qquad \beta^* = -1.$$

Therefore the line is $y = 2 - t$.

4. (15 points) Let us consider the following decomposition $A = Q\,R$

$$\begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 3 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{3} & -\sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{6} \end{pmatrix}$$

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be the three column vectors of A and $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be the column vectors of Q.

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ with respect to the Euclidean inner product. (Hint: you can do this working directly with Q).

This question is equivalent to showing that the matrix Q is orthogonal, that is, that $Q^T Q = I$.

(b) Write $\mathbf{w}_1$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$. Do the same for $\mathbf{w}_2$ and $\mathbf{w}_3$.

Looking at the matrix decomposition $A = Q\,R$, we have,

$$\mathbf{w}_1 = \sqrt{3}\,\mathbf{u}_1, \qquad \mathbf{w}_2 = -\sqrt{3}\,\mathbf{u}_1 + \sqrt{2}\mathbf{u}_2, \qquad w_3 = 2\sqrt{3}\,\mathbf{u}_1 + \sqrt{6}\,\mathbf{u}_3.$$

This information is coded in the QR decomposition and there's nothing else to do. Otherwise, we know that because the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ form an orthonormal basis of $\mathbb{R}^3$ (with respect to the Euclidean inner product), we can write for any vector

$$\mathbf{b} = (\mathbf{u}_1 \cdot \mathbf{b})\,\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{b})\,\mathbf{u}_2 + (\mathbf{u}_3 \cdot \mathbf{b})\,\mathbf{u}_3,$$

so computing these decompositions involves doing some inner products.

(c) Give an orthonormal basis for $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

The $Q\,R$ decomposition also says that $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is the answer. Again we didn't need any computation for this.

Applying the Gram–Schmidt method to $\{\mathbf{w}_1, \mathbf{w}_2\}$ will give us exactly the same result. Problem (b) also gives another reason, since from the two first relations, we can easily prove that $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

5. (20 points) Consider the following inner product in $\mathbb{R}^3$

$$\langle \mathbf{v}, \mathbf{w} \rangle = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

(a) Compute the Gram matrix associated to the vectors $\{(1,1,0), (0,1,-1)\}$. Show that it is positive definite (Hint: you are allowed to use theoretical results).

We only need to carry out this small computation

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 3 & 3 \end{pmatrix}.$$

Gram matrices are positive definite if they are associated to linearly independent sets of vectors, which is the case, since $(1,1,0)$ and $(0,1,-1)$ are not proportional to each other and are therefore linearly independent.

(b) Find the distance of $(1,0,-1)$ to the subspace $\operatorname{span}\{(1,1,0), (0,1,-1)\}$, understanding that the distance is measured with respect to the norm associated to the inner product above.

The closest point is $\mathbf{w} = c_1\,(1,1,0) + c_2\,(0,1,-1)$ where

$$\begin{pmatrix} 7 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

The solution to this system is $c_1 = 0, c_2 = 2/3$ (which can be found by visual inspection, since the right–hand side is just a multiple of the second column of the matrix. Therefore, the closest point is $\frac{2}{3}(0,1,-1) = (0, \frac{2}{3}, -\frac{2}{3})$. The distance of to original point to the subspace is the distance to the closest point

$$\|(1,0,-1) - (0, \tfrac{2}{3}, -\tfrac{2}{3})\| = \|(1, -\tfrac{2}{3}, -\tfrac{1}{3})\| = \sqrt{8/3}.$$

The last norm has to be computed using the inner product of the problem

$$\|(1,-\tfrac{2}{3},-\tfrac{1}{3})\|^2 = \frac{1}{9}\begin{pmatrix} 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = \frac{24}{9}.$$

6. (20 points) Consider the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(a) Without doing any computation, give a reason why this system is compatible.

The rank of the matrix is two (there are clearly two linearly independent rows), which means that there's no zero row in the upper echelon form (the number of pivots equals the number of rows) and the system is compatible. Also we can argue that the range of the matrix is the whole of $\mathbb{R}^2$, which makes every system with this matrix compatible.

(b) Find all the solutions to this system.

Using Gaussian elimination we easily obtain

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

Therefore, the solution to the system is $x = 1 - z$, $y = 2$ with z a free variable. We can also write this as

$$(1, 2, 0) + z\,(-1, 0, 1).$$

(c) Find an orthonormal basis (with respect to the Euclidean inner product) for the corange of the matrix.

We can apply Gram–Schmidt's method to any of the two available bases for the corange, either to $\{(1,1,1),(1,0,1)\}$ or to $\{(1,0,1),(0,1,0)\}$. The second one has the advantage of being already orthogonal. In the first case, we obtain

$$\mathbf{v}_1 = (1,1,1), \qquad \mathbf{v}_2 = (1,0,1) - \frac{(1,0,1)\cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\,\mathbf{v}_1 = (1,0,1) - \frac{2}{3}(1,1,1) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right).$$

We now have to normalize both vectors and we obtain an orthonormal basis for the corange

$$\mathbf{u}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \qquad \mathbf{u}_2 = \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right).$$

(d) Find the solution of the system that has the smallest norm.

The solution with smallest norm is the orthogonal projection onto the corange of any solution of the system (all of them give the same result). We can take as particular solution $(1,2,0)$ and obtain the projection using the formula for the case when we have an orthonormal basis (see (c))

$$\Big((1,2,0)\cdot \mathbf{u}_1\Big)\mathbf{u}_1 + \Big((1,2,0)\cdot \mathbf{u}_2\Big)\mathbf{u}_2 = \left(\frac{1}{2}, 2, \frac{1}{2}\right).$$