Traveling fronts bifurcating from stable layers in the presence of conservation laws

Alin Pogan
Miami University
Department of Mathematics
301 S. Patterson Ave.
Oxford, OH 45056, USA

Arnd Scheel
University of Minnesota
School of Mathematics
206 Church St. S.E.
Minneapolis, MN 55455, USA

Abstract

We study traveling waves bifurcating from stable standing layers in systems where a reaction-diffusion equation couples to a scalar conservation law. We prove the existence of weekly decaying traveling fronts that emerge in the presence of a weakly stable direction on a center manifold. Moreover, we show the existence of bifurcating traveling waves of constant mass. The main difficulty is to prove the smoothness of the ansatz in exponentially weighted spaces required to apply the Lyapunov-Schmidt methods. [1]

1 Introduction

In this paper we prove the existence traveling fronts bifurcating from (standing) layers in a class of parabolic systems that couple a scalar conservation law with a scalar reaction–diffusion equation. Our focus here is on a systems of the form

$$\begin{cases} u_t = [a(u)u_x - b(u)v_x]_x, \\ v_t = v_{xx} + \delta u + g(v), \end{cases} \tag{1.1}$$

on the real line $x \in \mathbb{R}$. Here $a, b, g \in C^3(\mathbb{R})$, $\delta \in \mathbb{R}$. Moreover, a is uniformly elliptic, that is, $a(u) \geq a_0 > 0$ for all $u \in \mathbb{R}$.

The system (1.1) encompasses a variety of interesting model problems, such as phase-field systems and the Keller-Segel model for chemotaxis with its generalizations [1, 4, 5, 8, 12, 13]. From a theoretical point of view, (1.1) is particularly interesting as a system just slightly more complex than a scalar equation: the steady-state problem can be readily seen to reduce to a scalar equation after integrating the first equation for u as a function of v and substituting the result into the second equation. On the other hand, stability properties of such stationary solutions are slightly more complex than in the scalar case, where only monotone solutions are stable; see [17, 18, 19, 20, 21]. Interesting dynamics of (1.1) are related to the fact that this system conserves mass $\int u$ with suitable decay conditions at $x = \pm\infty$. This induces a constraint that, in some circumstances, stabilizes energetically unstable solutions [21], but, on the other hand, complicates the analysis by introducing a "neutral mode". Technically, the

[1] **AMS MSC 2000 Mathematics Subject Classification:** 37L15
Keywords: traveling waves, bifurcation, conservation law, Lyapunov-Schmidt reduction.

linearization at stationary solutions always possesses a neutral eigenfunction related to the constraint, creating in particular neutral essential spectrum for linearized operators.

In previous work, we have analyzed periodic patterns, spikes (homoclinic), and layer (heteroclinic) stationary solutions of (1.1). While spikes and periodic solutions are always unstable on the real line, layers can be stable in some circumstances. For a layer solution $(u_L^*(x), v_L^*(x))$, we denote by $(u_L^\pm, v_L^\pm)$ its limits at $x = \pm\infty$. Typically, $u_L^+ \neq u_L^-$, so that layers typically separate spatial regions with different "mass" u. Varying system parameters, one finds codimension-one situations where $u_L^+ = u_L^-$. In such a situation, necessarily $b(u_L^\pm) = 0$ and $u_L(x) \equiv u_L^\pm$ and stability properties of layers change upon perturbing away from this degenerate point. We therefore consider (1.1) ε-dependent cross-term

$$b(u) \mapsto b(u) + \varepsilon.$$

In fact, assuming $\delta > 0$, we showed in [20] that the spectrum of the linearization at a layer solution is contained in $\mathrm{Re}\,\lambda \leq 0$ only if $(u_L^+ - u_L^-)(v_L^+ - v_L^-) \geq 0$. On the other hand, (1.1) possesses a Lyapunov function whenever $b > 0$ [17], so that the boundary of stability can also be seen as the boundary of gradient-like behavior.

Of course, changes of stability are expected to be accompanied by bifurcation of nontrivial solutions. Here, it turned out that the structure of (1.1) prevents a generic saddle-node of layer solutions and layer solutions can typically be continued through such a degenerate point. We emphasize that the change of stability is caused by an eigenvalue crossing from $\mathrm{Re}\,\lambda > 0$ into the essential spectrum, $\mathrm{Re}\,\lambda \leq 0$, upon increasing (or decreasing) ε through 0. It is therefore not immediately clear what type of bifurcation to expect.

In different circumstances, crossing of a zero eigenvalue at the linearization at a standing layer induces bifurcation of traveling fronts; see [2, 3, 6, 9, 14]. In this context, stationary layers are forced by a reflection symmetry in a reaction-diffusion system, and instabilities can occur in a non-variational context. Not surprisingly, given the symmetry, traveling fronts bifurcate in a pitchfork bifurcation with speed $s \sim \varepsilon$, where ε denotes a typical bifurcation parameter.

In the present context, stationary layers are not enforced by symmetry and there is no a priori reason to expect pitchfork bifurcations. Arguing somewhat intuitively, layers separate regions of different mass concentrations u. Since mass transport is primarily diffusive rather than reactive, it cannot propagate at finite speed. Not surprisingly, traveling front solutions $(u(x - st), v(x - st))$ therefore have equal asymptotic mass $u^+ = u^-$. This can be readily seen by integrating the first equation; see Lemma 2.1 for details. As a consequence, traveling fronts may limit on layers with $u^+ = u^-$ in limits where the speed vanishes.

The purpose of this paper is to analyze this somewhat vague and intuitive picture rigorously. Our approach is based on direct Lyapunov-Schmidt methods. We eliminate essential spectrum by the use of exponential weights, which induce negative Fredholm indices. Those can be compensated for by suitable far-field corrections. Complicating the situation compared to previous work [17, 20] is the emergence of a weakly stable direction on a center-manifold. We incorporate this weakly stable direction by explicitly correcting in the far-field via a center-manifold solution. In order to preserve differentiability in this ansatz, we use scales of exponential weights related to the proof of smoothness of center manifolds and stable foliations. A similar approach was used in [10, 11], albeit exploiting algebraic weights.

The remainder of this introduction will present the main results in a precise formulation. We denote by $H^k_\eta(\mathbb{R})$ the Hilbert space of functions u for which $u(\cdot)\cosh(\eta\cdot) \in H^k(\mathbb{R})$, the usual Hilbert space with square integrable derivatives up to order k.

Hypothesis 1. *Throughout this paper we assume that* (2.8) *has an exponentially convergent layer solution* (u^*_L, v^*_L) *with* $u^+_L = u^-_L = u^\infty_L$ *for* $\delta = \delta_0 \neq 0$.

Our main bifurcation result is summarized in the following theorem:

Theorem 1.1. *Assuming Hypothesis 1, there exists a locally unique family of traveling fronts with weak decay, parameterized by* $\pm s \in [0, s_0)$ *(speed) and* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *(bifurcation parameter), bifurcating from the standing layer. The traveling front profile is of the form*

$$
\begin{aligned}
u^*_\pm(\cdot; s, \varepsilon) &= \mu^*_\pm(s,\varepsilon)\chi_\mp + \chi_\pm u^\pm_c(\cdot; \mu^*_\pm(s,\varepsilon), \omega^*_\pm(s,\varepsilon), s, \varepsilon) + \varphi^*(\cdot; s, \varepsilon),\\
v^*_\pm(\cdot; s, \varepsilon) &= v^*_L + \chi_\mp\big(v^\mp(\mu^*_\pm(s,\varepsilon)) - v^\mp_L\big) + \chi_\pm\big(v^\pm_c(\cdot; \mu^*_\pm(s,\varepsilon), \omega^*_\pm(s,\varepsilon), s, \varepsilon) - v^\pm_L\big) + \psi^*(\cdot; s, \varepsilon)
\end{aligned}
$$

for $\pm s \geq 0$. *Here,* $v^\pm(\mu)$ *solve the equation* $g(v) + \delta_0\mu = 0$ *in the neighborhood of* $v^\pm_L$*, respectively, and* $\chi_\pm$ *are smooth with* $\chi_\pm(x) = 1$ *for* $\pm x > 2$, $\chi_\pm(x) = 0$ *for* $\mp x > 2$. *The functions* $\varphi^*(\cdot; s, \varepsilon)$, $\psi^*(\cdot; s, \varepsilon)$ *vary smoothly in* $(s, \varepsilon) \in \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)$ *as elements of* $H^1_\eta(\mathbb{R})$ *and* $H^2_\eta(\mathbb{R})$*, respectively, for some* $\eta > 0$ *sufficiently small. Finally,* $(u^\pm_c, v^\pm_c)(\cdot; \mu, \omega, s, \varepsilon)$ *are the center manifold solutions of the traveling waves ODE associated to* (1.1).

In the following corollary we compute the expansions of the real valued functions $\mu^*_\pm$ and $\omega^*_\pm$ and the first order derivatives of the profiles $u^*_\pm$ and $v^*_\pm$.

Corollary 1.2. *Assume Hypothesis 1 and let* $c_\infty = \frac{b'(u^\infty_L)}{a(u^\infty_L)}$. *The real valued functions* $\mu^*_\pm$ *and* $\omega^*_\pm$ *have the expansion*

$$
\begin{aligned}
\mu^*_\pm(s,\varepsilon) &= u^\infty_L + \frac{c_\infty e^{-c_\infty v^\pm_L}\|(v^*_L)_x\|_2^2}{\delta_0(e^{-c_\infty v^+_L} - e^{-c_\infty v^-_L})}s - \Big[\frac{e^{-c_\infty v^\pm_L}(v^+_L - v^-_L)}{a(u^\infty_L)(e^{-c_\infty v^+_L} - e^{-c_\infty v^-_L})} + \\
&\quad + \frac{1}{a(u^\infty_L)c_\infty}\Big]\varepsilon + \mathcal{O}(2; s, \varepsilon), \quad \textit{if} \quad c_\infty \neq 0,\\
\omega^*_\pm(s,\varepsilon) &= u^\infty_L + \frac{c_\infty e^{-c_\infty v^\mp_L}\|(v^*_L)_x\|_2^2}{\delta_0(e^{-c_\infty v^+_L} - e^{-c_\infty v^-_L})}s - \Big[\frac{e^{-c_\infty v^\mp_L}(v^+_L - v^-_L)}{a(u^\infty_L)(e^{-c_\infty v^+_L} - e^{-c_\infty v^-_L})} + \\
&\quad + \frac{1}{a(u^\infty_L)c_\infty}\Big]\varepsilon + \mathcal{O}(2; s, \varepsilon), \quad \textit{if} \quad c_\infty \neq 0,\\
\mu^*_\pm(s,\varepsilon) &= u^\infty_L - \frac{\|(v^*_L)_x\|_2^2}{\delta_0(v^+_L - v^-_L)}s - \frac{v^\pm_L - v^\mp_L}{2a(u^\infty_L)}\varepsilon + \mathcal{O}(2; s, \varepsilon), \quad \textit{if} \quad c_\infty = 0,\\
\omega^*_\pm(s,\varepsilon) &= u^\infty_L - \frac{\|(v^*_L)_x\|_2^2}{\delta_0(v^+_L - v^-_L)}s + \frac{v^\pm_L - v^\mp_L}{2a(u^\infty_L)}\varepsilon + \mathcal{O}(2; s, \varepsilon), \quad \textit{if} \quad c_\infty = 0.
\end{aligned}
\tag{1.2}
$$

In addition, we have that

$$
\partial_s u^*_\pm(\cdot; 0, 0) = -\kappa_1(c_\infty)e^{c_\infty v^*_L}, \quad \partial_\varepsilon u^*_\pm(\cdot; 0, 0) = \delta_0\kappa_2(c_\infty)(v^*_L)^{1-|\mathrm{sign}(c_\infty)|} + \delta_0\kappa_3(c_\infty)e^{c_\infty v^*_L},
$$

$$\partial_s v_\pm^*(\cdot;0,0) = W^*, \quad \partial_\varepsilon v_\pm^*(\cdot;0,0) = Z^*, \tag{1.3}$$

where W^* and Z^* solve the equations

$$\begin{aligned} &W^*_{xx} + g'(v_L^*)W^* = \delta_0\kappa_1(c_\infty)e^{c_\infty v_L^*} - (v_L^*)_x, \\ &Z^*_{xx} + g'(v_L^*)Z^* = -\delta_0\kappa_2(c_\infty)(v_L^*)^{1-|\mathrm{sign}(c_\infty)|} - \delta_0\kappa_3(c_\infty)e^{c_\infty v_L^*}. \end{aligned} \tag{1.4}$$

Here the functions $\kappa_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, 3$, are defined by

$$\kappa_1(c_\infty) = \begin{cases} \frac{c_\infty \|(v_L^*)_x\|_2^2}{\delta_0(e^{c_\infty v_L^+} - e^{c_\infty v_L^-})} & \text{if } c_\infty \neq 0 \\ \frac{\|(v_L^*)_x\|_2^2}{\delta_0(v_L^+ - v_L^-)} & \text{if } c_\infty = 0 \end{cases}, \quad \kappa_2(c_\infty) = \begin{cases} -\frac{1}{a(u_L^\infty)c_\infty} & \text{if } c_\infty \neq 0 \\ \frac{1}{a(u_L^\infty)} & \text{if } c_\infty = 0 \end{cases},$$

$$\kappa_3(c_\infty) = \begin{cases} \frac{v_L^+ - v_L^-}{a(u_L^\infty)(e^{c_\infty v_L^+} - e^{c_\infty v_L^-})} & \text{if } c_\infty \neq 0 \\ -\frac{(v_L^+ + v_L^-)}{2a(u_L^\infty)} & \text{if } c_\infty = 0 \end{cases}. \tag{1.5}$$

Next, we point out that there exists another class of traveling waves bifurcating from the standing layer with constant mass u_L^∞, under the additional assumption that $b'(u_L^\infty) \neq 0$. Indeed, one can prove that it is possible to find traveling waves with constant mass bifurcating from the standing layer (u_L^∞, v_L^*) under the same perturbation $b(u) \to b_\varepsilon(u) = b(u) + \varepsilon$.

Theorem 1.3. *Assume Hypothesis 1 and suppose that $b'(u_L^\infty) \neq 0$. Then, there exists a locally unique family of traveling fronts with constant mass u, parameterized by $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, bifurcating from the standing layer. The traveling front profile is of the form*

$$\begin{aligned} &\overline{u}(\cdot;\varepsilon) \equiv \overline{\mu}(\varepsilon) \\ &\overline{v}(\cdot;\varepsilon) = v^+(\overline{\mu}(\varepsilon))\chi_+ + v^-(\overline{\mu}(\varepsilon))\chi_- + \overline{\psi}(\cdot;\varepsilon) \end{aligned}$$

The functions $\overline{\mu}(\cdot)$ and $\overline{\psi}(\cdot;s,\varepsilon)$ vary smoothly in $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ as elements of $\mathbb{R}$ and $H^2_\eta(\mathbb{R})$, respectively, for some $\eta > 0$ sufficiently small. The speed of the traveling waves is given by a function $s = \overline{s}(\varepsilon)$, which has the expansion

$$\overline{s}(\varepsilon) = \frac{\delta_0(v_L^+ - v_L^-)}{b'(u_L^\infty)\|(v_L^*)_x\|_2^2}\varepsilon + \mathcal{O}(\varepsilon^2). \tag{1.6}$$

In addition, we have that $\partial_\varepsilon \overline{u}(\cdot;0) = \overline{\mu}'(0) = -\frac{1}{b'(u_L^\infty)}$ and $\partial_\varepsilon \overline{v}(\cdot;0) = Y^*$, where Y^* solves the equation

$$Y^*_{xx} + g'(v_L^*)Y^* = \frac{\delta_0}{b'(u_L^\infty)} - \frac{\delta_0(v_L^+ - v_L^-)}{b'(u_L^\infty)\|(v_L^*)_x\|_2^2}(v_L^*)_x. \tag{1.7}$$

Using the existence result from Theorem 1.3 we can study the stability of bifurcating traveling front solution. We compute the spectrum of the linearization of (1.1) in the moving frame at a traveling front $(\overline{u}(\cdot;\varepsilon), \overline{v}(\cdot;\varepsilon))$ obtained by Theorem 1.1,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \overline{\mathcal{L}}(\varepsilon)\begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.8}$$

where

$$\overline{\mathcal{L}}(\varepsilon) = \begin{bmatrix} \partial_x\Big(a(\overline{\mu}(\varepsilon))\partial_x - b'(\overline{\mu}(\varepsilon))\overline{v}_x(\cdot;\varepsilon)\Big) + \overline{s}(\varepsilon)\partial_x & 0 \\ \delta_0 & \partial_x^2 + \overline{s}(\varepsilon)\partial_x + g'(\overline{v}(\cdot;\varepsilon)) \end{bmatrix}. \tag{1.9}$$

We point out, that for any $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ the linear operator $\overline{\mathcal{L}}(\varepsilon)$ can be considered as as a closed linear operator on any space $L^2_\nu(\mathbb{R}, \mathbb{C}^2)$ for any $\nu \in \mathbb{R}$.

Corollary 1.4. *Assume Hypothesis 1 and suppose $b'(u_L^\infty) \neq 0$. Then, the bifurcating traveling fronts obtained by Theorem 1.3 are stable. More precisely, the following assertions hold true:*

(i) $\sup \mathrm{Re}\,\big(\sigma_{\mathrm{ess}}(\overline{\mathcal{L}}(\varepsilon))\big) = 0$ *and* $\sigma_{\mathrm{ess}}(\overline{\mathcal{L}}(\varepsilon)) \cap \mathrm{i}\mathbb{R} = \{0\}$ *for all* $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$*;*

(ii) The linear operator $\overline{\mathcal{L}}(\varepsilon)$ has no eigenvalue with positive real part;

Outline: In Section 2, we prepare the proofs in a sequence of lemmas, in particular setting up a nonlinear equation with far-field corrections, analyzing Fredholm properties of the linearization, and establishing smoothness and thus preparing for Lyapunov-Schmidt reduction. Section 3 exploits those results to prove our main bifurcation result, Theorem 1.1 and expansions in Corollary 1.2. Section 4 contains proofs for constant-mass traveling waves, Theorem 1.3 and Corollary 1.4.

Notations: For an operator T on a Hilbert space X we use T^*, $\mathrm{dom}(T)$, $\ker T$, $\mathrm{im}\, T$, $\sigma(T)$, $\rho(T)$ and $T_{|Y}$ to denote the adjoint, domain, kernel, range, spectrum, resolvent set and the restriction of T on a subspace Y of X. We divide the spectrum of T into two disjoint sets: $\sigma_{\mathrm{point}}(T)$, the union of eigenvalues λ for which $T - \lambda$ is Fredholm with index 0, and $\sigma_{\mathrm{ess}}(T)$ its complement in $\sigma(T)$. The Morse index of a hyperbolic matrix A, denoted $i(A)$, is the dimension of its unstable subspace, which is the generalized eigenspace associated with all eigenvalues λ of A that have Re $\lambda > 0$. The usual Lebesgue spaces, the space of bounded uniformly continuous functions and the weighted Lebesgue spaces of vector valued functions are denoted by $L^p(\mathbb{R}, \mathbb{C}^N)$, $BUC(\mathbb{R}, \mathbb{C}^N)$ and $L^p(\mathbb{R}, \mathbb{C}^N; \omega(x)dx)$ respectively. If $\omega(x) = e^{2\eta|x|}$ for all $x \in \mathbb{R}$ we denote the L^p-weighted space by $L^p_\eta(\mathbb{R}, \mathbb{C}^N)$. Similarly, we define weighted Sobolev spaces $W^{k,p}_\eta(\mathbb{R}, \mathbb{C}^N)$ and $H^k_\eta(\mathbb{R}, \mathbb{C}^N)$.

Acknowledgment: AS acknowledges support under grant NSF DMS-1311740. AP acknowledges support by through a Summer Research Grant by College of Arts and Science, Miami University.

2 Setting up the bifurcation problem — weakly decaying traveling fronts

In this section we set up a nonlinear bifurcation problem for the existence of traveling fronts in (1.1). The key steps are to identify far-field corrections, (2.13), differentiability of the nonlinearity in spaces that gain exponential localization, Lemmas 2.6 and 2.7, Fredholm properties of the linearized operator and its bordered version with far-field corrections, Lemmas 2.8–2.11, and differentiability of far-field contributions in spaces that loose exponential localization 2.12.

We start by looking for traveling front solutions of the form $(u(x-st), v(x-st))$ of the system (1.1) under the perturbation $b(u) \mapsto b_\varepsilon(u) = b(u) + \varepsilon$. The scalar functions (u, v) satisfy the system

$$\begin{cases} [a(u)u' - b_\varepsilon(u)v']' + su' = 0, \\ v'' + \delta u + g(v) + sv' = 0. \end{cases} \tag{2.1}$$

We note that the first equation of this system can be integrated once, to obtain the equation

$$a(u)u' - b_\varepsilon(u)v' + su = s\mu, \tag{2.2}$$

where $\mu \in \mathbb{R}$ is a constant. In the next lemma we obtain a necessary condition for any exponentially converging solution of (2.1).

Lemma 2.1. *Assume that* (u, v) *satisfies* (2.1) *for a fixed* $s, \varepsilon \in \mathbb{R}$ *and that* u, v *are exponentially converging, that is, there exist* $c, \eta > 0$, $u^\pm, v^\pm \in \mathbb{R}$ *such that*

$$|u(x) - u^\pm| \leq ce^{-\eta|x|}, \quad |v(x) - v^\pm| \leq ce^{-\eta|x|}, \quad \text{for all} \quad x \in \mathbb{R}_\pm. \tag{2.3}$$

Then, $u^+ = u^- = \mu$, $g(v^\pm) + \delta\mu = 0$, *and* $\lim_{x\to\pm\infty} u'(x) = \lim_{x\to\pm\infty} v'(x) = 0$.

Proof. We define the functions $w_\pm : \mathbb{R} \to \mathbb{R}$ by

$$w_\pm(x) = v(x) - s\int_x^{\pm\infty} (v(y) - v^\pm)\mathrm{d}y. \tag{2.4}$$

From (2.3), we infer that the functions $w_\pm$ are well-defined, of class C^2, and

$$w_\pm' = v' + s(v - v^\pm), \quad w_\pm'' = v'' + sv'.$$

Using the second equation of the system (2.1), we obtain that $w_\pm'' = -\delta u - g(v)$, which implies that $w_\pm''$ is bounded. Since, in addition, $\lim_{x\to\pm\infty} w_\pm(x)$ is finite, using Taylor's theorem we infer that $\lim_{x\to\pm\infty} w_\pm'(x) = 0$. From (2.3), we conclude that

$$v'(x) = w_\pm'(x) - s(v(x) - v^\pm) \to 0 \quad \text{as} \quad x \to \pm\infty. \tag{2.5}$$

Solving for u' in (2.2), we have that

$$u' = \frac{b_\varepsilon(u)}{a(u)}v' + \frac{s\mu - su}{a(u)}. \tag{2.6}$$

Since $\lim_{x\to\pm\infty} u(x)$ is finite, we infer from (2.5) and (2.6) that $\lim_{x\to\pm\infty} u'(x)$ is finite. From l'Hospital Theorem it follows that the last limit cannot be anything else but 0, that is

$$u'(x) \to 0 \quad \text{as} \quad x \to \pm\infty. \tag{2.7}$$

Passing to the limit as $x \to \pm\infty$ in (2.6), we obtain that $u^+ = u^- = \mu$. Similarly, passing to the limit as $x \to \pm\infty$ in the second equation of (2.1), we have that $\delta\mu + g(v^\pm) = 0$. $\square$

We are interested in finding traveling waves solutions of (1.1) whose profile at $s=\varepsilon=0$ is a heteroclinic solution of the system

$$\begin{cases} [a(u)u_x - b(u)v_x]_x = 0, \\ v_{xx} + \delta u + g(v) = 0. \end{cases} \tag{2.8}$$

In the next remark we collect a few results that follow immediately from Hypothesis 1; proofs are carried out in [20, Section 2].

Remark 2.2. *The following assertions are true:*

(i) $u_L^* \equiv u_L^\infty$, $b(u_L^\infty) = 0$*;*

(ii) $g(v_L^\pm) + \delta_0 u_L^\infty = 0$*;*

(iii) $g'(v_L^\pm) < 0$*;*

(iv) $\displaystyle\int_{v_L^-}^{v_L^+} g(v)\mathrm{d}v = -\delta_0 u_L^\infty (v_L^+ - v_L^-)$*.*

From Lemma 2.1, we note that (2.1) is equivalent to the system

$$\begin{cases} u' = \frac{b(u)+\varepsilon}{a(u)} v' + \frac{s(\mu-u)}{a(u)}, \\ v'' = -\delta_0 u - g(v) - sv', \\ \lim\limits_{x\to\pm\infty} u(x) = \mu. \end{cases} \tag{2.9}$$

Next, we note that the equilibria $v_L^\pm$ are robust under small perturbations and the profile of the traveling front satisfies the conditions required by Lemma 2.1.

Remark 2.3. *There exists* $\mu_0 > 0$ *and smooth functions* $v^\pm : (u_L^\infty - \mu_0, u_L^\infty + \mu_0) \to \mathbb{R}$ *such that*

(i) $v^\pm(u_L^\infty) = v_L^\pm$*;*

(ii) For each $\mu \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)$*, in a neighborhood of* $v_L^\pm$*, respectively, the equation* $g(v) + \delta_0\mu = 0$ *has the unique solution* $v = v^\pm(\mu)$*.*

The conclusions of the remark follow by applying the Implicit Functions Theorem to the function $H : \mathbb{R}^2 \to \mathbb{R}$ defined by $H(v,\mu) = g(v) + \delta_0\mu$. From Remark 2.2 it follows that

$$H(v_L^\pm, u_L^\infty) = g(v_L^\pm) + \delta_0 u_L^\infty = 0, \quad \partial_v H(v_L^\pm, u_L^\infty) = g'(v_L^\pm).$$

Next, we note that (2.9) can be rewritten as the first order system

$$\begin{cases} u' = \frac{b(u)+\varepsilon}{a(u)} w + \frac{s(\mu-u)}{a(u)}, \\ v' = w, \\ w' = -\delta_0 u - g(v) - sw. \end{cases} \tag{2.10}$$

For $s = 0$, (2.10) possesses an equilibrium at $u = \mu$, $v = v^{\pm}(\mu)$, $w = 0$, for any $\mu \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)$ and any $\varepsilon \in \mathbb{R}$. The Jacobian of the left-hand side of (2.10) at any of the equilibria described above is given by

$$J^{\pm}(\mu,\varepsilon) = \begin{bmatrix} 0 & 0 & \frac{b(\mu)+\varepsilon}{a(\mu)} \\ 0 & 0 & 1 \\ -\delta_0 & -g'(v^{\pm}(\mu)) & 0 \end{bmatrix} \quad \text{for any} \quad \mu \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0), \quad \varepsilon \in \mathbb{R}.$$

For any $\mu \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)$ and $\varepsilon \in \mathbb{R}$ small enough the matrix $J^{\pm}(\mu,\varepsilon)$ has three algebraically simple eigenvalues given by $\lambda = 0$ and $\lambda = \pm\sqrt{-g'(v^{\pm}(\mu)) - \delta_0 \frac{b(\mu)+\varepsilon}{a(\mu)}}$. Moreover, one can readily check that $\ker J^{\pm}(\mu,\varepsilon) = \mathrm{Span}\{(1, -\frac{1}{\delta_0} g'(v^{\pm}(\mu)), 0)^{\mathrm{T}}\}$ and $\ker J^{\pm}(\mu,\varepsilon)^* = \mathrm{Span}\{(1, -\frac{b(\mu)+\varepsilon}{a(\mu)}, 0)^{\mathrm{T}}\}$. We infer that for any $\mu \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)$ and $\varepsilon \in \mathbb{R}$ small enough, system (2.10) has a center manifold $\mathcal{W}^{\pm}(\mu,\varepsilon)$ at $(\mu, v^{\pm}(\mu), 0)^{\mathrm{T}}$, which at $s = 0$ is simply given by the curve of equilibria $\delta_0 u + g(v) = 0$, $w = 0$. The dynamics on the center manifold $\mathcal{W}^{\pm}(\mu,\varepsilon)$ are hence determined by the dynamics of the u-component,

$$u' = s\Big[\frac{\mu - u}{a(u_L^\infty)} + \mathcal{O}((u-\mu)^2)\Big]. \tag{2.11}$$

We denote by $u_c^{\pm}(\cdot;\mu,\omega,s,\varepsilon)$ the solution of (2.11) defined for $\pm x \geq 1$ with initial condition $u_c^{\pm}(\pm 1;\mu,\omega,s,\varepsilon) = \omega$. This solution possesses the expansion

$$u_c^{\pm}(x;\mu,\omega,s,\varepsilon) = \mu + e^{\frac{-s(x\mp 1)}{a(u_L^\infty)}}\Big[(\omega-\mu) + \mathcal{O}((\omega-\mu)^2)\Big], \ \pm x \geq 1, \mu,\omega \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0), \varepsilon \in \mathbb{R}. \tag{2.12}$$

The other components of the solution of (2.10) on the center manifold $\mathcal{W}^{\pm}(\mu,\varepsilon)$ satisfy the following expansions:

$$v_c^{\pm}(x;\mu,\omega,s,\varepsilon) = v^{\pm}(u_c^{\pm}(x;\mu,\omega,s,\varepsilon)) + \mathcal{O}(s), \quad w_c^{\pm}(x;\mu,\omega,s,\varepsilon) = \partial_x v_c^{\pm}(x;\mu,\omega,s,\varepsilon) = \mathcal{O}(2). \tag{2.13}$$

Next, we collect some of the properties of the center manifold solutions $(u_c^{\pm}, v_c^{\pm}, w_c^{\pm})$ needed in the sequel. We are especially interested in the boundedness and growth properties of these solutions for $\mu,\omega \sim u_L^\infty$ and $s,\varepsilon \sim 0$.

Remark 2.4. *Differentiating in* (2.11)–(2.13) *we have that the following assertions hold true:*

(i) There exists $\varepsilon_0 > 0$ and $s_0 > 0$ such that the functions $(u_c^{\pm}, v_c^{\pm}, w_c^{\pm})(\cdot,\mu,\omega,s,\varepsilon)$, and $(\partial_x u_c^{\pm}, \partial_x v_c^{\pm}, \partial_x w_c^{\pm})(\cdot,\mu,\omega,s,\varepsilon) \in L^\infty$, uniformly for all $\mu,\omega \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0), \varepsilon \in (-\varepsilon_0,\varepsilon_0)$ and $0 \leq \pm s < s_0$;

(ii) For any $q \in \{\mu,\omega,s,\varepsilon\}$ the partial derivatives $(\partial_q u_c^{\pm}, \partial_q v_c^{\pm}, \partial_q w_c^{\pm})(x,\mu,\omega,s,\varepsilon)$ grow polynomially as $x \to \pm\infty$, uniformly for all $\mu,\omega \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0), \varepsilon \in (-\varepsilon_0,\varepsilon_0)$ and $0 \leq \pm s < s_0$;

(iii) Moreover, we have that

$$\left\{\begin{array}{l} \partial_\omega u_c^{\pm}(\cdot;\mu,\omega,0,0) = 1, \\ \partial_\omega v_c^{\pm}(\cdot;\mu,\omega,0,0) = -\frac{\delta_0}{g'(v^{\pm}(\omega))}, \\ \partial_\omega w_c^{\pm}(\cdot;\mu,\omega,0,0) = 0, \end{array}\right. \quad \left\{\begin{array}{l} \partial_\mu u_c^{\pm}(\cdot;\mu,\omega,0,0) = 0, \\ \partial_\mu v_c^{\pm}(\cdot;\mu,\omega,0,0) = 0, \\ \partial_\mu w_c^{\pm}(\cdot;\mu,\omega,0,0) = 0, \end{array}\right. \tag{2.14}$$

$$\begin{cases} \partial_s u_c^{\pm}(x;\mu,\omega,0,0) = \frac{\mu-\omega}{a(\omega)}(x\pm 1), \\ \partial_s v_c^{\pm}(x;\mu,\omega,0,0) = 0, \\ \partial_s w_c^{\pm}(x;\mu,\omega,0,0) = 0, \end{cases} \qquad \begin{cases} \partial_\varepsilon u_c^{\pm}(\cdot;\mu,\omega,0,0) = 0, \\ \partial_\varepsilon v_c^{\pm}(\cdot;\mu,\omega,0,0) = 0, \\ \partial_\varepsilon w_c^{\pm}(\cdot;\mu,\omega,0,0) = 0. \end{cases} \tag{2.15}$$

Since for any $\mu \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the equilibrium $(\mu, v^{\pm}(\mu), 0)^{\mathrm{T}}$ is stable within the center manifold $\mathcal{W}^{\pm}(\mu,\varepsilon)$, solutions that converge to the equilibrium converge with uniform exponential rate for $x \to \pm\infty$. Therefore, when $s \geq 0$, we use the ansatz

$$\begin{pmatrix} u(x) \\ v(x) \\ w(x) \end{pmatrix} = \begin{pmatrix} u_L^\infty \\ v_L^*(x) \\ (v_L^*)'(x) \end{pmatrix} + \chi_-(x) \begin{pmatrix} \mu - u_L^\infty \\ v^-(\mu) - v_L^- \\ 0 \end{pmatrix} + \chi_+(x) \begin{pmatrix} u_c^+(x;\mu,\omega,s,\varepsilon) - u_L^\infty \\ v_c^+(x;\mu,\omega,s,\varepsilon) - v_L^+ \\ w_c^+(x;\mu,\omega,s,\varepsilon) \end{pmatrix} + \begin{pmatrix} \varphi(x) \\ \psi(x) \\ \phi(x) \end{pmatrix}, \tag{2.16}$$

while for the case $s \leq 0$, we use the ansatz

$$\begin{pmatrix} u(x) \\ v(x) \\ w(x) \end{pmatrix} = \begin{pmatrix} u_L^\infty \\ v_L^*(x) \\ (v_L^*)'(x) \end{pmatrix} + \chi_-(x) \begin{pmatrix} u_c^-(x;\mu,\omega,s,\varepsilon) - u_L^\infty \\ v_c^-(x;\mu,\omega,s,\varepsilon) - v_L^- \\ w_c^-(x;\mu,\omega,s,\varepsilon) \end{pmatrix} + \chi_+(x) \begin{pmatrix} \mu - u_L^\infty \\ v^+(\mu) - v_L^+ \\ 0 \end{pmatrix} + \begin{pmatrix} \varphi(x) \\ \psi(x) \\ \phi(x) \end{pmatrix}. \tag{2.17}$$

Here we used the definition $\chi_\pm = \frac{1}{2}(1 \pm \rho)$, with $\rho \in C^\infty(\mathbb{R})$ such that $-1 \leq \rho \leq 1$, $\rho(x) = -1$ for all $x \leq -2$ and $\rho(x) = 1$ for all $x \geq 2$. The functions $v^{\pm}(\cdot)$ are defined in Lemma 2.3, $\varphi, \psi, \phi \in H_\eta^1(\mathbb{R},\mathbb{R})$, $\eta > 0$ is a small exponential weight chosen such that

$$\tilde{v}_L^* := v_L^* - v_L^+ \chi_+ - v_L^- \chi_- \in H_\eta^2(\mathbb{R},\mathbb{R}). \tag{2.18}$$

Substituting the ansatz (2.16)–(2.17) into (2.10) we obtain two equations $\mathcal{F}_\pm(\varphi,\psi,\phi,\mu,\omega,s,\varepsilon) = 0$, $\pm s \geq 0$, where $\mathcal{F}_\pm : H_\eta^1(\mathbb{R},\mathbb{R}^3) \times (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0,\varepsilon_0) \to L_\eta^2(\mathbb{R},\mathbb{R}^3)$ is defined by[2]

$$\begin{aligned} \mathcal{F}_\pm(\varphi,\psi,\phi,\mu,\omega,s,\varepsilon) = \begin{pmatrix} \varphi' \\ \psi' \\ \phi' \end{pmatrix} - \begin{pmatrix} \frac{b(\chi_\mp \mu + \chi_\pm u_c^\pm + \varphi) + \varepsilon}{a(\chi_\mp \mu + \chi_\pm u_c^\pm + \varphi)} \Big((v_L^*)_x + \chi_\pm w_c^\pm + \phi \Big) \\ \phi \\ -\delta_0(\chi_\mp \mu + \varphi) - g\big(\tilde{v}_L^* + \chi_\mp v^\mp(\mu) + \chi_\pm v_c^\pm + \psi\big) \end{pmatrix} \\ + \begin{pmatrix} \chi_\mp' \mu + \chi_\pm' u_c^\pm + \chi_\pm (u_c^\pm)_x + \frac{s(\chi_\mp \mu + \chi_\pm u_c^\pm + \varphi - \mu)}{a(\chi_\mp \mu + \chi_\pm u_c^\pm + \varphi)} \\ \chi_\mp'(v^\mp(\mu) - v_L^\mp) + \chi_\pm'(v_c^\pm - v_L^\pm) \\ s\Big((v_L^*)_x + \phi\Big) + (v_L^*)_{xx} + \chi_\pm' w_c^\pm - \chi_\pm g(v_c^\pm) \end{pmatrix}. \end{aligned} \tag{2.19}$$

We note that the functions $\mathcal{F}_\pm$ are not of class C^1. To overcome this issue, we formally expand the functions $\mathcal{F}_\pm$ as follows:

$$\begin{aligned} \mathcal{F}_\pm(\varphi,\psi,\phi,\mu,\omega,s,\varepsilon) = L_\pm(\mu,\omega,s,\varepsilon) \begin{pmatrix} \varphi \\ \psi \\ \phi \end{pmatrix} + Q_\mu^\pm(\mu - u_L^\infty) + Q_\omega^\pm(\omega - u_L^\infty) + Q_s^\pm s + Q_\varepsilon^\pm \varepsilon \\ + R_\pm(\mu,\omega,s,\varepsilon) + N_\pm(\varphi,\psi,\phi,\mu,\omega,s,\varepsilon). \end{aligned} \tag{2.20}$$

[2] Here we abreviate $(u_c^\pm, v_c^\pm, w_c^\pm) = (u_c^\pm, v_c^\pm, w_c^\pm)(\cdot,\mu,\omega,s,\varepsilon)$.

Here $L_\pm : (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}\big(H^1_\eta(\mathbb{R}, \mathbb{R}^3), L^2_\eta(\mathbb{R}, \mathbb{R}^3)\big)$ are defined by

$$L_\pm(\mu,\omega,s,\varepsilon)\begin{pmatrix}\varphi\\ \psi\\ \phi\end{pmatrix} = \begin{pmatrix}\varphi'\\ \psi'\\ \phi'\end{pmatrix} - \begin{pmatrix}\big(\frac{b+\varepsilon}{a}\big)'(\chi_\mp\mu+\chi_\pm u_c^\pm)\Big((v_L^*)_x+\chi_\pm w_c^\pm\Big)\varphi\\ \phi\\ -\delta_0\varphi - g'\big(\tilde v_L^* + \chi_\mp v^\mp(\mu)+\chi_\pm v_c^\pm\big)\psi\end{pmatrix}$$
$$+\begin{pmatrix} s\frac{a(\chi_\mp\mu+\chi_\pm u_c^\pm)-\chi_\pm(u_c^\pm+\mu)a'(\chi_\mp\mu+\chi_\pm u_c^\pm)}{a^2(\chi_\mp\mu+\chi_\pm u_c^\pm)}(\chi_\mp\mu+\chi_\pm u_c^\pm)\varphi\\ 0\\ s\phi\end{pmatrix}; \tag{2.21}$$

$$\begin{aligned}
Q_\mu^\pm &= \Big(-c_\infty\chi_\mp(v_L^*)_x+\chi_\mp', -\frac{\delta_0}{g'(v_L^\mp)}\chi_\mp', \delta_0\chi_\mp - \delta_0\frac{g'(v_L^*)}{g'(v_L^\mp)}\chi_\mp\Big)^{\mathrm{T}},\\
Q_\omega^\pm &= \Big(-c_\infty\chi_\pm(v_L^*)_x+\chi_\pm', -\frac{\delta_0}{g'(v_L^\pm)}\chi_\pm', \delta_0\chi_\pm - \delta_0\frac{g'(v_L^*)}{g'(v_L^\pm)}\chi_\pm\Big)^{\mathrm{T}},\\
Q_s^\pm &= \Big(0,0,(v_L^*)_x\Big)^{\mathrm{T}},\quad Q_\varepsilon^\pm = \Big(-\frac{1}{a(u_L^\infty)}(v_L^*)_x,0,0\Big)^{\mathrm{T}}.
\end{aligned}\tag{2.22}$$

Moreover, $R_\pm : (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to L^2_\eta(\mathbb{R}, \mathbb{R}^3)$ is defined by

$$R_\pm(\mu,\omega,s,\varepsilon) = \mathcal{F}_\pm(0,0,0,\mu,\omega,s,\varepsilon) - Q_\mu^\pm(\mu-u_L^\infty) - Q_\omega^\pm(\omega-u_L^\infty) - Q_s^\pm s - Q_\varepsilon^\pm\varepsilon, \tag{2.23}$$

while $N_\pm : H^1_\eta(\mathbb{R}, \mathbb{R}^3) \times (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to L^2_\eta(\mathbb{R}, \mathbb{R}^3)$ is the remainder, satisfying the condtion

$$N_\pm(\varphi,\psi,\phi,\mu,\omega,s,\varepsilon) = \mathcal{O}(2;\varphi,\psi,\phi). \tag{2.24}$$

Next, we are focusing our attention on the properties of the functions from the decomposition (2.20).

Remark 2.5. *Since the layer (u_L^∞, v_L^*) converges exponentially at $\pm\infty$, we infer that there exists $\gamma_0 > 0$ such that $Q_\mu^\pm, Q_\omega^\pm, Q_s^\pm, Q_\varepsilon^\pm \in H^1_{\eta+\gamma}$ for any $\gamma \in (0, \gamma_0)$.*

Lemma 2.6. *Then, the functions $R_\pm : (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3)$ are of class C^1 for any $\gamma > 0$. Moreover, we have that*

$$R_\pm(u_L^\infty, u_L^\infty, 0, 0) = \partial_q R_\pm(u_L^\infty, u_L^\infty, 0, 0) = 0 \quad \textit{for any} \quad q \in \{\mu, \omega, s, \varepsilon\}. \tag{2.25}$$

Proof. Since the functions $(u_c^\pm, v_c^\pm, w_c^\pm)$ are the center manifold solutions of (2.10) used in the ansatz (2.16)–(2.17), we conclude that $\mathcal{F}_\pm(0,0,0,\mu,\omega,s,\varepsilon)$ is a smooth function with compact support. Thus, we have that there exist $\tau^\pm \in C^\infty(\mathbb{R})$ with compact support and $f_1^\pm, f_2^\pm, f_3^\pm : \mathbb{R} \times (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ such that

$$R_\pm(\mu,\omega,s,\varepsilon) = \begin{pmatrix} f_1^\pm(\cdot;\mu,\omega,s,\varepsilon)\\ f_2^\pm(\cdot;\mu,\omega,s,\varepsilon)\\ f_3^\pm(\cdot;\mu,\omega,s,\varepsilon)\end{pmatrix}\tau^\pm(\cdot) - Q_\mu^\pm(\mu-u_L^\infty) - Q_\omega^\pm(\omega-u_L^\infty) - Q_s^\pm s - Q_\varepsilon^\pm\varepsilon. \tag{2.26}$$

The functions $f_j^{\pm}$, $j = 1, 2, 3$, can be expressed in terms of the functions a, b, g, the center manifold solutions $(u_c^{\pm}, v_c^{\pm}, w_c^{\pm})$, the cut-off functions $\chi_{\pm}$ and the variables μ, ω, s, ε. From Remark 2.4(i) we have that

$$f_j^{\pm}, \partial_x f_j^{\pm} \in L^{\infty}\Big(\mathbb{R} \times (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)\Big), \quad j = 1, 2, 3. \tag{2.27}$$

In addition, from Remark 2.4(i) we conclude that for any $q \in \{\mu, \omega, s, \varepsilon\}$ the partial derivatives $\partial_q f_j^{\pm}$ grow polynomially for $x \to \pm\infty$. Since the center manifold solutions $(u_c^{\pm}, v_c^{\pm}, w_c^{\pm})$ are solutions of (2.10), it follows that for any $q \in \{\mu, \omega, s, \varepsilon\}$ the partial derivatives $\partial_x \partial_q f_j^{\pm}$ grow polynomially for $x \to \pm\infty$. We infer that for any $\theta > 0$ there exits $M_\theta > 0$ such that

$$|\partial_q f_j^{\pm}(x; \mu, \omega, s, \varepsilon)| + |\partial_x \partial_q f_j^{\pm}(x; \mu, \omega, s, \varepsilon)| \leq M_\theta \, e^{\theta|x|} \tag{2.28}$$

for any $x \in \mathbb{R}$, $\mu, \omega \in (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $0 \leq \pm s \leq s_0$, $q \in \{\mu, \omega, s, \varepsilon\}$, $j = 1, 2, 3$. From (2.23), (2.27), (2.28), Remark 2.5 and Lemma A.1 we obtain that the functions $R_{\pm} : (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3)$ are of class C^1 for any $\gamma > 0$. Assertion (2.25) follows from Remark 2.4(iii) and the definitions of the functions $\mathcal{F}_{\pm}$ and $R_{\pm}$ in (2.19) and (2.23), respectively. $\square$

Lemma 2.7. *The functions $N_{\pm} : H^1_\eta(\mathbb{R}, \mathbb{R}^3) \times (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to H^1_{2\eta-\gamma}(\mathbb{R}, \mathbb{R}^3)$ are of class C^1 for any $\gamma > 0$.*

Proof. Since the functions $N_{\pm}$ are defined as the second order remainder in the decomposition (2.20), we have that

$$N_{\pm}(\varphi, \psi, \phi, \mu, \omega, s, \varepsilon) = \sum_{(k,i,j) \in \mathcal{Z}_2} \varphi^k \psi^i \phi^j \Big[\alpha_{kij}^{\pm}(f_{kij}^{\pm}(\cdot; \mu, \omega, s, \varepsilon) + \varphi) + \beta_{kij}^{\pm}(g_{kij}^{\pm}(\cdot; \mu, \omega, s, \varepsilon) + \psi)\Big]. \tag{2.29}$$

Here $\mathcal{Z}_2$ is defined by $\mathcal{Z}_2 = \{(k, i, j) \in \mathbb{Z}_+^3 : k + i + j = 2\}$. The functions $\alpha_{kij}^{\pm}, \beta_{kij}^{\pm} : \mathbb{R} \to \mathbb{R}$ are C^3 functions for any $(k, i, j) \in \mathcal{Z}_2$. Similar to the previous lemma, the functions $f_{kij}^{\pm}, g_{kij}^{\pm} : \mathbb{R} \times (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ can be expressed in terms of the functions a, b, g, the center manifold solutions $(u_c^{\pm}, v_c^{\pm}, w_c^{\pm})$, the cut-off functions $\chi_{\pm}$ and the variables μ, ω, s, ε. Therefore, from Remark 2.4 we infer that

$$f_{kij}^{\pm}, \partial_x f_{kij}^{\pm}, g_{kij}^{\pm}, \partial_x g_{kij}^{\pm} \in L^{\infty}\Big(\mathbb{R} \times (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)\Big) \tag{2.30}$$

for any $(k, i, j) \in \mathcal{Z}_2$. In addition, we have that for any $\theta > 0$ there exits $M_\theta > 0$ such that

$$\begin{aligned} &|\partial_q f_{kij}^{\pm}(x; \mu, \omega, s, \varepsilon)| + |\partial_x \partial_q f_{kij}^{\pm}(x; \mu, \omega, s, \varepsilon)| \leq M_\theta \, e^{\theta|x|}, \\ &|\partial_q g_{kij}^{\pm}(x; \mu, \omega, s, \varepsilon)| + |\partial_x \partial_q g_{kij}^{\pm}(x; \mu, \omega, s, \varepsilon)| \leq M_\theta \, e^{\theta|x|} \end{aligned} \tag{2.31}$$

for any $x \in \mathbb{R}$, $\mu, \omega \in (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $0 \leq \pm s \leq s_0$, $q \in \{\mu, \omega, s, \varepsilon\}$, $(k, i, j) \in \mathcal{Z}_2$. From (2.30), (2.31) and Lemma A.1 we conclude that the functions $f_{kij}^{\pm}(\cdot; \mu, \omega, s, \varepsilon)$, $g_{kij}^{\pm}(\cdot; \mu, \omega, s, \varepsilon)$ are of class C^1 for any $(k, i, j) \in \mathcal{Z}_2$. Together with (2.30), from Lemma A.3 we conclude that the functions $F^{\pm}_{kij}, G^{\pm}_{kij} : H^1_\eta(\mathbb{R})\times(u_L^\infty-\mu_0, u_L^\infty+\mu_0)^2\times\mathbb{R}_\pm\cap(-s_0,s_0)\times(-\varepsilon_0,\varepsilon_0)\to H^1_{-\gamma}(\mathbb{R})$ defined by

$$F^{\pm}_{kij}(\varphi,\mu,\omega,s,\varepsilon)=\alpha^{\pm}_{kij}(f^{\pm}_{kij}(\cdot;\mu,\omega,s,\varepsilon)+\varphi),\ G^{\pm}_{kij}(\varphi,\mu,\omega,s,\varepsilon)=\beta^{\pm}_{kij}(g^{\pm}_{kij}(\cdot;\mu,\omega,s,\varepsilon)+\psi) \tag{2.32}$$

are of class C^1 for any $(k,i,j)\in\mathcal{Z}_2$. The lemma follows shortly from (2.29) and (2.32). $\square$

Next, we study the Fredholm properties of the linear operators $L_\pm(\mu,\omega,s,\varepsilon)$. First, we note that

$$\mathcal{T}:=L_+(u_L^\infty,u_L^\infty,0,0)=L_-(u_L^\infty,u_L^\infty,0,0)=\frac{\mathrm{d}}{\mathrm{d}x}-A(x):H^1_\eta(\mathbb{R},\mathbb{R}^3)\to L^2_\eta(\mathbb{R},\mathbb{R}^3), \tag{2.33}$$

where

$$A(x)=\begin{bmatrix}\frac{b'(u_L^\infty)}{a(u_L^\infty)}(v_L^*)'(x) & 0 & 0\\ 0 & 0 & 1\\ -\delta_0 & -g'(v_L^*(x)) & 0\end{bmatrix}. \tag{2.34}$$

Using that $v_L^*(x)\to v_L^\pm$ and $(v_L^*)'(x)\to 0$ as $x\to\pm\infty$ we obtain that $A(x)\to A_\pm$ as $x\to\pm\infty$, where

$$A_\pm=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ -\delta_0 & -g'(v_L^\pm) & 0\end{bmatrix}. \tag{2.35}$$

In the next lemma we show that the operator $\mathcal{T}$ is Fredholm and we compute its index.

Lemma 2.8. *There exists $\eta^*>0$ such that $\mathcal{T}$ is Fredholm and* $\mathrm{ind}(\mathcal{T})=-1$ *on $L^2_\eta(\mathbb{R},\mathbb{R}^3)$ for all $\eta\in(0,\eta^*)$.*

Proof. First, we introduce $h_\eta\in C^\infty(\mathbb{R})$ a smooth function satisfying the properties: $h_\eta(x)=e^{-\eta|x|}$ for all $x\in\mathbb{R}$ with $|x|\ge 1$ and $\inf_{|x|\le 1}h_\eta(x)>0$. Then, one immediately checks that $L^2_\eta(\mathbb{R},\mathbb{R}^3)=L^2(\mathbb{R},\mathbb{R}^3,h_\eta(x)^{-2}\mathrm{d}x)$ with equivalent norms $\|\cdot\|_{L^2_\eta}$ and $\|\cdot\|_{L^2_{h_\eta^{-2}}}$.

Next, we define $U_\eta : L^2(\mathbb{R},\mathbb{R}^3)\to L^2_\eta(\mathbb{R},\mathbb{R}^3)$ by $U_\eta\underline{w}=h_\eta\underline{w}$. The operator U_η is bounded, invertible with bounded inverse. Thus, $\mathcal{T}$ is Fredholm on $L^2_\eta(\mathbb{R},\mathbb{R}^3)$ if and only if $\mathcal{T}_\eta=U_\eta^{-1}\mathcal{T}U_\eta$ is Fredholm on $L^2(\mathbb{R},\mathbb{R}^3)$ and $\mathrm{ind}(\mathcal{T})=\mathrm{ind}(\mathcal{T}_\eta)$. From (2.33) it follows that

$$\mathcal{T}_\eta=\frac{\mathrm{d}}{\mathrm{d}x}+\frac{h'_\eta}{h_\eta}-A(\cdot)=\frac{\mathrm{d}}{\mathrm{d}x}-A_\eta(\cdot), \tag{2.36}$$

where $A_\eta(x)=A(x)-\frac{h'_\eta(x)}{h_\eta(x)}$. Since $A(x)\to A_\pm$ as $x\to\pm\infty$ we have that

$$A_{\pm,\eta}=\lim_{x\to\pm\infty}A_\eta(x)=A_\pm\pm\eta I_3.$$

The matrix $A_\pm$ has eigenvalues 0, $\sqrt{-g'(v_L^\pm)}$ and $-\sqrt{-g'(v_L^\pm)}$, all with multiplicity 1. Letting $\eta^*=\frac{1}{2}\min\left\{\sqrt{-g'(v_L^+)},\sqrt{-g'(v_L^-)}\right\}$, we infer that for each $\eta\in(0,\eta^*)$ the matrices $A_{\pm,\eta}$ are hyperbolic with Morse indices $i(A_{-,\eta}) = 1$ and $i(A_{+,\eta}) = 2$. From Palmer's classical result, [15, 16] we conclude that $\mathcal{T}_\eta$ is Fredholm on $L^2(\mathbb{R}, \mathbb{R}^3)$ and

$$\operatorname{ind}(\mathcal{T}_\eta) = i(A_{-,\eta}) - i(A_{+,\eta}) = -1,$$

proving the lemma. $\square$

In the next lemma we describe the kernels of $\mathcal{T}$ and of $\mathcal{T}^*$, the L^2-adjoint of $\mathcal{T}$. Here, we consider the operator $\mathcal{T}^*$ as a closed, densely defined linear operator on $L^2_{-\eta}(\mathbb{R}, \mathbb{R}^3)$, with $\eta \in (0, \eta^*)$.

Lemma 2.9. *Let $c_\infty = \frac{b'(u_L^\infty)}{a(u_L^\infty)}$. The following assertions are true:*

(i) The kernel of $\mathcal{T}$ is spanned by $(0, (v_L^*)_x, (v_L^*)_{xx})^{\mathrm{T}}$;

(ii) If $c_\infty \neq 0$ the kernel of $\mathcal{T}^*$ is spanned by $(e^{-c_\infty v_L^*}, 0, 0)^{\mathrm{T}}$ and $(\frac{\delta_0}{c_\infty}, -(v_L^*)_{xx}, (v_L^*)_x)^{\mathrm{T}}$;

(iii) If $c_\infty = 0$ the kernel of $\mathcal{T}^*$ is spanned by $(1, 0, 0)^{\mathrm{T}}$ and $(\delta_0 v_L^*, -(v_L^*)_{xx}, (v_L^*)_x)^{\mathrm{T}}$.

Proof. (i) To find $\ker \mathcal{T}$ we solve the system

$$\begin{cases} \varphi' = c_\infty (v_L^*)_x \varphi, \\ \psi' = \phi, \\ \phi' = -\delta_0 \varphi - g'(v_L^*)\psi. \end{cases} \tag{2.37}$$

Solving the first equation of (2.37) we obtain that $\varphi = ce^{c_\infty v_L^*}$, for some $c \in \mathbb{R}$. Since $v_L^*(x) \to v_L^\pm \in \mathbb{R}$ as $x \to \pm\infty$ we have that $ce^{c_\infty v_L^*} \in H^1_\eta(\mathbb{R})$ if and only if $c = 0$. Thus, if $(\varphi, \psi, \phi) \in \ker \mathcal{T}$ then $\varphi = 0$. From the second and third equations of (2.37) we obtain the equation

$$\psi'' + g'(v_L^*)\psi = 0. \tag{2.38}$$

Since equation (2.38) is the variational equation of

$$v'' + \delta_0 u_L^\infty + g(v) = 0, \tag{2.39}$$

and since $\psi \in H^2_\eta(\mathbb{R})$, it follows that $\psi = d(v_L^*)_x$ for some $d \in \mathbb{R}$. Finally, $\phi = \psi' = \beta(v_L^*)_{xx}$. Similarly, to find $\ker \mathcal{T}^*$ we solve the system

$$\begin{pmatrix} \varphi \\ \psi \\ \phi \end{pmatrix}' = -A(x)^{\mathrm{T}} \begin{pmatrix} \varphi \\ \psi \\ \phi \end{pmatrix}. \tag{2.40}$$

Here A^{T} denotes the transpose of the matrix A. This system is equivalent to

$$\begin{cases} \varphi' = -c_\infty (v_L^*)_x \varphi + \delta_0 \phi, \\ \psi' = g'(v_L^*)\phi, \\ \phi' = -\psi. \end{cases} \tag{2.41}$$

From the second and the third equation we obtain that

$$\phi'' + g'(v_L^*)\phi = 0, \tag{2.42}$$

that is ϕ is an exponentially localized solution of (2.38), the variational equation of (2.39). It follows $\phi = d(v_L^*)_x$ and $\psi = -d(v_L^*)_{xx}$, for some $d \in \mathbb{R}$. We conclude that φ satisfies the first order differential equation

$$\varphi' = -c_\infty (v_L^*)_x \varphi + \delta_0 \beta (v_L^*)_x. \tag{2.43}$$

We infer that there exists $c \in \mathbb{R}$ such that

$$\varphi = \begin{cases} ce^{-c_\infty v_L^*} + d\frac{\delta_0}{c_\infty}, \text{ if } c_\infty \neq 0 \\ c + \delta_0 d v_L^*, \text{ if } c_\infty = 0 \end{cases},$$

proving the lemma. □

Next, we introduce the functions $\mathcal{L}_\pm : (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}(H_\eta^1(\mathbb{R}, \mathbb{R}^3) \times \mathbb{R}^2, L_\eta^2(\mathbb{R}, \mathbb{R}^3))$ defined by

$$\mathcal{L}_\pm(\mu, \omega, s, \varepsilon)\big(\varphi, \psi, \phi, z_1, z_2\big)^{\mathrm{T}} = L_\pm(\mu, \omega, s, \varepsilon)(\varphi, \psi, \phi)^{\mathrm{T}} + Q_\mu^\pm z_1 + Q_\omega^\pm z_2. \tag{2.44}$$

In the next lemma we enumerate the properties of the linear operator $\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0)$.

Lemma 2.10. *We have that $\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0)$ are Fredholm operators with index 1 on $L_\eta^2(\mathbb{R}, \mathbb{R}^3)$ for any $\eta \in (0, \eta^*)$. Moreover, $\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0)$ are onto and their kernel is spanned by $\big(0, (v_L^*)_x, (v_L^*)_{xx}, 0, 0\big)^{\mathrm{T}}$.*

Proof. First, we prove that the linear operators $\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0)$ are onto. From (2.44) one readily checks that

$$\operatorname{Im} \mathcal{T} \subset \operatorname{Im} \mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0) \quad \text{and} \quad Q_\mu^\pm, Q_\omega^\pm \in \operatorname{Im} \mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0). \tag{2.45}$$

Thus, to prove that the linear operators $\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0)$ are onto, it is enough to show that

$$L_\eta^2(\mathbb{R}, \mathbb{R}^3) = \operatorname{Im} \mathcal{T} \oplus \operatorname{Span}\{Q_\mu^\pm, Q_\omega^\pm\}. \tag{2.46}$$

From Lemma 2.9(ii)–(iii) we have that

$$\operatorname{Im} \mathcal{T} = \{\underline{U} \in L_\eta^2(\mathbb{R}, \mathbb{R}^3) : \langle \underline{U}, \underline{U}_1 \rangle_{L^2} = \langle \underline{U}, \underline{U}_2 \rangle_{L^2} = 0\}, \tag{2.47}$$

where

$$\underline{U}_1 = \begin{cases} \Big(e^{-c_\infty v_L^*}, 0, 0\Big)^{\mathrm{T}}, \text{ if } c_\infty \neq 0 \\ \Big(1, 0, 0\Big)^{\mathrm{T}}, \text{ if } c_\infty = 0 \end{cases}, \quad \underline{U}_2 = \begin{cases} \Big(\frac{\delta_0}{c_\infty}, -(v_L^*)_{xx}, (v_L^*)_x\Big)^{\mathrm{T}}, \text{ if } c_\infty \neq 0 \\ \Big(\delta_0 v_L^*, -(v_L^*)_{xx}, (v_L^*)_x\Big)^{\mathrm{T}}, \text{ if } c_\infty = 0 \end{cases}. \tag{2.48}$$

From (2.47) we conclude that (2.46) holds true provided that the matrix

$$\mathcal{Q}_\pm = \begin{pmatrix} \langle Q_\mu^\pm, \underline{U}_1 \rangle_{L^2} & \langle Q_\omega^\pm, \underline{U}_1 \rangle_{L^2} \\ \langle Q_\mu^\pm, \underline{U}_2 \rangle_{L^2} & \langle Q_\omega^\pm, \underline{U}_2 \rangle_{L^2} \end{pmatrix} \quad \text{is invertible.} \tag{2.49}$$

In order to evaluate the scalar products above, we use (2.22) and (2.48). We distinguish between the two cases: $c_\infty = \frac{b'(u_L^\infty)}{a(u_L^\infty)} \neq 0$ and $c_\infty = 0$.

<u>Case 1</u>. $c_\infty \neq 0$.

In this case the two vectors that span $\ker \mathcal{T}^*$ on $L^2_{-\eta}(\mathbb{R}, \mathbb{R}^3)$ are $\underline{U}_1 = (e^{-c_\infty v_L^*}, 0, 0)^{\mathrm{T}}$ and $\underline{U}_2 = (\frac{\delta_0}{c_\infty}, -(v_L^*)_{xx}, (v_L^*)_x)^{\mathrm{T}}$. It follows that

$$
\begin{aligned}
\langle Q_\mu^\pm, \underline{U}_1\rangle_{L^2} &= \int_{\mathbb{R}} e^{-c_\infty v_L^*}\Big(\chi'_\mp - c_\infty (v_L^*)_x \chi_\mp\Big) = \chi_\mp e^{-c_\infty v_L^*}\Big|_{-\infty}^{\infty} = \mp e^{-c_\infty v_L^\mp};\\
\langle Q_\mu^\pm, \underline{U}_2\rangle_{L^2} &= \frac{\delta_0}{c_\infty}\int_{\mathbb{R}} \chi'_\mp - \int_{\mathbb{R}}\Big(c_\infty (v_L^*)_x \chi_\mp \frac{\delta_0}{c_\infty}\Big) + \frac{\delta_0}{g'(v_L^\mp)}\int_{\mathbb{R}} \chi'_\mp (v_L^*)_{xx}\\
&\qquad + \delta_0 \int_{\mathbb{R}} \chi_\mp (v_L^*)_x - \frac{\delta_0}{g'(v_L^\mp)}\int_{\mathbb{R}} g'(v_L^*)\chi_\mp (v_L^*)_x\\
&= \frac{\delta_0}{c_\infty}\chi_\mp\Big|_{-\infty}^{\infty} + \frac{\delta_0}{g'(v_L^\mp)}\int_{\mathbb{R}}\Big(\chi'_\mp (v_L^*)_{xx} + \chi_\mp (v_L^*)_{xxx}\Big)\\
&= \mp\frac{\delta_0}{c_\infty} + \frac{\delta_0}{g'(v_L^\mp)}\chi_\mp (v_L^*)_{xx}\Big|_{-\infty}^{\infty} = \mp\frac{\delta_0}{c_\infty};\\
\langle Q_\omega^\pm, \underline{U}_1\rangle_{L^2} &= \int_{\mathbb{R}} e^{-c_\infty v_L^*}\Big(\chi'_\pm - c_\infty (v_L^*)_x \chi_\pm\Big) = \chi_\pm e^{-c_\infty v_L^*}\Big|_{-\infty}^{\infty} = \pm e^{-c_\infty v_L^\pm};\\
\langle Q_\omega^\pm, \underline{U}_2\rangle_{L^2} &= \frac{\delta_0}{c_\infty}\int_{\mathbb{R}} \chi'_\pm - \int_{\mathbb{R}}\Big(c_\infty (v_L^*)_x \chi_\pm \frac{\delta_0}{c_\infty}\Big) + \frac{\delta_0}{g'(v_L^\pm)}\int_{\mathbb{R}} \chi'_\pm (v_L^*)_{xx}\\
&\qquad + \delta_0 \int_{\mathbb{R}} \chi_\pm (v_L^*)_x - \frac{\delta_0}{g'(v_L^\pm)}\int_{\mathbb{R}} g'(v_L^*)\chi_\pm (v_L^*)_x\\
&= \frac{\delta_0}{c_\infty}\chi_\pm\Big|_{-\infty}^{\infty} + \frac{\delta_0}{g'(v_L^\pm)}\int_{\mathbb{R}}\Big(\chi'_\pm (v_L^*)_{xx} + \chi_\pm (v_L^*)_{xxx}\Big)\\
&= \pm\frac{\delta_0}{c_\infty} + \frac{\delta_0}{g'(v_L^\pm)}\chi_\pm (v_L^*)_{xx}\Big|_{-\infty}^{\infty} = \pm\frac{\delta_0}{c_\infty}.
\end{aligned}
\tag{2.50}
$$

We conclude that

$$
\mathrm{Det}(\mathcal{Q}_\pm) = \begin{vmatrix} \mp e^{-c_\infty v_L^\mp} & \pm e^{-c_\infty v_L^\pm} \\ \mp\frac{\delta_0}{c_\infty} & \pm\frac{\delta_0}{c_\infty} \end{vmatrix} = \frac{\delta_0}{c_\infty}(e^{-c_\infty v_L^\pm} - e^{-c_\infty v_L^\mp}) \neq 0. \tag{2.51}
$$

<u>Case 2</u>. $c_\infty = 0$.

This case is similar to Case 1. From (2.48) we have that $\ker \mathcal{T}^*$ is spanned by $\underline{U}_1 = (1, 0, 0)^{\mathrm{T}}$ and $\underline{U}_2 = (\delta_0, -(v_L^*)_{xx}, (v_L^*)_x)^{\mathrm{T}}$. Next, we compute

$$
\begin{aligned}
\langle Q_\mu^\pm, \underline{U}_1\rangle_{L^2} &= \int_{\mathbb{R}} \chi'_\mp = \chi_\mp\Big|_{-\infty}^{\infty} = \mp 1;\\
\langle Q_\mu^\pm, \underline{U}_2\rangle_{L^2} &= \delta_0\int_{\mathbb{R}} \chi'_\mp v_L^* + \frac{\delta_0}{g'(v_L^\mp)}\int_{\mathbb{R}} \chi'_\mp (v_L^*)_{xx} + \delta_0\int_{\mathbb{R}} \chi_\mp (v_L^*)_x - \frac{\delta_0}{g'(v_L^\mp)}\int_{\mathbb{R}} g'(v_L^*)\chi_\mp (v_L^*)_x\\
&= \delta_0 \chi_\mp v_L^*\Big|_{-\infty}^{\infty} + \frac{\delta_0}{g'(v_L^\mp)}\int_{\mathbb{R}}\Big(\chi'_\mp (v_L^*)_{xx} + \chi_\mp (v_L^*)_{xxx}\Big)\\
&= \mp\delta_0 v_L^\mp + \frac{\delta_0}{g'(v_L^\mp)}\chi_\mp (v_L^*)_{xx}\Big|_{-\infty}^{\infty} = \mp\delta_0 v_L^\mp;
\end{aligned}
$$

$$
\begin{aligned}
\langle Q^{\pm}_{\omega}, \underline{U}_1\rangle_{L^2} &= \int_{\mathbb{R}} \chi'_{\pm} = \chi_{\pm}\Big|_{-\infty}^{\infty} = \pm 1;\\
\langle Q^{\pm}_{\omega}, \underline{U}_2\rangle_{L^2} &= \delta_0 \int_{\mathbb{R}} \chi'_{\pm} v^*_L + \frac{\delta_0}{g'(v^{\pm}_L)} \int_{\mathbb{R}} \chi'_{\pm}(v^*_L)_{xx} + \delta_0 \int_{\mathbb{R}} \chi_{\pm}(v^*_L)_x - \frac{\delta_0}{g'(v^{\pm}_L)} \int_{\mathbb{R}} g'(v^*_L)\chi_{\pm}(v^*_L)_x\\
&= \delta_0 \chi_{\pm} v^*_L\Big|_{-\infty}^{\infty} + \frac{\delta_0}{g'(v^{\pm}_L)} \int_{\mathbb{R}} \Big(\chi'_{\pm}(v^*_L)_{xx} + \chi_{\pm}(v^*_L)_{xxx}\Big)\\
&= \pm\delta_0 v^{\pm}_L + \frac{\delta_0}{g'(v^{\pm}_L)} \chi_{\pm}(v^*_L)_{xx}\Big|_{-\infty}^{\infty} = \pm\delta_0 v^{\pm}_L.
\end{aligned}
\tag{2.52}
$$

Again, we conclude that

$$
\operatorname{Det}(\mathcal{Q}_{\pm}) = \begin{vmatrix} \mp 1 & \pm 1 \\ \mp\delta_0 v^{\mp}_L & \pm\delta_0 v^{\pm}_L \end{vmatrix} = -\delta_0(v^{\pm}_L - v^{\mp}_L) \neq 0. \tag{2.53}
$$

From (2.51) and (2.53) we obtain that (2.49) holds true, which implies that the linear operators $\mathcal{L}_{\pm}(u^{\infty}_L, u^{\infty}_L, 0, 0)$ are onto. To finish the proof of lemma we show that $\ker \mathcal{L}_{\pm}(u^{\infty}_L, u^{\infty}_L, 0, 0)$ is one dimensional. Indeed, from (2.44), (2.46) and Lemma 2.9(i) one readily checks that

$$
\big(\varphi, \psi, \phi, z_1, z_2\big)^{\mathrm{T}} \in \ker \mathcal{L}_{\pm}(u^{\infty}_L, u^{\infty}_L, 0, 0) \quad \text{if and only if} \quad (\varphi, \psi, \phi)^{\mathrm{T}} \in \ker \mathcal{T},\ z_1 = z_2 = 0,
$$

which implies that $\ker \mathcal{L}_{\pm}(u^{\infty}_L, u^{\infty}_L, 0, 0) = \operatorname{Span}\big\{\big(0, (v^*_L)_x, (v^*_L)_{xx}, 0, 0\big)^{\mathrm{T}}\big\}$. □

Next, we are going to analyze the linear operators $\mathcal{L}_{\pm}(\mu, \omega, s, \varepsilon)$ in further detail. First, we note that

$$
\mathcal{L}_{\pm}(\mu, \omega, s, \varepsilon)\big(\varphi, \psi, \phi, z_1, z_2\big)^{\mathrm{T}} = \mathcal{L}_{\pm}(u^{\infty}_L, u^{\infty}_L, 0, 0)\big(\varphi, \psi, \phi, z_1, z_2\big)^{\mathrm{T}} + \mathcal{L}^{\dagger}_{\pm}(\mu, \omega, s, \varepsilon)\big(\varphi, \psi, \phi\big)^{\mathrm{T}} \tag{2.54}
$$

for any $\big(\varphi, \psi, \phi, z_1, z_2\big)^{\mathrm{T}} \in H^1_{\eta}(\mathbb{R}, \mathbb{R}^3) \times \mathbb{R}^2$, $(\mu, \omega, s, \varepsilon) \in (u^{\infty}_L - \mu_0, u^{\infty}_L + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)$, where the functions $\mathcal{L}^{\dagger}_{\pm} : (u^{\infty}_L - \mu_0, u^{\infty}_L + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}(L^2_{\eta}(\mathbb{R}, \mathbb{R}^3), L^2_{\eta}(\mathbb{R}, \mathbb{R}^3))$ are defined by

$$
\mathcal{L}^{\dagger}_{\pm}(\mu, \omega, s, \varepsilon)\big(\varphi, \psi, \phi\big)^{\mathrm{T}} = \big(k^{\pm}_1(\cdot; \mu, \omega, s, \varepsilon)\varphi, 0, k^{\pm}_2(\cdot; \mu, \omega, s, \varepsilon)\psi + s\phi\big)^{\mathrm{T}}; \tag{2.55}
$$

$$
\begin{aligned}
k^{\pm}_1(\cdot; \mu, \omega, s, \varepsilon) &= -\Big(\frac{b + \varepsilon}{a}\Big)'\Big(\chi_{\mp}\mu + \chi_{\pm}u^{\pm}_c\Big)\Big((v^*_L)_x + +\chi_{\pm}w^{\pm}_c\Big) + \frac{b'(u^{\infty}_L)}{a(u^{\infty}_L)}(v^*_L)_x +\\
&\quad + s\frac{a(\chi_{\mp}\mu + \chi_{\pm}u^{\pm}_c) - \chi_{\pm}(u^{\pm}_c + \mu)a'(\chi_{\mp}\mu + \chi_{\pm}u^{\pm}_c)}{a^2(\chi_{\mp}\mu + \chi_{\pm}u^{\pm}_c)}(\chi_{\mp}\mu + \chi_{\pm}u^{\pm}_c);\\
k^{\pm}_2(\cdot; \mu, \omega, s, \varepsilon) &= g'\big(\tilde{v}^*_L + \chi_{\mp}v^{\mp}(\mu) + \chi_{\pm}v^{\pm}_c\big) - g'(v^*_L).
\end{aligned}
\tag{2.56}
$$

In the next lemma we prove the invertibility of the linear operators $\mathcal{L}_{\pm}(\mu, \omega, s, \varepsilon)$ with $\mu, \omega \sim u^{\infty}_L$ and $s, \varepsilon \sim 0$. To formulate this result we introduce the Hilbert spaces

$$
\mathcal{H}_{\eta} := H^1_{\eta}(\mathbb{R}, \mathbb{R}^3) \ominus \operatorname{Span}\big\{\big(0, (v^*_L)_x, (v^*_L)_{xx}\big)^{\mathrm{T}}\big\}, \quad \eta > 0, \tag{2.57}
$$

where the symbol $V \ominus W = Z$ refers to a the (arbitrary) choice of a complement Z of W in V.

Lemma 2.11. *There exist $\mu_0 > 0$, $s_0 > 0$ and $\varepsilon_0 > 0$ small enough such that $\mathcal{L}_\pm(\mu, \omega, s, \varepsilon)$ is invertible from $\mathcal{H}_\eta \times \mathbb{R}^2$ to $L^2_\eta(\mathbb{R}, \mathbb{R}^3)$ and*

$$\sup\Big\{\|(\mathcal{L}_\pm(\mu, \omega, s, \varepsilon))^{-1}\| : (\mu, \omega, s, \varepsilon) \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)\Big\} < \infty.$$

Proof. Since the functions a, b, g are of class C^3 and the functions $\chi_\pm$ and v_L^* are bounded on $\mathbb{R}$, from Remark 2.4(i) and (2.56) we infer that $k_j^\pm(\cdot; \mu, \omega, s, \varepsilon) \in L^\infty(\mathbb{R})$ and

$$\|k_j^\pm(\cdot; \mu, \omega, s, \varepsilon)\|_\infty = \mathcal{O}(\mu - u_L^\infty, \omega - u_L^\infty, s, \varepsilon), \ j = 1, 2 \tag{2.58}$$

for any $(\mu, \omega, s, \varepsilon) \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)$. From Lemma 2.10 we obtain that the operators $\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0)$ are invertible from $\mathcal{H}_\eta \times \mathbb{R}^2$ to $L^2_\eta(\mathbb{R}, \mathbb{R}^3)$ and that their inverse are bounded. From (2.58) we conclude that

$$\left\|\Big(\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0)\Big)^{-1}\mathcal{L}^\dagger_\pm(\mu, \omega, s, \varepsilon)\right\|_{L^2_\eta(\mathbb{R},\mathbb{R}^3)\to\mathcal{H}_\eta\times\mathbb{R}^2} = \mathcal{O}(\mu - u_L^\infty, \omega - u_L^\infty, s, \varepsilon). \tag{2.59}$$

Choosing $\mu_0 > 0$, $s_0 > 0$ and $\varepsilon_0 > 0$ small enough, the lemma follows shortly from (2.54), (2.59) and Lemma 2.10. ☐

From (2.20) and (2.44), we note that the equations $\mathcal{F}_\pm(\varphi, \psi, \phi, \mu, \omega, s, \varepsilon) = 0$ are equivalent to, respectively

$$\mathcal{L}_\pm(\mu, \omega, s, \varepsilon)(\varphi, \psi, \phi, \mu - u_L^\infty, \omega - u_L^\infty)^{\mathrm{T}} + Q_s^\pm s + Q_\varepsilon^\pm \varepsilon + R_\pm(\mu, \omega, s, \varepsilon) + N_\pm(\varphi, \psi, \phi, \mu, \omega, s, \varepsilon) = 0. \tag{2.60}$$

Furthermore, from Lemma 2.11 we infer that by multiplying with $\big(\mathcal{L}_\pm(\mu, \omega, s, \varepsilon)\big)^{-1}$, $(\mu, \omega, s, \varepsilon) \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)$, equation (2.60) is equivalent to

$$\begin{aligned}(\varphi, \psi, \phi, \mu - u_L^\infty, \omega - u_L^\infty)^{\mathrm{T}} + \Big(\mathcal{L}_\pm(\mu, \omega, s, \varepsilon)\Big)^{-1}\Big[Q_s^\pm s + Q_\varepsilon^\pm \varepsilon + R_\pm(\mu, \omega, s, \varepsilon) +& \\ + N_\pm(\varphi, \psi, \phi, \mu, \omega, s, \varepsilon)\Big] = 0.&\end{aligned} \tag{2.61}$$

Next, we fix $\eta \in (0, \eta^*)$ and choose $\gamma > 0$ such that $\gamma < \max\{\eta^* - \eta, \frac{\eta}{4}\}$. We introduce the functions $V_\pm : H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3) \times (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{H}_\eta \times \mathbb{R}^2$ and $\mathcal{N}_\pm : H^1_\eta(\mathbb{R}, \mathbb{R}^3) \times (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3)$ defined by

$$V_\pm(\underline{f}, \mu, \omega, s, \varepsilon) = \Big(\mathcal{L}_\pm(\mu, \omega, s, \varepsilon)\Big)^{-1}\underline{f}, \tag{2.62}$$

$$\mathcal{N}_\pm(\varphi, \psi, \phi, \mu, \omega, s, \varepsilon) = Q_s^\pm s + Q_\varepsilon^\pm \varepsilon + R_\pm(\mu, \omega, s, \varepsilon) + N_\pm(\varphi, \psi, \phi, \mu, \omega, s, \varepsilon). \tag{2.63}$$

In the next lemma we prove the smoothness properties of the functions $V_\pm$ defined in (2.62).

Lemma 2.12. *Let $\eta \in (0, \eta^*)$ and $0 < \gamma < \max\{\eta^* - \eta, \frac{\eta}{4}\}$. Then, the functions $V_\pm : H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3) \times (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{H}_\eta \times \mathbb{R}^2$ are of class C^1.*

Proof. First, we note that from (2.56) we conclude that the functions $k_1^{\pm}, k_2^{\pm} : \mathbb{R} \times (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ can be expressed in terms of the functions a, b, g, the center manifold solutions $(u_c^{\pm}, v_c^{\pm}, w_c^{\pm})$, the cut-off functions $\chi_{\pm}$ and the variables μ, ω, s, ε. Thus, from Remark 2.4 we infer that

$$k_j^{\pm}, \partial_x k_j^{\pm} \in L^{\infty}\Big(\mathbb{R} \times (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)\Big) \tag{2.64}$$

for any $j = 1, 2$. In addition, we have that for any $\theta > 0$ there exists $M_\theta > 0$ such that

$$|\partial_q k_j^{\pm}(x; \mu, \omega, s, \varepsilon)| + |\partial_x \partial_q k_j^{\pm}(x; \mu, \omega, s, \varepsilon)| \leq M_\theta \, e^{\theta |x|}, \tag{2.65}$$

for any $x \in \mathbb{R}$, $\mu, \omega \in (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $0 \leq \pm s \leq s_0$, $q \in \{\mu, \omega, s, \varepsilon\}$, $j = 1, 2$. From (2.55) and Lemma A.4 we conclude that the functions

$$\mathcal{L}_{\pm}^{\dagger} : (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}(H^1_{\nu_1}(\mathbb{R}, \mathbb{R}^3), H^1_{\nu_2}(\mathbb{R}, \mathbb{R}^3)) \tag{2.66}$$

are of class C^1 for any of the following pairs $(\nu_1, \nu_2) = (\eta + \gamma, \eta)$, $(\nu_1, \nu_2) = (\eta + \gamma, \eta + \frac{\gamma}{2})$ and $(\nu_1, \nu_2) = (\eta + \frac{\gamma}{2}, \eta)$. Next, we prove that the function $\mathcal{L}_{\pm}^{-1} : (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3), \mathcal{H}_\eta \times \mathbb{R}^2)$ is of class C^1. We note that

$$\begin{aligned}\big(\mathcal{L}_{\pm}(y_1)\big)^{-1} - \big(\mathcal{L}_{\pm}(y_2)\big)^{-1} &= \big(\mathcal{L}_{\pm}(y_1)\big)^{-1}\Big(\mathcal{L}_{\pm}(y_2) - \mathcal{L}_{\pm}(y_1)\Big)\big(\mathcal{L}_{\pm}(y_2)\big)^{-1} \\ &= \big(\mathcal{L}_{\pm}(y_1)\big)^{-1}\Big(\mathcal{L}^{\dagger}_{\pm}(y_2) - \mathcal{L}^{\dagger}_{\pm}(y_1)\Big)\Pi_\infty\big(\mathcal{L}_{\pm}(y_2)\big)^{-1},\end{aligned} \tag{2.67}$$

where $y_1 = (\mu_1, \omega_1, s_1, \varepsilon_1), y_2 = (\mu_2, \omega_2, s_2, \varepsilon_2) \in (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)$ and $\Pi_\infty : L^2_\eta(\mathbb{R}, \mathbb{R}^3) \times \mathbb{R}^2 \to L^2_\eta(\mathbb{R}, \mathbb{R}^3)$ is defined by $\Pi_\infty\big(\varphi, \psi, \phi, z_1, z_2\big)^{\mathrm{T}} = \big(\varphi, \psi, \phi\big)^{\mathrm{T}}$. Since $\eta, \eta + \gamma \in (0, \eta^*)$, from Lemma 2.11 it follows that

$$\begin{aligned}&\big(\mathcal{L}_{\pm}(\mu, \omega, s, \varepsilon)\big)^{-1} H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3) \subset \big(\mathcal{L}_{\pm}(\mu, \omega, s, \varepsilon)\big)^{-1} L^2_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3) = \mathcal{H}_{\eta+\gamma} \times \mathbb{R}^2, \\ &\big(\mathcal{L}_{\pm}(\mu, \omega, s, \varepsilon)\big)^{-1} H^1_{\eta}(\mathbb{R}, \mathbb{R}^3) \subset \big(\mathcal{L}_{\pm}(\mu, \omega, s, \varepsilon)\big)^{-1} L^2_{\eta}(\mathbb{R}, \mathbb{R}^3) = \mathcal{H}_{\eta} \times \mathbb{R}^2.\end{aligned} \tag{2.68}$$

Moreover, we recall that if V, W and Z are Banach spaces, $T \in \mathcal{B}(V, W)$ and $Z \hookrightarrow V$ then

$$T \in \mathcal{B}(Z, W) \quad \text{and} \quad \Big\|T\Big\|_{Z \to W} \leq \Big\|T\Big\|_{V \to W}. \tag{2.69}$$

Since $\|\Pi_\infty\| = 1$ from (2.67), (2.68), (2.69) and Lemma 2.11 we obtain that

$$\begin{aligned}&\Big\|\big(\mathcal{L}_{\pm}(y_1)\big)^{-1} - \big(\mathcal{L}_{\pm}(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma} \to \mathcal{H}_\eta \times \mathbb{R}^2} \leq \\ &\leq \Big\|\big(\mathcal{L}_{\pm}(y_1)\big)^{-1}\Big\|_{H^1_\eta \to \mathcal{H}_\eta \times \mathbb{R}^2} \Big\|\mathcal{L}^{\dagger}_{\pm}(y_2) - \mathcal{L}^{\dagger}_{\pm}(y_1)\Big\|_{H^1_{\eta+\gamma} \to H^1_\eta} \Big\|\big(\mathcal{L}_{\pm}(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma} \to \mathcal{H}_{\eta+\gamma} \times \mathbb{R}^2} \\ &\leq \Big\|\big(\mathcal{L}_{\pm}(y_1)\big)^{-1}\Big\|_{L^2_\eta \to \mathcal{H}_\eta \times \mathbb{R}^2} \Big\|\mathcal{L}^{\dagger}_{\pm}(y_2) - \mathcal{L}^{\dagger}_{\pm}(y_1)\Big\|_{H^1_{\eta+\gamma} \to H^1_\eta} \Big\|\big(\mathcal{L}_{\pm}(y_2)\big)^{-1}\Big\|_{L^2_{\eta+\gamma} \to \mathcal{H}_{\eta+\gamma} \times \mathbb{R}^2} \\ &\leq K_\eta K_{\eta+\gamma} \Big\|\mathcal{L}^{\dagger}_{\pm}(y_2) - \mathcal{L}^{\dagger}_{\pm}(y_1)\Big\|_{H^1_{\eta+\gamma} \to H^1_\eta},\end{aligned} \tag{2.70}$$

for any $y_1, y_2 \in (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)$, where

$$K_\nu = \sup\Big\{\Big\|\big(\mathcal{L}_{\pm}(y)\big)^{-1}\Big\|_{L^2_\nu \to \mathcal{H}_\nu \times \mathbb{R}^2} : y \in (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)\Big\}. \tag{2.71}$$

From (2.66) and (2.70) we infer that the functions $\mathcal{L}_{\pm}^{-1} : (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3), \mathcal{H}_\eta \times \mathbb{R}^2)$ are continuous. Moreover, one can readily check (using (2.66)) that the same conclusion is true if we change γ to $\frac{\gamma}{2}$ and/or η to $\eta + \frac{\gamma}{2}$. Thus, we conclude that the functions

$$\mathcal{L}_{\pm}^{-1} : (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}(H^1_{\nu_1}(\mathbb{R}, \mathbb{R}^3), \mathcal{H}_{\nu_2} \times \mathbb{R}^2) \tag{2.72}$$

are continuous for any of the following pairs $(\nu_1, \nu_2) = (\eta + \gamma, \eta)$, $(\nu_1, \nu_2) = (\eta + \gamma, \eta + \frac{\gamma}{2})$ and $(\nu_1, \nu_2) = (\eta + \frac{\gamma}{2}, \eta)$. Next, we prove that the functions $\mathcal{L}_{\pm}^{-1}$ are differentiable in any direction $\underline{y} \in \mathbb{R}^4$. Let $y \in (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)$. Denoting by $\partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y)$ the derivative of $\mathcal{L}_{\pm}^{\dagger}$ in the direction $\underline{y}$ at y, from (2.67), (2.68), (2.69), (2.71) and Lemma 2.11 we conclude that

$$\begin{aligned}
&\Big\| \frac{1}{t}\Big[\big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} - \big(\mathcal{L}_{\pm}(y)\big)^{-1}\Big] + \big(\mathcal{L}_{\pm}(y)\big)^{-1} \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Pi_\infty \big(\mathcal{L}_{\pm}(y)\big)^{-1} \Big\|_{H^1_{\eta+\gamma} \to \mathcal{H}_\eta \times \mathbb{R}^2} = \\
&\Big\| \Big[\frac{1}{t} \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big(\mathcal{L}_{\pm}^{\dagger}(y) - \mathcal{L}_{\pm}^{\dagger}(y + t\underline{y}) \Big) + \big(\mathcal{L}_{\pm}(y)\big)^{-1} \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big] \Pi_\infty \big(\mathcal{L}_{\pm}(y)\big)^{-1} \Big\|_{H^1_{\eta+\gamma} \to \mathcal{H}_\eta \times \mathbb{R}^2} \\
&\leq \Bigg[\Big\| \frac{1}{t} \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big(\mathcal{L}_{\pm}^{\dagger}(y) - \mathcal{L}_{\pm}^{\dagger}(y + t\underline{y}) \Big) + \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big\|_{H^1_{\eta+\gamma} \to \mathcal{H}_\eta \times \mathbb{R}^2} \\
&\quad + \Big\| \Big[\big(\mathcal{L}_{\pm}(y)\big)^{-1} - \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big] \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big\|_{H^1_{\eta+\gamma} \to \mathcal{H}_\eta \times \mathbb{R}^2} \Bigg] \times \Big\| \big(\mathcal{L}_{\pm}(y)\big)^{-1} \Big\|_{H^1_{\eta+\gamma} \to \mathcal{H}_{\eta+\gamma} \times \mathbb{R}^2} \\
&\leq \Bigg[\Big\| \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big[\frac{1}{t} \Big(\mathcal{L}_{\pm}^{\dagger}(y) - \mathcal{L}_{\pm}^{\dagger}(y + t\underline{y}) \Big) + \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big] \Big\|_{H^1_{\eta+\gamma} \to \mathcal{H}_\eta \times \mathbb{R}^2} \\
&\quad + \Big\| \Big[\big(\mathcal{L}_{\pm}(y)\big)^{-1} - \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big] \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big\|_{H^1_{\eta+\gamma} \to \mathcal{H}_\eta \times \mathbb{R}^2} \Bigg] \times \Big\| \big(\mathcal{L}_{\pm}(y)\big)^{-1} \Big\|_{L^2_{\eta+\gamma} \to \mathcal{H}_{\eta+\gamma} \times \mathbb{R}^2} \\
&\leq K_{\eta+\gamma} \Bigg[\Big\| \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big\|_{H^1_{\eta+\frac{\gamma}{2}} \to \mathcal{H}_\eta \times \mathbb{R}^2} \Big\| \frac{1}{t} \Big(\mathcal{L}_{\pm}^{\dagger}(y) - \mathcal{L}_{\pm}^{\dagger}(y + t\underline{y}) \Big) + \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big\|_{H^1_{\eta+\gamma} \to H^1_{\eta+\frac{\gamma}{2}}} \\
&\quad + \Big\| \big(\mathcal{L}_{\pm}(y)\big)^{-1} - \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big\|_{H^1_{\eta+\frac{\gamma}{2}} \to \mathcal{H}_\eta \times \mathbb{R}^2} \Big\| \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big\|_{H^1_{\eta+\gamma} \to H^1_{\eta+\frac{\gamma}{2}}} \Bigg] \\
&\leq K_{\eta+\gamma} \Bigg[\Big\| \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big\|_{L^2_{\eta} \to \mathcal{H}_\eta \times \mathbb{R}^2} \Big\| \frac{1}{t} \Big(\mathcal{L}_{\pm}^{\dagger}(y) - \mathcal{L}_{\pm}^{\dagger}(y + t\underline{y}) \Big) + \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big\|_{H^1_{\eta+\gamma} \to H^1_{\eta+\frac{\gamma}{2}}} \\
&\quad + \Big\| \big(\mathcal{L}_{\pm}(y)\big)^{-1} - \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big\|_{H^1_{\eta+\frac{\gamma}{2}} \to \mathcal{H}_\eta \times \mathbb{R}^2} \Big\| \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big\|_{H^1_{\eta+\gamma} \to H^1_{\eta+\frac{\gamma}{2}}} \Bigg] \\
&\leq K_{\eta+\gamma} K_\eta \Big\| \frac{1}{t} \Big(\mathcal{L}_{\pm}^{\dagger}(y) - \mathcal{L}_{\pm}^{\dagger}(y + t\underline{y}) \Big) + \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big\|_{H^1_{\eta+\gamma} \to H^1_{\eta+\frac{\gamma}{2}}} \\
&\quad + K_{\eta+\gamma} \Big\| \big(\mathcal{L}_{\pm}(y)\big)^{-1} - \big(\mathcal{L}_{\pm}(y + t\underline{y})\big)^{-1} \Big\|_{H^1_{\eta+\frac{\gamma}{2}} \to \mathcal{H}_\eta \times \mathbb{R}^2} \Big\| \partial_{\underline{y}} \mathcal{L}_{\pm}^{\dagger}(y) \Big\|_{H^1_{\eta+\gamma} \to H^1_{\eta+\frac{\gamma}{2}}}.
\end{aligned} \tag{2.73}$$

From (2.66) and (2.72) it follows that the functions

$$\mathcal{L}_{\pm}^{-1} : (u_L^{\infty} - \mu_0, u_L^{\infty} + \mu_0)^2 \times \mathbb{R}_{\pm} \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3), \mathcal{H}_\eta \times \mathbb{R}^2) \tag{2.74}$$

are differentiable and

$$\partial_{\underline{y}}(\mathcal{L}_\pm)^{-1}(y) = -\big(\mathcal{L}_\pm(y)\big)^{-1}\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y)\Pi_\infty\big(\mathcal{L}_\pm(y)\big)^{-1} \quad \text{in} \quad \mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R},\mathbb{R}^3), \mathcal{H}_\eta \times \mathbb{R}^2) \tag{2.75}$$

for all $y \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)$ and $\underline{y} \in \mathbb{R}^4$. Analyzing the argument from (2.73) in detail, we infer that the operator $\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y)$ in (2.75) is understood as a derivative (limit) in the $\mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R},\mathbb{R}^3), H^1_{\eta+\frac{\gamma}{2}}(\mathbb{R},\mathbb{R}^3))$ topology. Since $\mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R},\mathbb{R}^3), H^1_{\eta+\frac{\gamma}{2}}(\mathbb{R},\mathbb{R}^3)) \hookrightarrow \mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R},\mathbb{R}^3), H^1_{\eta}(\mathbb{R},\mathbb{R}^3))$ we have that this operator can be understood as the derivative (limit) in the $\mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R},\mathbb{R}^3), H^1_{\eta}(\mathbb{R},\mathbb{R}^3))$ topology. However, from (2.66) we know that the derivative exists also in the $\mathcal{B}(H^1_{\eta+\frac{\gamma}{2}}(\mathbb{R},\mathbb{R}^3), H^1_{\eta}(\mathbb{R},\mathbb{R}^3))$ topology. Therefore we have that

$$\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y)_{|H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta} \quad \text{is an extension of the operator} \quad \partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y)_{|H^1_{\eta+\gamma}\to H^1_{\eta+\frac{\gamma}{2}}}.$$

From (2.68) we conclude that

$$\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y)_{|H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}\Pi_\infty\big(\mathcal{L}_\pm(y)\big)^{-1} = \partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y)_{|H^1_{\eta+\gamma}\to H^1_{\eta+\frac{\gamma}{2}}}\Pi_\infty\big(\mathcal{L}_\pm(y)\big)^{-1}. \tag{2.76}$$

for all $y \in (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0)$ and $\underline{y} \in \mathbb{R}^4$. Next, we prove that $\partial_{\underline{y}}(\mathcal{L}_\pm)^{-1}$ is continuous for any $\underline{y} \in \mathbb{R}^4$. From (2.67), (2.68), (2.69), (2.71), and Lemma 2.11 it follows that

$$\begin{aligned}
&\Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1}\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_1)\Pi_\infty\big(\mathcal{L}_\pm(y_1)\big)^{-1} - \big(\mathcal{L}_\pm(y_2)\big)^{-1}\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_2)\Pi_\infty\big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_\eta\times\mathbb{R}^2} \\
&\le \Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1}\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_1)\Pi_\infty\Big[\big(\mathcal{L}_\pm(y_1)\big)^{-1} - \big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big]\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_\eta\times\mathbb{R}^2} \\
&\quad + \Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1}\Big[\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_1) - \partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_2)\Big]\Pi_\infty\big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_\eta\times\mathbb{R}^2} \\
&\quad + \Big\|\Big[\big(\mathcal{L}_\pm(y_1)\big)^{-1} - \big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big]\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_2)\Pi_\infty\big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_\eta\times\mathbb{R}^2} \\
&\le \Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1}\Big\|_{H^1_\eta\to\mathcal{H}_\eta\times\mathbb{R}^2}\Big\|\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_1)\Big\|_{H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}\Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1} - \big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_{\eta+\frac{\gamma}{2}}\times\mathbb{R}^2} \\
&\quad + \Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1}\Big\|_{H^1_\eta\to\mathcal{H}_\eta\times\mathbb{R}^2}\Big\|\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_1) - \partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_2)\Big\|_{H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}\|\big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_{\eta+\frac{\gamma}{2}}\times\mathbb{R}^2} \\
&\quad + \Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1} - \big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}\Big\|\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_2)\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_{\eta+\frac{\gamma}{2}}}\Big\|\big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_{\eta+\gamma}\times\mathbb{R}^2} \\
&\le \Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1}\Big\|_{L^2_\eta\to\mathcal{H}_\eta\times\mathbb{R}^2}\Big\|\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_1)\Big\|_{H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}\Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1} - \big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_{\eta+\frac{\gamma}{2}}\times\mathbb{R}^2} \\
&\quad + \Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1}\Big\|_{L^2_\eta\to\mathcal{H}_\eta\times\mathbb{R}^2}\Big\|\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_1) - \partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_2)\Big\|_{H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}\|\big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{L^2_{\eta+\frac{\gamma}{2}}\to\mathcal{H}_{\eta+\frac{\gamma}{2}}\times\mathbb{R}^2} \\
&\quad + \Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1} - \big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}\Big\|\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_2)\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_{\eta+\frac{\gamma}{2}}}\Big\|\big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{L^2_{\eta+\gamma}\to\mathcal{H}_{\eta+\gamma}\times\mathbb{R}^2} \\
&\le K_\eta\Big\|\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_1)\Big\|_{H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}\Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1} - \big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_{\eta+\frac{\gamma}{2}}\times\mathbb{R}^2} \\
&\quad + K_\eta K_{\eta+\frac{\gamma}{2}}\Big\|\partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_1) - \partial_{\underline{y}}\mathcal{L}_\pm^\dagger(y_2)\Big\|_{H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}
\end{aligned}$$

$$+ K_{\eta+\gamma}\Big\|\big(\mathcal{L}_\pm(y_1)\big)^{-1} - \big(\mathcal{L}_\pm(y_2)\big)^{-1}\Big\|_{H^1_{\eta+\frac{\gamma}{2}}\to H^1_\eta}\Big\|\partial_{\underline{y}}\mathcal{L}^\dagger_\pm(y_2)\Big\|_{H^1_{\eta+\gamma}\to\mathcal{H}_{\eta+\frac{\gamma}{2}}}. \tag{2.77}$$

From (2.66) and (2.72) it follows that the functions

$$\partial_{\underline{y}}\mathcal{L}^{-1}_\pm : (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3), \mathcal{H}_\eta \times \mathbb{R}^2) \tag{2.78}$$

are continuous for any $\underline{y} \in \mathbb{R}^4$. From (2.72), (2.74) and (2.78) we conclude that

$$\mathcal{L}^{-1}_\pm : (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3), \mathcal{H}_\eta \times \mathbb{R}^2) \tag{2.79}$$

are of class C^1. From (2.62) we obtain that the functions $V_\pm : H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3) \times (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{H}_\eta \times \mathbb{R}^2$ are of class C^1, proving the lemma. □

3 Existence of weakly decaying fronts — proof of Theorem 1.1 and Corollary 1.2

We prove the main result of this paper on bifurcation of traveling waves from standing layers. To prove Theorem 1.1, it is enough to show that the equations $\mathcal{F}_\pm(\varphi, \psi, \phi, \mu, \omega, s, \varepsilon) = 0$ can be solved for $\varphi, \psi, \phi \in H^1_\eta(\mathbb{R})$ and $\mu, \omega \sim u_L^\infty$, in terms of $s, \varepsilon \sim 0$. In the previous section, we showed that these equations are equivalent to equations (2.61). From (2.62) and (2.63) we have that equations (2.61) are equivalent to

$$(\varphi, \psi, \phi, \mu - u_L^\infty, \omega - u_L^\infty)^{\mathrm{T}} + V_\pm\Big(\mathcal{N}_\pm(\varphi, \psi, \phi, \mu, \omega, s, \varepsilon), \mu, \omega, s, \varepsilon\Big) = 0. \tag{3.1}$$

Next, we relabel the variables as follows:

$$\mathbf{u} = (\varphi, \psi, \phi)^{\mathrm{T}}, \mathbf{p} = (\mu - u_L^\infty, \omega - u_L^\infty)^{\mathrm{T}}, \mathbf{q} = (s, \varepsilon)^{\mathrm{T}}. \tag{3.2}$$

Furthermore, we introduce the functions $\Gamma_\pm : H^1_\eta(\mathbb{R}, \mathbb{R}^3) \times (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to H^1_\eta(\mathbb{R}, \mathbb{R}^3) \times \mathbb{R}^2$ defined by

$$\Gamma_\pm(\mathbf{u}, \mathbf{p}, \mathbf{q}) = (\mathbf{u}, \mathbf{p})^{\mathrm{T}} + V_\pm(\mathcal{N}_\pm(\mathbf{u}, \mathbf{p}, \mathbf{q}), \mathbf{p}, \mathbf{q}). \tag{3.3}$$

From Remark 2.5, Lemma 2.6 and Lemma 2.7 we conclude that the functions

$$\mathcal{N}_\pm : H^1_\eta(\mathbb{R}, \mathbb{R}^3) \times (u_L^\infty - \mu_0, u_L^\infty + \mu_0)^2 \times \mathbb{R}_\pm \cap (-s_0, s_0) \times (-\varepsilon_0, \varepsilon_0) \to H^1_{\eta+\gamma}(\mathbb{R}, \mathbb{R}^3) \tag{3.4}$$

are well-defined and of class C^1. From (3.4) and Lemma 2.12 we conclude that the functions $\Gamma_\pm$ are of class C^1. Moreover, if we denote by Id_η the identity operator on $H^1_\eta(\mathbb{R}, \mathbb{R}^3)$, from (2.24), (2.25), (2.62) and (2.63), we obtain that

$$\begin{aligned}
\partial_{\mathbf{u}}\Gamma_\pm(0,0,0) &= \mathrm{diag}(\mathrm{Id}_\eta, 0) + \partial_{\underline{f}}V_\pm(\mathcal{N}_\pm(0,0,0),0,0)\partial_{\mathbf{u}}\mathcal{N}_\pm(0,0,0) \\
&= \mathrm{diag}(\mathrm{Id}_\eta, 0) + \big(\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0)\big)^{-1}\partial_{\mathbf{u}}N_\pm(0,0,0) = \mathrm{diag}(\mathrm{Id}_\eta, 0) \\
\partial_{\mathbf{p}}\Gamma_\pm(0,0,0) &= \mathrm{diag}(0, I_2) + \partial_{\underline{f}}V_\pm(\mathcal{N}_\pm(0,0,0),0,0)\partial_{\mathbf{p}}\mathcal{N}_\pm(0,0,0) + \partial_{\mathbf{p}}V_\pm(\mathcal{N}_\pm(0,0,0),0,0) \\
&= \mathrm{diag}(0, I_2) + \big(\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0, 0)\big)^{-1}\partial_{\mathbf{p}}\mathcal{N}_\pm(0,0,0) + \big(\partial_{\mathbf{p}}\mathcal{L}^{-1}_\pm\big)(0,0)\mathcal{N}_\pm(0,0,0)
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{diag}(0, I_2) + \big(\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0,0)\big)^{-1}\big(\partial_{\mathbf{p}} R_\pm(0,0) + \partial_{\mathbf{p}}\mathcal{N}_\pm(0,0,0)\big)\\
&= \operatorname{diag}(0, I_2).
\end{aligned}\tag{3.5}
$$

It follows that $\partial_{(\mathbf{u},\mathbf{p})}\Gamma_\pm(0,0,0) = \operatorname{diag}(\mathrm{Id}_\eta, I_2)$ is the identity operator on $H^1_\eta(\mathbb{R},\mathbb{R}^3)\times\mathbb{R}^2$. Since, one can readily check that $\Gamma_\pm(0,0,0) = 0$, we infer from the Implicit Function Theorem that there exist $\mu_0 > 0$, $s_0 > 0$ and $\varepsilon_0 > 0$ small enough and C^1 functions $(\mathbf{u}^*_\pm, \mathbf{p}^*_\pm) : \mathbb{R}_\pm \cap (-s_0, s_0)\times(-\varepsilon_0,\varepsilon_0) \to H^1_\eta(\mathbb{R},\mathbb{R}^3)\times\mathbb{R}^2$ such that,

(i) $(\mathbf{u}^*_\pm, \mathbf{p}^*_\pm)(0,0) = (0,0,0)^{\mathrm{T}}$;

(ii) locally, the equation $\Gamma_\pm(\mathbf{u},\mathbf{p},\mathbf{q}) = 0$ has the unique solution $(\mathbf{u},\mathbf{p})^{\mathrm{T}} = (\mathbf{u}^*_\pm(\mathbf{q}), \mathbf{p}^*_\pm(\mathbf{q}))$.

Let $\mathbf{u}^*_\pm(\cdot; s,\varepsilon) = (\varphi^*_\pm(\cdot, s,\varepsilon), \psi^*_\pm(\cdot, s,\varepsilon), \phi^*_\pm(\cdot, s,\varepsilon))^{\mathrm{T}} \in H^1_\eta(\mathbb{R},\mathbb{R}^3)$ and $\mathbf{p}^*_\pm(s,\varepsilon) = (\mu^*_\pm(s,\varepsilon) - u_L^\infty, \omega^*_\pm(s,\varepsilon) - u_L^\infty)^{\mathrm{T}}$ be the local solution of bifurcation equations (3.1) and

$$
\begin{aligned}
u^*_\pm(\cdot; s,\varepsilon) &= \mu^*_\pm(s,\varepsilon)\chi_\mp + \chi_\pm u_c^\pm(\cdot; \mu^*_\pm(s,\varepsilon), \omega^*_\pm(s,\varepsilon), s,\varepsilon) + \varphi^*_\pm(\cdot; s,\varepsilon), \qquad (3.6)\\
v^*_\pm(\cdot; s,\varepsilon) &= v_L^* + \chi_\mp\big(v^\mp(\mu^*_\pm(s,\varepsilon)) - v_L^\mp\big) + \chi_\pm\big(v_c^\pm(\cdot; \mu^*_\pm(s,\varepsilon), \omega^*_\pm(s,\varepsilon), s,\varepsilon) - v_L^\pm\big) + \psi^*_\pm(\cdot; s,\varepsilon).
\end{aligned}
$$

Since equations (3.1) are equivalent to equations $\Gamma_\pm(\mathbf{u},\mathbf{p},\mathbf{q}) = 0$ via substitutions (3.2), Theorem 1.1 is proved.

To start the proof of Corollary 1.2, we differentiate $\Gamma_\pm$ with respect to $\mathbf{q}$. From (2.24), (2.25), (2.62) and (2.63) we have that

$$
\begin{aligned}
\partial_{\mathbf{q}}\Gamma_\pm(0,0,0) &= \partial_{\underline{f}} V_\pm(\mathcal{N}_\pm(0,0,0),0,0)\partial_{\mathbf{q}}\mathcal{N}_\pm(0,0,0) + \partial_{\mathbf{q}} V_\pm(\mathcal{N}_\pm(0,0,0),0,0)\\
&= \big(\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0,0)\big)^{-1}\partial_{\mathbf{q}}\mathcal{N}_\pm(0,0,0) + \big(\partial_{\mathbf{q}}\mathcal{L}_\pm^{-1}\big)(0,0)\mathcal{N}_\pm(0,0,0)\\
&= \big(\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0,0)\big)^{-1}\big(Q_s^\pm \otimes Q_\varepsilon^\pm + \partial_{\mathbf{q}} R_\pm(0,0) + \partial_{\mathbf{q}}\mathcal{N}_\pm(0,0,0)\big)\\
&= \big(\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0,0)\big)^{-1}\big(Q_s^\pm \otimes Q_\varepsilon^\pm\big).
\end{aligned}\tag{3.7}
$$

Since $\partial_{(\mathbf{u},\mathbf{p})}\Gamma_\pm(0,0,0) = \operatorname{diag}(\mathrm{Id}_\eta, I_2)$ we conclude that

$$
\big(\partial_{\mathbf{q}}\mathbf{u}^*_\pm(0,0), \partial_{\mathbf{q}}\mathbf{p}^*_\pm(0,0)\big)^{\mathrm{T}} = -\big(\mathcal{L}_\pm(u_L^\infty, u_L^\infty, 0,0)\big)^{-1}\big(Q_s^\pm \otimes Q_\varepsilon^\pm\big).\tag{3.8}
$$

Introducing the notation

$$
\begin{aligned}
(\Phi_s^\pm, \Psi_s^\pm, \Upsilon_s^\pm)^{\mathrm{T}} &= \big(\partial_s\varphi^*_\pm(\cdot; 0,0), \partial_s\psi^*_\pm(\cdot; 0,0), \partial_s\phi^*_\pm(\cdot; 0,0)\big)^{\mathrm{T}} \in H^1_\eta(\mathbb{R},\mathbb{R}^3),\\
(\Phi_\varepsilon^\pm, \Psi_\varepsilon^\pm, \Upsilon_\varepsilon^\pm)^{\mathrm{T}} &= \big(\partial_\varepsilon\varphi^*_\pm(\cdot; 0,0), \partial_\varepsilon\psi^*_\pm(\cdot; 0,0), \partial_\varepsilon\phi^*_\pm(\cdot; 0,0)\big)^{\mathrm{T}} \in H^1_\eta(\mathbb{R},\mathbb{R}^3),
\end{aligned}
$$

we obtain from (3.8) that

$$
\begin{aligned}
\mathcal{T}(\Phi_s^\pm, \Psi_s^\pm, \Upsilon_s^\pm)^{\mathrm{T}} + \partial_s\mu^*_\pm(0,0)Q_\mu^\pm + \partial_s\omega^*_\pm(0,0)Q_\omega^\pm &= -Q_s^\pm\\
\mathcal{T}(\Phi_\varepsilon^\pm, \Psi_\varepsilon^\pm, \Upsilon_\varepsilon^\pm)^{\mathrm{T}} + \partial_\varepsilon\mu^*_\pm(0,0)Q_\mu^\pm + \partial_\varepsilon\omega^*_\pm(0,0)Q_\omega^\pm &= -Q_\varepsilon^\pm.
\end{aligned}\tag{3.9}
$$

Taking the L^2-scalar product with $\underline{U}_j$, $j = 1,2$, defined in (2.48), we conclude from (2.47) that

$$
\langle Q_\mu^\pm, \underline{U}_j\rangle_{L^2}\partial_s\mu^*_\pm(0,0) + \langle Q_\omega^\pm, \underline{U}_j\rangle_{L^2}\partial_s\omega^*_\pm(0,0) = -\langle Q_s^\pm, \underline{U}_j\rangle_{L^2}, \quad j = 1,2,
$$

$$\langle Q_\mu^\pm, \underline{U}_j\rangle_{L^2}\partial_\varepsilon\mu_\pm^*(0,0) + \langle Q_\omega^\pm, \underline{U}_j\rangle_{L^2}\partial_\varepsilon\omega_\pm^*(0,0) = -\langle Q_\varepsilon^\pm, \underline{U}_j\rangle_{L^2}, \quad j = 1,2. \tag{3.10}$$

Using the definition of the invertible matrices $\mathcal{Q}_\pm$ given in (2.49), it follows that

$$\begin{aligned}
\mathcal{Q}_\pm\big(\partial_s\mu_\pm^*(0,0), \partial_s\omega_\pm^*(0,0)\big)^{\mathrm{T}} &= -\big(\langle Q_s^\pm, \underline{U}_1\rangle_{L^2}, \langle Q_s^\pm, \underline{U}_2\rangle_{L^2}\big)^{\mathrm{T}},\\
\mathcal{Q}_\pm\big(\partial_\varepsilon\mu_\pm^*(0,0), \partial_\varepsilon\omega_\pm^*(0,0)\big)^{\mathrm{T}} &= -\big(\langle Q_\varepsilon^\pm, \underline{U}_1\rangle_{L^2}, \langle Q_\varepsilon^\pm, \underline{U}_2\rangle_{L^2}\big)^{\mathrm{T}}.
\end{aligned} \tag{3.11}$$

Next, we evaluate the L^2 scalar products from the right hand side of (3.11). From (2.22) and (2.48) we obtain that

$$\langle Q_s^\pm, \underline{U}_1\rangle_{L^2} = 0, \quad \langle Q_s^\pm, \underline{U}_2\rangle_{L^2} = \|(v_L^*)_x\|_2^2. \tag{3.12}$$

$$\begin{aligned}
\langle Q_\varepsilon^\pm, \underline{U}_1\rangle_{L^2} &= -\frac{1}{a(u_L^\infty)}\int_{\mathbb{R}} e^{-c_\infty v_L^*}(v_L^*)_x = \begin{cases} \frac{1}{a(u_L^\infty)c_\infty}(e^{-c_\infty v_L^+} - e^{-c_\infty v_L^-}), \text{ if } c_\infty \neq 0\\ -\frac{1}{a(u_L^\infty)}(v_L^+ - v_L^-), \text{ if } c_\infty = 0\end{cases},\\
\langle Q_\varepsilon^\pm, \underline{U}_2\rangle_{L^2} &= \begin{cases} -\frac{\delta_0}{a(u_L^\infty)c_\infty}\int_{\mathbb{R}}(v_L^*)_x, \text{ if } c_\infty \neq 0\\ -\frac{\delta_0}{a(u_L^\infty)}\int_{\mathbb{R}} v_L^*(v_L^*)_x, \text{ if } c_\infty = 0\end{cases} = \begin{cases} -\frac{\delta_0}{a(u_L^\infty)c_\infty}(v_L^+ - v_L^-), \text{ if } c_\infty \neq 0\\ -\frac{\delta_0}{2a(u_L^\infty)}\Big((v_L^+)^2 - (v_L^-)^2\Big), \text{ if } c_\infty = 0\end{cases}.
\end{aligned} \tag{3.13}$$

Since the entries of the matrices $\mathcal{Q}_\pm$ were evaluated in (2.50) and (2.52), we infer that expansions (1.2) follow immediately from (3.11), (3.12) and (3.13). Next, we prove the conditions satisfied by the partial derivatives $\partial_{s,\varepsilon}u_\pm^*(\cdot;0,0)$ and $\partial_{s,\varepsilon}v_\pm^*(\cdot;0,0)$. Using the definition of $\mathcal{T}$ from (2.33)–(2.34), we obtain from (2.22) and (3.9) that

$$\begin{aligned}
(\Phi_s^\pm)' &= c_\infty(v_L^*)_x\Phi_s^\pm - \partial_s\mu_\pm^*(0,0)\big(\chi_\mp' - c_\infty(v_L^*)_x\chi_\mp\big) - \partial_s\omega_\pm^*(0,0)\big(\chi_\pm' - c_\infty(v_L^*)_x\chi_\pm\big),\\
(\Psi_s^\pm)' &= \Upsilon_s^\pm + \frac{\delta_0}{g'(v_L^\mp)}\partial_s\mu_\pm^*(0,0)\chi_\mp' + \frac{\delta_0}{g'(v_L^\pm)}\partial_s\omega_\pm^*(0,0)\chi_\pm',\\
(\Upsilon_s^\pm)' + \delta_0\Phi_s^\pm + g'(v_L^*)\Psi_s^\pm &= -(v_L^*)_x + \delta_0\partial_s\mu_\pm^*(0,0)\chi_\mp\big(\frac{g'(v_L^*)}{g'(v_L^\mp)} - 1\big) +\\
&\quad + \delta_0\partial_s\omega_\pm^*(0,0)\chi_\pm\big(\frac{g'(v_L^*)}{g'(v_L^\pm)} - 1\big).
\end{aligned} \tag{3.14}$$

By multiplying the first equation by $e^{-c_\infty v_L^*}$ and integrating it, we obtain that

$$\Phi_s^\pm = -\partial_s\mu_\pm^*(0,0)\chi_\mp - \partial_s\omega_\pm^*(0,0)\chi_\pm + ce^{c_\infty v_L^*}, \tag{3.15}$$

for some constant $c \in \mathbb{R}$. Since $\Phi_s^\pm \in H_\eta^1(\mathbb{R})$, it follows that $\lim_{x\to\pm\infty}\Phi_s^\pm(x) = 0$, which implies that $c = e^{-c_\infty v_L^\mp}\partial_s\mu_\pm^*(0,0) = e^{-c_\infty v_L^\pm}\partial_s\omega_\pm^*(0,0) = -\kappa_1(c_\infty)$. From (3.15) we conclude that

$$\Phi_s^\pm = -\partial_s\mu_\pm^*(0,0)\chi_\mp - \partial_s\omega_\pm^*(0,0)\chi_\pm - \kappa_1(c_\infty)e^{c_\infty v_L^*}. \tag{3.16}$$

Solving for $\Upsilon_s^\pm$ in the second equation of (3.14) and substituting the result into the third equation of (3.14), we obtain from (3.16) that

$$(\Psi_s^\pm)'' + g'(v_L^*)\Psi_s^\pm - \partial_s\mu_\pm^*(0,0)\chi_\mp''\frac{\delta_0}{g'(v_L^\mp)} - \partial_s\omega_\pm^*(0,0)\chi_\pm''\frac{\delta_0}{g'(v_L^\pm)} = -(v_L^*)_x + \delta_0\kappa_1(c_\infty)e^{c_\infty v_L^*}$$

$$+ g'(v_L^*)\Big(\partial_s\mu_\pm^*(0,0)\chi_\mp\frac{\delta_0}{g'(v_L^\mp)} + \partial_s\omega_\pm^*(0,0)\chi_\pm\frac{\delta_0}{g'(v_L^\pm)}\Big). \tag{3.17}$$

Next, we note that $\Psi_s^\pm - \partial_s\mu_\pm^*(0,0)\chi_\mp\frac{\delta_0}{g'(v_L^\mp)} - \partial_s\omega_\pm^*(0,0)\chi_\pm\frac{\delta_0}{g'(v_L^\pm)}$ satisfies the equation

$$W_{xx} + g'(v_L^*)W = \delta_0\kappa_1(c_\infty)e^{c_\infty v_L^*} - (v_L^*)_x. \tag{3.18}$$

From the definition of the function κ_1 in (1.5), one readily checks that

$$\langle \delta_0\kappa_1(c_\infty)e^{c_\infty v_L^*} - (v_L^*)_x, (v_L^*)_x\rangle_{L^2} = 0,$$

which proves that equation (3.18) has a unique solution denoted W^*. It follows that

$$\Psi_s^\pm = \partial_s\mu_\pm^*(0,0)\chi_\mp\frac{\delta_0}{g'(v_L^\mp)} + \partial_s\omega_\pm^*(0,0)\chi_\pm\frac{\delta_0}{g'(v_L^\pm)} + W^*. \tag{3.19}$$

Finally, from the second equation of (3.14) we conclude that $\Upsilon_s^\pm = W_x^*$. Differentiating with respect to s in (3.6), we conclude from (2.14), (2.15), (3.16), and(3.19) that

$$\begin{aligned}
\partial_s u_\pm^*(\cdot;0,0) &= \chi_\pm\Big[\partial_\mu u_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\partial_s\mu_\pm^*(0,0) + \partial_\omega u_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\partial_s\omega_\pm^*(0,0)\\
&\qquad + \partial_s u_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\Big] + \partial_s\mu_\pm^*(0,0)\chi_\mp + \partial_s\varphi_\pm^*(\cdot;0,0)\\
&= \partial_s\mu_\pm^*(0,0)\chi_\mp + \partial_s\omega_\pm^*(0,0)\chi_\pm + \Phi_s^\pm = -\kappa_1(c_\infty)e^{c_\infty v_L^*},\\
\partial_s v_\pm^*(\cdot;0,0) &= \chi_\pm\Big[\partial_\mu v_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\partial_s\mu_\pm^*(0,0) + \partial_\omega v_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\partial_s\omega_\pm^*(0,0)\\
&\qquad + \partial_s v_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\Big] + \partial_s\mu_\pm^*(0,0)(v^\pm)'(u_L^\infty)\chi_\mp + \partial_s\psi_\pm^*(\cdot;0,0)\\
&= -\frac{\delta_0}{g'(v_L^\mp)}\partial_s\mu_\pm^*(0,0)\chi_\mp - \frac{\delta_0}{g'(v_L^\pm)}\partial_s\omega_\pm^*(0,0)\chi_\pm + \Psi_s^\pm = W^*.
\end{aligned} \tag{3.20}$$

To finish the proof of the corollary we compute $(\Phi_\varepsilon^\pm, \Psi_\varepsilon^\pm, \Upsilon_\varepsilon^\pm)^{\mathrm{T}}$. Using again the definition of $\mathcal{T}$ from (2.33)–(2.34) from (2.22) and (3.9) we obtain that

$$\begin{aligned}
&(\Phi_\varepsilon^\pm)' = c_\infty(v_L^*)_x\Phi_\varepsilon^\pm - \partial_\varepsilon\mu_\pm^*(0,0)\big(\chi_\mp' - c_\infty(v_L^*)_x\chi_\mp\big) - \partial_\varepsilon\omega_\pm^*(0,0)\big(\chi_\pm' - c_\infty(v_L^*)_x\chi_\pm\big) + \frac{(v_L^*)_x}{a(u_L^\infty)},\\
&(\Psi_\varepsilon^\pm)' = \Upsilon_\varepsilon^\pm + \frac{\delta_0}{g'(v_L^\mp)}\partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp' + \frac{\delta_0}{g'(v_L^\pm)}\partial_\varepsilon\omega_\pm^*(0,0)\chi_\pm',\\
&(\Upsilon_\varepsilon^\pm)' + \delta_0\Phi_\varepsilon^\pm + g'(v_L^*)\Psi_\varepsilon^\pm = \delta_0\partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp\big(\frac{g'(v_L^*)}{g'(v_L^\mp)} - 1\big) + \delta_0\partial_\varepsilon\omega_\pm^*(0,0)\chi_\pm\big(\frac{g'(v_L^*)}{g'(v_L^\pm)} - 1\big).
\end{aligned} \tag{3.21}$$

We note that the system (3.21) is almost identical to (3.14), the only difference being the term $\frac{(v_L^*)_x}{a(u_L^\infty)}$ from the first equation. Multiplying again the first equation by $e^{-c_\infty v_L^*}$, integrating it and arguing as in (3.15)–(3.16), we conclude that

$$\Phi_\varepsilon^\pm = -\partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp - \partial_\varepsilon\omega_\pm^*(0,0)\chi_\pm + \kappa_2(c_\infty)(v_L^*)^{1-|\mathrm{sign}(c_\infty)|} + \kappa_3(c_\infty)e^{c_\infty v_L^*}, \tag{3.22}$$

where κ_2 and κ_3 are defined in (1.5). Solving for $\Upsilon_\varepsilon^\pm$ in the second equation of (3.21) and substituting the result into the third equation of (3.21), we obtain from (3.22) that

$$(\Psi_\varepsilon^\pm)'' + g'(v_L^*)\Psi_\varepsilon^\pm - \partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp''\frac{\delta_0}{g'(v_L^\mp)} - \partial_\varepsilon\omega_\pm^*(0,0)\chi_\pm''\frac{\delta_0}{g'(v_L^\pm)} = -\delta_0\kappa_3(c_\infty)e^{c_\infty v_L^*}$$
$$- \delta_0\kappa_2(c_\infty)(v_L^*)^{1-|\mathrm{sign}(c_\infty)|} + g'(v_L^*)\Big(\partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp\frac{\delta_0}{g'(v_L^\mp)} + \partial_\varepsilon\omega_\pm^*(0,0)\chi_\pm\frac{\delta_0}{g'(v_L^\pm)}\Big). \tag{3.23}$$

Similar to (3.18), we note that $\Psi_\varepsilon^\pm - \partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp\frac{\delta_0}{g'(v_L^\mp)} - \partial_\varepsilon\omega_\pm^*(0,0)\chi_\pm\frac{\delta_0}{g'(v_L^\pm)}$ satisfies the equation

$$Z_{xx} + g'(v_L^*)Z = -\delta_0\kappa_2(c_\infty)(v_L^*)^{1-|\mathrm{sign}(c_\infty)|} - \delta_0\kappa_3(c_\infty)e^{c_\infty v_L^*}. \tag{3.24}$$

Moreover, from the definition of the function κ_2 and κ_3 in (1.5) we infer that

$$\langle \delta_0\kappa_1(c_\infty)e^{c_\infty v_L^*} - (v_L^*)_x, (v_L^*)_x\rangle_{L^2} = 0,$$

which proves that equation (3.24) has a unique solution denoted Z^*. Thus, we have that

$$\Psi_\varepsilon^\pm = \partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp\frac{\delta_0}{g'(v_L^\mp)} + \partial_\varepsilon\omega_\pm^*(0,0)\chi_\pm\frac{\delta_0}{g'(v_L^\pm)} + Z^*. \tag{3.25}$$

Using again the second equation of (3.14), it follows that $\Upsilon_\varepsilon^\pm = Z_x^*$. Differentiating with respect to ε in (3.6), we conclude from (2.14), (2.15), (3.22) and(3.25) that

$$\begin{aligned}
\partial_\varepsilon u_\pm^*(\cdot;0,0) &= \chi_\pm\Big[\partial_\mu u_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\partial_\varepsilon\mu_\pm^*(0,0) + \partial_\omega u_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\partial_\varepsilon\omega_\pm^*(0,0)\\
&\qquad + \partial_\varepsilon u_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\Big] + \partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp + \partial_\varepsilon\varphi_\pm^*(\cdot;0,0)\\
&= \partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp + \partial_\varepsilon\omega_\pm^*(0,0)\chi_\pm + \Phi_\varepsilon^\pm = \kappa_2(c_\infty)(v_L^*)^{1-|\mathrm{sign}(c_\infty)|} + \kappa_3(c_\infty)e^{c_\infty v_L^*},\\
\partial_\varepsilon v_\pm^*(\cdot;0,0) &= \chi_\pm\Big[\partial_\mu v_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\partial_\varepsilon\mu_\pm^*(0,0) + \partial_\omega v_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\partial_\varepsilon\omega_\pm^*(0,0)\\
&\qquad + \partial_\varepsilon v_c^\pm(\cdot;u_L^\infty,u_L^\infty,0,0)\Big] + \partial_\varepsilon\mu_\pm^*(0,0)(v^\pm)'(u_L^\infty)\chi_\mp + \partial_\varepsilon\psi_\pm^*(\cdot;0,0)\\
&= -\frac{\delta_0}{g'(v_L^\mp)}\partial_\varepsilon\mu_\pm^*(0,0)\chi_\mp - \frac{\delta_0}{g'(v_L^\pm)}\partial_\varepsilon\omega_\pm^*(0,0)\chi_\pm + \Psi_\varepsilon^\pm = Z^*.
\end{aligned} \tag{3.26}$$

Assertions (3.20) and (3.26) show that (1.3) hold true, thus proving the corollary.

4 Existence of traveling fronts with constant mass profile

In this section we prove the existence of traveling fronts of (1.1) with constant mass u under the perturbation $b(u) \to b_\varepsilon(u) = b(u) + \varepsilon$. That is, we are looking for solutions of the system (2.1) with constant profile u. Once this existence result is proved, we prove that the bifurcating traveling fronts are stable. Throughout this section we assume in addition that $b'(u_L^\infty) \neq 0$. To start, we notice that the first equation of system (2.1) is satisfied by a profile $u(x) \equiv \mu$ if and only if $b_\varepsilon(\mu) = 0$. Since $b(u_L^\infty) = 0$, $b'(u_L^\infty) \neq 0$ and $b(\cdot)$ is a C^3 function, this equation can be solved locally using the Implicit Function Theorem. That is, there exists $\varepsilon_1 > 0$ and a C^3 function $\overline{\mu} : (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$ such that $\overline{\mu}(0) = u_L^\infty$ and

$$\text{for any} \quad \varepsilon \in (-\varepsilon_1, \varepsilon_1), \quad b_\varepsilon(\mu) = 0 \quad \text{if and only if} \quad \mu = \overline{\mu}(\varepsilon). \tag{4.1}$$

Next, we substitute $u(x) \equiv \overline{\mu}(\varepsilon)$ into the second equation of (2.1) to obtain the equation

$$v'' + sv' + \delta_0 \overline{\mu}(\varepsilon) + g(v) = 0. \tag{4.2}$$

To prove the existence result we need to show that there exists a smooth function $\overline{s} : (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$ and a smoothly varying solution $\overline{v}(\cdot; \varepsilon)$ of (4.2) for $s = \overline{s}(\varepsilon)$. We use the ansatz

$$v = v^+(\overline{\mu}(\varepsilon))\chi_+ + v^-(\overline{\mu}(\varepsilon))\chi_- + \psi, \quad \psi \in H^2_\eta(\mathbb{R}), \tag{4.3}$$

where $\eta > 0$ small enough, $v^\pm(\mu)$ are defined in Remark 2.3(ii) and $\chi_\pm$ are as in Theorem 1.1. Substituting this ansatz into (4.2), we obtain the equation $\mathcal{G}(s, \varepsilon, \psi) = 0$, where the function $\mathcal{G} : \mathbb{R} \times (-\varepsilon_1, \varepsilon_1) \times H^2_\eta(\mathbb{R}) \to L^2_\eta(\mathbb{R})$ is defined by

$$\begin{aligned} \mathcal{G}(s, \varepsilon, \psi) = \psi'' + s\psi' + g\Big(v^+(\overline{\mu}(\varepsilon))\chi_+ + v^-(\overline{\mu}(\varepsilon))\chi_- + \psi\Big) + \delta_0\overline{\mu}(\varepsilon) \\ + v^+(\overline{\mu}(\varepsilon))\chi_+'' + v^-(\overline{\mu}(\varepsilon))\chi_-'' + sv^+(\overline{\mu}(\varepsilon))\chi_+' + sv^-(\overline{\mu}(\varepsilon))\chi_-'. \end{aligned} \tag{4.4}$$

Lemma 4.1. *Assume $b'(u_L^\infty) \neq 0$. Then, the function $\mathcal{G} : \mathbb{R} \times (-\varepsilon_1, \varepsilon_1) \times H^2_\eta(\mathbb{R}) \to L^2_\eta(\mathbb{R})$ defined in (4.4) is well-defined and of class C^1.*

Proof. First, we prove that the function $\mathcal{G}$ is well-defined. Since the functions $\chi_\pm'$ and $\chi_\pm''$ have compact support, it is enough to show that

$$g\Big(v^+(\overline{\mu}(\varepsilon))\chi_+ + v^-(\overline{\mu}(\varepsilon))\chi_- + \psi\Big) + \delta_0\overline{\mu}(\varepsilon) \in L^2_\eta(\mathbb{R}) \quad \text{for any} \quad \varepsilon \in (-\varepsilon_1, \varepsilon_1), \psi \in H^2_\eta(\mathbb{R}). \tag{4.5}$$

Since any function $\psi \in H^2_\eta(\mathbb{R})$ is bounded, $0 \leq \chi_\pm \leq 1$ and g is a function of class C^3, we infer that there exists a constant $C = C(\psi) > 0$ such that

$$|g\Big(v^+(\overline{\mu}(\varepsilon))\chi_+(x) + v^-(\overline{\mu}(\varepsilon))\chi_-(x) + \psi(x)\Big) - g\Big(v^+(\overline{\mu}(\varepsilon))\chi_+(x) + v^-(\overline{\mu}(\varepsilon))\chi_-(x)\Big)| \leq C(\psi)|\psi(x)|$$

for all $x \in \mathbb{R}$, $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, which implies that

$$g\Big(v^+(\overline{\mu}(\varepsilon))\chi_+ + v^-(\overline{\mu}(\varepsilon))\chi_- + \psi\Big) - g\Big(v^+(\overline{\mu}(\varepsilon))\chi_+ + v^-(\overline{\mu}(\varepsilon))\chi_-\Big) \in L^2_\eta(\mathbb{R}) \tag{4.6}$$

for any $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\psi \in H^2_\eta(\mathbb{R})$. Using the fact that $\chi_\pm(x) = 1$ for $\pm x \geq \pm 2$ and $\chi_\pm(x) = 0$ for $\mp x \geq \pm 2$, from Remark 2.3 we obtain that

$$g\Big(v^+(\overline{\mu}(\varepsilon))\chi_+(x) + v^-(\overline{\mu}(\varepsilon))\chi_-(x)\Big) + \delta_0\overline{\mu}(\varepsilon) = 0 \quad \text{if} \quad |x| \geq 2, \varepsilon \in (-\varepsilon_1, \varepsilon_1), \tag{4.7}$$

which shows that $g\Big(v^+(\overline{\mu}(\varepsilon))\chi_+ + v^-(\overline{\mu}(\varepsilon))\chi_-\Big) + \delta_0\overline{\mu}(\varepsilon)$ is a smooth function with compact support. From (4.6) and (4.7) we conclude that assertion (4.5) holds true, proving that the function $\mathcal{G}$ is well defined.

From Remark 2.3 we have that the functions $v^{\pm}$ are of class C^1. Moreover, the function $\overline{\mu}$ defined uniquely by (4.1) is of class C^3. Since the functions $\chi'_{\pm}$ and $\chi''_{\pm}$ have compact support, from (4.4) it follows that to prove the lemma it is enough to show that the function $\mathcal{G}_1 : (-\varepsilon_1, \varepsilon_1) \times H^2_\eta(\mathbb{R}) \to L^2_\eta(\mathbb{R})$ defined by

$$\mathcal{G}_1(\varepsilon, \psi) = g\Big(v^+(\overline{\mu}(\varepsilon))\chi_+ + v^-(\overline{\mu}(\varepsilon))\chi_- + \psi\Big) + \delta_0\overline{\mu}(\varepsilon) \tag{4.8}$$

is of class C^1. We note that the well-posedness of $\mathcal{G}_1$ was proved in (4.5). If we denote by χ_Ω the characteristic function of the set Ω, from the definition of the functions $\chi_{\pm}$ we have that

$$\begin{aligned}\mathcal{G}_1(\varepsilon, \psi) &= \mathcal{G}_1(\varepsilon, \psi)\chi_{(-\infty,-2]} + \mathcal{G}_1(\varepsilon, \psi)\chi_{[-2,2]} + \mathcal{G}_1(\varepsilon, \psi)\chi_{[2,\infty)} \\ &= \Big(g\big(v^-(\overline{\mu}(\varepsilon)) + \psi\big) - g\big(v^-(\overline{\mu}(\varepsilon))\big)\Big)\chi_{(-\infty,-2]} + \mathcal{G}_1(\varepsilon, \psi)\chi_{[-2,2]} + \\ &\quad + \Big(g\big(v^+(\overline{\mu}(\varepsilon)) + \psi\big) - g\big(v^+(\overline{\mu}(\varepsilon))\big)\Big)\chi_{[2,\infty)}.\end{aligned} \tag{4.9}$$

Using again that $v^{\pm}$ and $\overline{\mu}$ are bounded, C^1 functions we infer that the functions $\mathcal{G}_1^{\pm} : (-\varepsilon_1, \varepsilon_1) \times H^2_\eta(\mathbb{R}) \to L^2_\eta(\mathbb{R})$ defined by

$$\mathcal{G}_1^{\pm}(\varepsilon, \psi) = g\big(v^{\pm}(\overline{\mu}(\varepsilon)) + \psi\big) - g\big(v^{\pm}(\overline{\mu}(\varepsilon)\big) \tag{4.10}$$

are of class C^1. Moreover, since the functions $\chi_{\pm}$ are bounded and $\chi'_{\pm}$ have compact support, from Lemma A.1 and Lemma A.3 we conclude that $\mathcal{G}_1 : (-\varepsilon_1, \varepsilon_1) \times H^2_\eta(\mathbb{R}) \to H^1_{-\eta}(\mathbb{R})$ is of class C^1. Since the operator of multiplication by $\chi_{[-2,2]}$ is a bounded linear operator from $H^1_{-\eta}(\mathbb{R})$ to $L^2_\eta(\mathbb{R})$ and since the functions $\mathcal{G}_1^{\pm}$ defined in (4.10) are of class C^1, from (4.9) we obtain that the function $\mathcal{G}_1 : (-\varepsilon_1, \varepsilon_1) \times H^2_\eta(\mathbb{R}) \to L^2_\eta(\mathbb{R})$ is of class C^1, proving the lemma. □

In the next lemma we study the Fredholm properties of the partial derivative $\partial_\psi \mathcal{G}(0, 0, \tilde{v}_L^*)$, describe its kernel and the kernel of its L^2-adjoint, considered as a closed, densely defined linear operator on $L^2_{-\eta}(\mathbb{R})$. We recall the definition of $\tilde{v}_L^* = v_L^* - v_L^+\chi_+ - v_L^-\chi_- \in H^2_\eta(\mathbb{R})$ given in (2.18).

Lemma 4.2. *Assume $b'(u_L^\infty) \neq 0$. Then, the following assertions hold true:*

- *(i) There exists $\overline{\eta} > 0$ such that the operator $\partial_\psi \mathcal{G}(0, 0, \tilde{v}_L^*)$ is Fredholm index 0 on $L^2_\eta(\mathbb{R})$ for any $\eta \in [0, \overline{\eta})$;*
- *(ii) The kernel of $\partial_\psi \mathcal{G}(0, 0, \tilde{v}_L^*)$ is spanned by $(v_L^*)_x$;*
- *(ii) The kernel of $\partial_\psi \mathcal{G}(0, 0, \tilde{v}_L^*)^*$ is spanned by $(v_L^*)_x$.*

Proof. (i) Differentiating with respect to ψ in (4.4) one can easily check that $\partial_\psi \mathcal{G}(0, 0, \tilde{v}_L^*) = \partial_x^2 + g'(v_L^*)$. From Remark 2.2(iii) we have that $g'(v_L^{\pm}) < 0$, which implies that there exist $\overline{\eta} > 0$ such that the limiting operators $\partial_x^2 + g'(v_L^{\pm})$ are invertible from $H^2_\eta(\mathbb{R})$ to $L^2_\eta(\mathbb{R})$ for any $\eta \in [0, \overline{\eta})$. Using the results from [23, Section 8], we can immediately conclude that that the operator $\partial_\psi \mathcal{G}(0, 0, \tilde{v}_L^*)$ is Fredholm index 0 on $L^2_\eta(\mathbb{R})$ for any $\eta \in [0, \overline{\eta})$, proving (i).

(ii) Fix, $\eta \in [0,\overline{\eta})$. We note that a function $\psi \in H^2_\eta(\mathbb{R})$ belongs to the kernel of $\partial_\psi \mathcal{G}(0,0,\tilde{v}_L^*)$ if it satisfies the equation $\psi'' + g'(v_L^*)\psi = 0$. We recall that the latter equation is the variational equation of equation $v'' + \delta_0 u_L^\infty + g(v) = 0$, therefore we have that $\psi = d(v_L^*)_x$ for some scalar d, proving (ii). Assertion (iii) follows immediately since the L^2-adjoint of $\partial_\psi \mathcal{G}(0,0,\tilde{v}_L^*)$ is given by $\partial_\psi \mathcal{G}(0,0,\tilde{v}_L^*)^* = \partial_x^2 + g'(v_L^*)$. $\square$

Proof of Theorem 1.3. Summarizing the results from this section, we conclude that to prove Theorem 1.3 it is enough to show that we can solve the equation $\mathcal{G}(s,\varepsilon,\psi) = 0$ for $s \sim 0$ and $\psi \sim \tilde{v}_L^*$ in terms of $\varepsilon \sim 0$. Differentiating with respect to s in (4.4) and using that $\overline{\mu}(0) = u_L^\infty$, we obtain that

$$\partial_s \mathcal{G}(0,0,\tilde{v}_L^*) = (\tilde{v}_L^*)_x + v_L^- \chi_-' + v_L^+ \chi_+' = (v_L^*)_x. \tag{4.11}$$

From Lemma 4.2 we have that the linear operator $\partial_\psi \mathcal{G}(0,0,(\tilde{v}_L^*)_x)$ is Fredholm index 0 and

$$\operatorname{im} \partial_\psi \mathcal{G}(0,0,(\tilde{v}_L^*)_x) = \{\psi \in L^2_\eta(\mathbb{R},\mathbb{R}^3) : \langle \psi, (v_L^*)_x \rangle_{L^2} = 0\},\ \ker \partial_\psi \mathcal{G}(0,0,(\tilde{v}_L^*)_x) = \operatorname{Span}\{(v_L^*)_x\}. \tag{4.12}$$

From (4.11) and (4.12) we infer that the linear operator $\partial_{(\psi,s)} \mathcal{G}(0,0,(\tilde{v}_L^*)_x) : H^2_\eta(\mathbb{R}) \times \mathbb{R} \to L^2_\eta(\mathbb{R})$ is onto, and its kernel is spanned by $((v_L^*)_x, 0)^{\mathrm{T}}$. From the Implicit Function Theorem it follows that there exists $\varepsilon_1 > 0$ small enough and $\overline{s} : (-\varepsilon_1,\varepsilon_1) \to \mathbb{R}$ and $\overline{\psi} : (-\varepsilon_1,\varepsilon_1) \to H^2_\eta(\mathbb{R}) \ominus \operatorname{Span}\{(v_L^*)_x\}$ two C^1 functions such that

(i) $\overline{s}(0) = 0$, $\overline{\psi}(0) = \tilde{v}_L^*$;

(ii) locally, the equation $\mathcal{G}(s,\varepsilon,\psi) = 0$, $\psi \in H^2_\eta(\mathbb{R}) \ominus \operatorname{Span}\{(v_L^*)_x\}$ has the unique solution $(s,\psi)^{\mathrm{T}} = (\overline{s}(\varepsilon), \overline{\psi}(\varepsilon))$;

see (2.57) for the definition of the symbol $\ominus$. To finish the proof of Theorem 1.3, we compute the partial derivatives of $\overline{s}$ and $\overline{\psi}$ with respect to ε. First, we compute $\partial_\varepsilon \mathcal{G}(0,0,(v_L^*)_x)$. Since $\overline{\mu}(0) = u_L^\infty$ and $\overline{\mu}'(0) = -\frac{1}{b'(u_L^\infty)}$, from Remark 2.3 and (4.4) we obtain that

$$\partial_\varepsilon \mathcal{G}(0,0,\tilde{v}_L^*) = \frac{\delta_0}{b'(u_L^\infty) g'(v_L^+)} \big(\chi_+'' + \chi_+ g'(v_L^*)\big) + \frac{\delta_0}{b'(u_L^\infty) g'(v_L^-)} \big(\chi_-'' + \chi_- g'(v_L^*)\big) - \frac{\delta_0}{b'(u_L^\infty)} \tag{4.13}$$

Differentiating with respect to ε in the equation $\mathcal{G}(\overline{s}(\varepsilon),\varepsilon,\overline{\psi}(\varepsilon)) = 0$, we have that

$$\overline{s}'(0) \partial_s \mathcal{G}(0,0,(\tilde{v}_L^*)_x) + \partial_\varepsilon \mathcal{G}(0,0,(\tilde{v}_L^*)_x) + \partial_\psi \mathcal{G}(0,0,(\tilde{v}_L^*)_x) \overline{\psi}'(0) = 0. \tag{4.14}$$

Taking L^2-scalar product with $(v_L^*)_x$, from (4.12) and (4.11) we conclude that

$$\overline{s}'(0) = -\frac{\langle \partial_\varepsilon \mathcal{G}(0,0,(\tilde{v}_L^*)_x), (v_L^*)_x \rangle_{L^2}}{\langle \partial_s \mathcal{G}(0,0,(\tilde{v}_L^*)_x), (v_L^*)_x \rangle_{L^2}} = -\frac{\langle \partial_\varepsilon \mathcal{G}(0,0,(\tilde{v}_L^*)_x), (v_L^*)_x \rangle_{L^2}}{\|(v_L^*)_x\|_2^2}. \tag{4.15}$$

To evaluate this scalar product we note that

$$\begin{aligned} \langle \chi_\pm'' + \chi_\pm g'(v_L^*), (v_L^*)_x \rangle_{L^2} &= \int_{\mathbb{R}} \Big(\chi_\pm''(v_L^*)_x + \chi_\pm g'(v_L^*)(v_L^*)_x \Big) \\ &= \chi_\pm'(v_L^*)_x \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} \chi_\pm'(v_L^*)_{xx} + \int_{\mathbb{R}} \chi_\pm \big(g(v_L^*)\big)' \end{aligned}$$

$$= -\int_{\mathbb{R}} \chi'_\pm (v_L^*)_{xx} - \int_{\mathbb{R}} \chi_\pm (v_L^*)_{xxx} = -\chi_\pm (v_L^*)_{xx}\Big|_{-\infty}^{\infty} = 0. \tag{4.16}$$

From (4.13), (4.15) and (4.16), we infer that

$$\overline{s}'(0) = \frac{\delta_0}{b'(u_L^\infty)\|(v_L^*)_x\|_2^2}\int_{\mathbb{R}} (v_L^*)_x = \frac{\delta_0(v_L^+ - v_L^-)}{b'(u_L^\infty)\|(v_L^*)_x\|_2^2}. \tag{4.17}$$

Denoting by $\Psi = \overline{\psi}'(0)$, from equation (4.13) we obtain that Ψ satisfies the following equation:

$$\begin{aligned}\Psi_{xx} + g'(v_L^*)\Psi = &-\frac{\delta_0}{b'(u_L^\infty)g'(v_L^+)}\big(\chi''_+ + \chi_+ g'(v_L^*)\big) - \frac{\delta_0}{b'(u_L^\infty)g'(v_L^-)}\big(\chi''_- + \chi_- g'(v_L^*)\big)\\ &+ \frac{\delta_0}{b'(u_L^\infty)} - \frac{\delta_0(v_L^+ - v_L^-)}{b'(u_L^\infty)\|(v_L^*)_x\|_2^2}(v_L^*)_x.\end{aligned} \tag{4.18}$$

It follows that $\Psi + \frac{\delta_0}{b'(u_L^\infty)g'(v_L^+)}\chi_+ + \frac{\delta_0}{b'(u_L^\infty)g'(v_L^-)}\chi_-$ satisfies the equation

$$Y_{xx} + g'(v_L^*)Y = \frac{\delta_0}{b'(u_L^\infty)} - \frac{\delta_0(v_L^+ - v_L^-)}{b'(u_L^\infty)\|(v_L^*)_x\|_2^2}(v_L^*)_x. \tag{4.19}$$

Next, we note that the L^2-scalar product of the right hand side of equation (4.19) versus $(v_L^*)_x$ is 0, which implies that equation (4.19) has a unique solution denoted Y^*. Thus,

$$\overline{\psi}'(0) = \Psi = Y^* - \frac{\delta_0}{b'(u_L^\infty)g'(v_L^+)}\chi_+ - \frac{\delta_0}{b'(u_L^\infty)g'(v_L^-)}\chi_-. \tag{4.20}$$

Since $\overline{\mu}(0) = u_L^\infty$ and $\overline{\mu}'(0) = -\frac{1}{b'(u_L^\infty)}$, from Remark 2.3, (4.3) and (4.20) we conclude that

$$\begin{aligned}\partial_\varepsilon \overline{v}(\cdot;0) &= (v^+)'(u_L^\infty)\overline{\mu}'(0)\chi_+ + (v^-)'(u_L^\infty)\overline{\mu}'(0)\chi_- + \overline{\psi}'(0)\\ &= \frac{\delta_0}{b'(u_L^\infty)g'(v_L^+)}\chi_+ + \frac{\delta_0}{b'(u_L^\infty)g'(v_L^-)}\chi_- + \Psi = Y^*.\end{aligned} \tag{4.21}$$

Proof of Corollary 1.4. First, we focus our attention on computing the essential spectrum. Using the results from Theorem 1.3 one can readily check that :

$$\overline{\mathcal{L}}(\varepsilon) = \overline{D}(x,\varepsilon)\partial_x^2 + \overline{M}(x,\varepsilon)\partial_x + \overline{N}(x,\varepsilon), \tag{4.22}$$

where the matrix–valued functions $\overline{D}(\cdot,\cdot)$, $\overline{M}(\cdot,\cdot)$ and $\overline{N}(\cdot,\cdot)$ are continuous and bounded. $\overline{D}(x,s)$ is a diagonal matrix and thus, invertible, and the matrix-valued function $\overline{D}^{-1}(\cdot,\cdot)$ is bounded. Moreover

$$\overline{D}(x,\varepsilon) \to \overline{D}^\infty(\varepsilon),\ \overline{M}(x,\varepsilon) \to \overline{s}(\varepsilon)I_2,\ \overline{N}(x,s) \to \overline{N}^\pm(\varepsilon), \quad \text{as} \quad x \to \pm\infty, \tag{4.23}$$

where

$$\overline{D}^\infty(\varepsilon) = \begin{bmatrix} a(\overline{\mu}(\varepsilon)) & 0\\ 0 & 1\end{bmatrix}, \quad N^\pm(s) = \begin{bmatrix} 0 & 0\\ \delta_0 & g'(v^\pm(\overline{\mu}(\varepsilon)))\end{bmatrix}. \tag{4.24}$$

Fix $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$. Since $\overline{D}(\cdot,\varepsilon)$ and $\overline{D}^{-1}(\cdot,\varepsilon)$ are continuous and bounded, we infer that $\overline{\mathcal{L}}(\varepsilon) - \lambda$ is Fredholm if and only if $\mathcal{M}_{(\overline{D}(\cdot,\varepsilon))^{-1}}(\overline{\mathcal{L}}(\varepsilon) - \lambda)$ is Fredholm. Here $\mathcal{M}_{(\overline{D}(\cdot,\varepsilon))^{-1}}$ denotes the operator of multiplication on $L^2(\mathbb{R},\mathbb{C}^2)$ by the matrix valued function $\overline{D}^{-1}(\cdot,\varepsilon)$. Since $\lim_{x\to\pm\infty}\overline{M}(x,\varepsilon)=\overline{s}(\varepsilon)I_2$ we have that $\mathcal{M}_{(\overline{D}(\cdot,\varepsilon))^{-1}}(\overline{\mathcal{L}}(\varepsilon)-\lambda)$ is a relatively compact perturbation of $I_2\partial_x^2+\mathcal{M}_{(\overline{D}(\cdot,\varepsilon))^{-1}}\big[\overline{s}(\varepsilon)I_2\partial_x+\overline{N}(\cdot,\varepsilon)-\lambda I_2\big]$. Fredholm properties of the latter can be inferred from [7, Chapter 5, Thm A2]: $\mathcal{L}(s)-\lambda$ is Fredholm if and only if

$$\det\Big(-\overline{D}^{\infty}(\varepsilon)\tau^2+\mathrm{i}\tau\overline{s}(\varepsilon)I_2+\overline{N}^{+}(\varepsilon)-\lambda I_2\Big)\neq 0 \text{ and } \det\Big(-\overline{D}^{\infty}(\varepsilon)\tau^2+\mathrm{i}\tau\overline{s}(\varepsilon)I_2+\overline{N}^{-}(s)-\lambda I_2\Big)\neq 0, \tag{4.25}$$

From (4.24) and (4.25) we conclude that the essential spectrum of $\overline{\mathcal{L}}(\varepsilon)$ consists of the union of three graphs

$$\sigma_{\mathrm{ess}}(\overline{\mathcal{L}}(\varepsilon))=\Big\{\overline{\lambda}_0(\tau;\varepsilon):\tau\in\mathbb{R}\Big\}\cup\Big\{\overline{\lambda}_+(\tau;\varepsilon):\tau\in\mathbb{R}\Big\}\cup\Big\{\overline{\lambda}_-(\tau;\varepsilon):\tau\in\mathbb{R}\Big\}, \tag{4.26}$$

where

$$\overline{\lambda}_0(\tau;\varepsilon)=-a(\overline{\mu}(\varepsilon))\tau^2+\mathrm{i}\overline{s}(\varepsilon)\tau,\ \overline{\lambda}_\pm(\tau;\varepsilon)=-\tau^2+\mathrm{i}\overline{s}(\varepsilon)\tau+g'(v^\pm(\overline{\mu}(\varepsilon))),\ \tau\in\mathbb{R}. \tag{4.27}$$

Since $g'(v_L^\pm)<0$ and the functions g', $v^\pm$ and $\overline{\mu}$ are continuous, it follows that we can choose $\varepsilon_1>0$ small enough such that $g'(v^\pm(\overline{\mu}(\varepsilon)))<\frac{1}{2}g'(v_L^\pm)<0$ for any $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$, which implies that $\mathrm{Re}\ \overline{\lambda}_\pm(\tau;\varepsilon)=-\tau^2+g'(v^\pm(\overline{\mu}(\varepsilon)))\leq g'(v^\pm(\overline{\mu}(\varepsilon)))<\frac{1}{2}g'(v_L^\pm)<0$ for any $\tau\in\mathbb{R}$ and $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$. Moreover, $\mathrm{Re}\ \overline{\lambda}_0(\tau;\varepsilon)=-a(\overline{\mu}(\varepsilon))\tau^2\leq 0$ for any $\tau\in\mathbb{R}$ and $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$. We note that $\lambda\in\sigma_{\mathrm{ess}}(\overline{\mathcal{L}}(\varepsilon))\cap\mathrm{i}\mathbb{R}$ if and only $\lambda=\overline{\lambda}_0(\tau;\varepsilon)$ and $\tau=0$, which implies that $\lambda=0$, proving (i).

To start the proof of (ii), we note that the operator $\overline{\mathcal{L}}(\varepsilon)$ has a lower-triangular block structure, which implies that the the eigenvalue problem $\overline{\mathcal{L}}(\varepsilon)(u,v)^{\mathrm{T}}=\lambda(u,v)^{\mathrm{T}}$ decouples as follows:

$$\overline{\mathcal{L}}(\varepsilon)(u,v)^{\mathrm{T}}=\lambda(u,v)^{\mathrm{T}} \text{ if and only if } \begin{cases}\partial_x\Big(a(\overline{\mu}(\varepsilon))\partial_x-b'(\overline{\mu}(\varepsilon))\overline{v}_x(\cdot;\varepsilon)+\overline{s}(\varepsilon)\Big)u=\lambda u,\\ v''+\overline{s}(\varepsilon)v'+\delta_0u+g'(\overline{v}(\cdot;\varepsilon))v=\lambda v,\end{cases} \tag{4.28}$$

Since the operator $\overline{\mathcal{L}}_{11}(\varepsilon)=\partial_x\Big(a(\overline{\mu}(\varepsilon))\partial_x-b'(\overline{\mu}(\varepsilon))\overline{v}_x(\cdot;\varepsilon)+\overline{s}(\varepsilon)\Big)$ is in divergence form, we have that $\overline{\mathcal{L}}_{11}(\varepsilon)$ has no eigenvalue with positive real part. Arguing for a contradiction, assume $\overline{\mathcal{L}}(\varepsilon)_{11}$ has an eigenvalue with positive real part. Thus, there exists a solution $\overline{u}$ of the equation $u_t=\overline{\mathcal{L}}_{11}(\varepsilon)u$ exponentially growing in time and exponentially localized in space, which implies that $\|\overline{u}(t)\|_{L^1}$ would be growing exponentially as $t\to\infty$. Using the fact that $\int u$ is conserved by splitting initial conditions into positive and negative parts, and exploiting positivity of the solution, we have that the semigroup generated by $\overline{\mathcal{L}}(\varepsilon)_{11}$ is a contraction on $L^1(\mathbb{R},\mathbb{C}^2)$. This is a contradiction, therefore, from the first equation we infer that $\mathrm{Re}\,\lambda\leq 0$ or $u=0$. If $u=0$ from the second equation of (4.28) we obtain that $\overline{\mathcal{L}}_{22}(\varepsilon)v:=v''+\overline{s}(\varepsilon)v'+g'(\overline{v}(\cdot;\varepsilon))v=\lambda v$. We note that $v\in\ker\overline{\mathcal{L}}_{22}(\varepsilon)$ if and only if

$$v''+\overline{s}(\varepsilon)v'+g'(\overline{v}(\cdot;\varepsilon))v=0. \tag{4.29}$$

Since equation (4.30) is the variational equation of (4.2), we obtain that

$$\ker\overline{\mathcal{L}}_{22}(\varepsilon)=\mathrm{Span}\{\overline{v}_x(\cdot;\varepsilon)\}. \tag{4.30}$$

Since $g'(v^{\pm}(\overline{\mu}(\varepsilon)))<0$ for any $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$, we have that (4.2) is a bistable second order scalar equation. Using phase-plane analysis, one can show that for each fixed $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$ the profile $\overline{v}(\cdot;\varepsilon)$ is monotone. Since the operator $\overline{\mathcal{L}}_{22}(\varepsilon)$ is Sturm-Liouvile and its kernel $\overline{v}_x(\cdot;\varepsilon)$ has no sign change because $\overline{v}(\cdot;\varepsilon)$ is monotone, we have that $\overline{\mathcal{L}}_{22}(\varepsilon)$ has no eigenvalue with positive real part. This shows that $\overline{\mathcal{L}}(\varepsilon)$ has no eigenvalues with positive real part, proving (ii).

A Appendix

Lemma A.1. *Let $I\subset\mathbb{R}^m$ be an interval and $f:\mathbb{R}\times I\to\mathbb{R}$ be a C^2 function satisfying the following properties:*

(i) $f,\partial_x f\in L^\infty(\mathbb{R}\times I)$;

(ii) For any $\theta>0$ there exits $M_\theta>0$ such that

$$|\partial_{y_j}f(x,y)|+|\partial_x\partial_{y_j}f(x,y)|\le M_\theta\, e^{\theta|x|}\quad\textit{for all}\quad x\in\mathbb{R},y\in I,j=1,\dots,m. \tag{A.1}$$

Then, the function $F:I\to H^1_{-\gamma}(\mathbb{R})$ defined by $F(y)=f(\cdot,y)$ is of class C^1 for any $\gamma>0$.

Proof. First, we prove that F is continuous on I. Fix $y\in I$ and let $\{y_n\}_{n\ge1}$ be a sequence of elements in I such that $y_n\to y$ as $n\to\infty$. Then,

$$\|F(y_n)-F(y)\|^2_{H^1_{-\gamma}}=\int_{\mathbb{R}}e^{-2\gamma|x|}\Big(|f(x,y_n)-f(x,y)|^2+|\partial_xf(x,y_n)-\partial_xf(x,y)|^2\Big)\,\mathrm{d}x.$$

Since $f\in C^2(\mathbb{R}\times I)$ from (i) and Lebesgue's Dominated Convergence Theorem we conclude that $\|F(y_n)-F(y)\|_{H^1_{-\gamma}}\to0$ as $n\to\infty$, proving that F is continuous on I. Next, we prove that all partial derivatives of F with respect to y_j, $j=1,\dots,m$, exist. Fix $y\in I$ again and let $\{t_n\}_{n\ge1}$ be a sequence of real numbers with $t_n\ne0$ for all $n\ge1$ and $t_n\to0$ as $n\to\infty$. For any $j=1,\dots,m$ we introduce the sequence of functions by $h_n^j=\frac{1}{t_n}(F(y+t_ne_j)-F(y))$ (Here e_j, $j=1,\dots,m$, denote the vectors of the canonical basis in $\mathbb{R}^m$). To prove that $h_n^j\to\partial_{y_j}f(\cdot,y)$ as $n\to\infty$ in $H^1_{-\gamma}(\mathbb{R})$ we use again Lebesgue's Dominated Convergence Theorem. Since $f\in C^2(\mathbb{R}\times I)$ we have that

$$\lim_{n\to\infty}h_n^j(x)=\partial_{y_j}f(x,y)\quad\text{and}\quad\lim_{n\to\infty}(h_n^j)'(x)=\partial_x\partial_{y_j}f(x,y)\quad\text{for all}\quad x\in\mathbb{R}. \tag{A.2}$$

Since $f\in C^2(\mathbb{R}\times I)$ we conclude that for any $x\in\mathbb{R}$ there exists $\overline{y}_n^j(x),\tilde{y}_n^j(x)\in I$ such that

$$h_n^j(x)=\partial_{y_j}f(x,\overline{y}_n^j(x))\quad\text{and}\quad(h_n^j)'(x)=\partial_x\partial_{y_j}f(x,\tilde{y}_n^j(x))$$

for all $x\in\mathbb{R}$, $n\ge1$ and $j=1,\dots,m$. From (ii) we obtain that

$$|h_n^j(x)|\le M_{\frac{\gamma}{2}}e^{\frac{\gamma}{2}|x|}\quad\text{and}\quad|(h_n^j)'(x)|\le M_{\frac{\gamma}{2}}e^{\frac{\gamma}{2}|x|} \tag{A.3}$$

for all $x\in\mathbb{R}$, $n\ge1$ and $j=1,\dots,m$. From (A.2) and (A.3) and Lebesgue's Dominated Convergence Theorem we obtain that $\frac{1}{t_n}(F(y+t_ne_j)-F(y))\to\partial_{y_j}f(\cdot,y)$ as $n\to\infty$ in $H^1_{-\gamma}(\mathbb{R})$, proving that the partial derivatives of F exist and $\partial_{y_j}F(y)=\partial_{y_j}f(\cdot,y)$ for all $y\in I$.

To finish the proof of lemma we have to prove that the partial derivatives of F are continuous. Let $y \in I$ and $\{y_n\}_{n\geq 1}$ such that $y_n \to y$ as $n \to \infty$. We note that

$$\begin{aligned}\|\partial_{y_j}F(y_n)-\partial_{y_j}F(y)\|^2_{H^1_{-\gamma}} = &\int_{\mathbb{R}} e^{-2\gamma|x|}|\partial_{y_j}f(x,y_n)-\partial_{y_j}f(x,y)|^2\,\mathrm{d}x\\ &+\int_{\mathbb{R}} e^{-2\gamma|x|}|\partial_x\partial_{y_j}f(x,y_n)-\partial_x\partial_{y_j}f(x,y)|^2\,\mathrm{d}x.\end{aligned}$$

From (ii) for $\theta = \frac{\gamma}{2}$ we have that

$$|\partial_{y_j}f(x,y_n)-\partial_{y_j}f(x,y)|^2 \leq 4M^2_{\frac{\gamma}{2}}e^{\gamma|x|} \text{ and } |\partial_x\partial_{y_j}f(x,y_n)-\partial_x\partial_{y_j}f(x,y)|^2 \leq 4M^2_{\frac{\gamma}{2}}e^{\gamma|x|} \quad \text{(A.4)}$$

for all $x \in \mathbb{R}$, $n \geq 1$ and $j = 1,\dots,m$. Since $f \in C^2(\mathbb{R}\times I)$ from (A.4) and Lebesgue's Dominated Convergence Theorem we obtain that $\partial_{y_j}F$ are continuous on I for all $j = 1,\dots,m$, proving the lemma. □

In what follows we denote by $c > 0$ a generic positive constant. To prove the next lemma we recall the following result.

Remark A.2. *For any functions $g \in H^1_\gamma(\mathbb{R})$ and $h \in H^1_\kappa(\mathbb{R})$ we have that $gh \in H^1_{\gamma+\kappa}(\mathbb{R})$ and*

$$\|gh\|_{H^1_{\gamma+\kappa}(\mathbb{R})} \leq c\|g\|_{H^1_\gamma(\mathbb{R})}\,\|h\|_{H^1_\kappa(\mathbb{R})}.$$

Lemma A.3. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a C^3 function, $I \subset \mathbb{R}^m$ a bounded interval and $G : I \to H^1_{-\frac{\gamma}{2}}(\mathbb{R})$ a function satisfying the properties:*

(i) *G is of class C^1;*

(ii) $\mathrm{Range}(G) \subset L^\infty(\mathbb{R})$*;*

(iii) *$M = \sup_{y\in I}\|G(y)\|_\infty < \infty$.*

Then, the function $F : H^1_\eta(\mathbb{R}) \times I \to H^1_{-\gamma}(\mathbb{R})$ defined by $F(\varphi, y) = \alpha(G(y) + \varphi)$ is of class C^1.

Proof. To prove that the function F is of class C^1 on $H^1_\eta(\mathbb{R}) \times I$ it is enough to prove that it is of class C^1 on

$$\mathcal{D}_p = \{\varphi \in H^1_\eta(\mathbb{R}) : \|\varphi\|_{H^1_\eta(\mathbb{R})} \leq p\} \quad \text{(A.5)}$$

for any $p \in \mathbb{Z}_+$. Since the function G is of class C^1 it follows that the function $\Lambda : H^1_\eta(\mathbb{R}) \times I \to H^1_{-\frac{\gamma}{2}}(\mathbb{R})$ defined by $\Lambda(\varphi, y) = G(y) + \varphi$ is of class C^1. Moreover, we have that

$$\mathrm{Range}(\Lambda) \subset \mathcal{A}_p := H^1_{-\frac{\gamma}{2}}(\mathbb{R}) \cap \{\psi \in L^\infty(\mathbb{R}) : \|\psi\|_\infty \leq M + p\}. \quad \text{(A.6)}$$

Next, we introduce the function $F_\alpha : \mathcal{A}_p \to H^1_{-\gamma}(\mathbb{R})$ by $F_\alpha(\psi) = \alpha \circ \psi$. Since $F_{|\mathcal{D}_p} = F_\alpha \circ \Lambda$, to prove the lemma it is enough to show that the function F_α is of class C^1. First, we prove that F_α is well-defined. Let $\psi \in \mathcal{A}_p$. Since the function α is of class C^3 on $\mathbb{R}$ and $\psi \in L^\infty(\mathbb{R})$ we have that

$$\alpha \circ \psi \in L^\infty(\mathbb{R}) \subset L^2_{-\gamma}(\mathbb{R}). \quad \text{(A.7)}$$

Since $\psi \in \mathcal{A}_p \subset H^1_{-\frac{\gamma}{2}}(\mathbb{R})$ we have that ψ is absolutely continuous on $\mathbb{R}$. Therefore, since α is of class C^3 on $\mathbb{R}$ we have that $\alpha \circ \psi$ is absolutely continuous on $\mathbb{R}$ and $(\alpha \circ \psi)' = (\alpha' \circ \psi)\, \psi'$. In addition, since $\psi \in \mathcal{A}_p$ we obtain that $\psi \in L^\infty(\mathbb{R})$ and $\psi' \in L^2_{-\gamma}(\mathbb{R})$, which implies that

$$(\alpha \circ \psi)' = (\alpha' \circ \psi)\, \psi' \in L^2_{-\gamma}(\mathbb{R}). \tag{A.8}$$

From (A.7) and (A.8) we conclude that F_α is well-defined. Moreover, one can modify the argument above to prove that

$$\alpha' \circ \psi \in H^1_{-\frac{\gamma}{2}}(\mathbb{R}) \quad \text{for all} \quad \psi \in \mathcal{A}_p. \tag{A.9}$$

Next, we show that the function F_α is differentiable. Fix $\psi_0 \in \mathcal{A}_p$ and let $\psi \in \mathcal{A}_p$ with $\|\psi - \psi_0\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})} \ll 1$. From (A.9) we infer that the $M_{\alpha' \circ \phi_0}$, the operator of multiplication by $\alpha' \circ \psi_0 \in H^1_{-\frac{\gamma}{2}}(\mathbb{R})$, is bounded from $H^1_{-\frac{\gamma}{2}}(\mathbb{R})$ to $H^1_{-\gamma}(\mathbb{R})$. We will show that F_α is differentiable at ψ_0 and $DF_\alpha(\psi_0) = M_{\alpha' \circ \psi_0}$. Since $\alpha \in C^3(\mathbb{R})$ we have that

$$|\alpha(z_1) - \alpha(z_2) - \alpha'(z_2)(z_1 - z_2)| \leq K_1 |z_1 - z_2|^2 \quad \text{for all} \quad z_1, z_2 \in [-M - p, M + p], \tag{A.10}$$

where $K_1 = \frac{1}{2} \sup_{|z| \leq M+p} |\alpha''(z)|$. Since $\psi, \psi_0 \in \mathcal{A}_p$ we have that $z_1 = \psi(x) \in [-M - p, M + p]$, $z_2 = \psi_0(x) \in [-M - p, M + p]$ for all $x \in \mathbb{R}$. Thus, from (A.10) we obtain that

$$|\alpha(\psi(x)) - \alpha(\psi_0(x)) - \alpha'(\psi_0(x))(\psi(x) - \psi_0(x))| \leq K_1 |\psi(x) - \psi_0(x)|^2 \quad \text{for all} \quad x \in \mathbb{R}.$$

Integrating with respect to x we infer that

$$\begin{aligned}
&\Big\| F_\alpha(\psi) - F_\alpha(\psi_0) - M_{\alpha' \circ \psi_0}(\psi - \psi_0) \Big\|^2_{L^2_{-\gamma}(\mathbb{R})} \leq c \int_{\mathbb{R}} e^{-2\gamma|x|} |\psi(x) - \psi_0(x)|^4 \mathrm{d}x \leq \\
&\leq c \Big\| e^{-\gamma|\cdot|} (\psi - \psi_0)^2 \Big\|_\infty \Big\| \psi - \psi_0 \Big\|^2_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})} \leq c \Big\| e^{-\frac{\gamma}{2}|\cdot|} (\psi - \psi_0) \Big\|^2_\infty \Big\| \psi - \psi_0 \Big\|^2_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})} \\
&\leq c \Big\| \psi - \psi_0 \Big\|^4_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})}.
\end{aligned} \tag{A.11}$$

Moreover, since

$$\begin{aligned}
&\Big(F_\alpha(\psi) - F_\alpha(\psi_0) - M_{\alpha' \circ \psi_0}(\psi - \psi_0) \Big)' = (\alpha' \circ \psi - \alpha' \circ \psi_0)\psi' - (\alpha'' \circ \psi_0)\psi_0'(\psi - \psi_0) \\
&= (\alpha' \circ \psi - \alpha' \circ \psi_0)(\psi' - \psi_0') + \Big[\alpha' \circ \psi - \alpha' \circ \psi_0 - (\alpha'' \circ \psi_0)(\psi - \psi_0) \Big] \psi_0',
\end{aligned}$$

we estimate that

$$\begin{aligned}
\Big\| \Big(F_\alpha(\psi) - F_\alpha(\psi_0) - M_{\alpha' \circ \psi_0}(\psi - \psi_0) \Big)' \Big\|_{L^2_{-\gamma}(\mathbb{R})} &\leq \Big\| (\alpha' \circ \psi - \alpha' \circ \psi_0)(\psi' - \psi_0') \Big\|_{L^2_{-\gamma}(\mathbb{R})} \\
&+ \Big\| \Big[\alpha' \circ \psi - \alpha' \circ \psi_0 - (\alpha'' \circ \psi_0)(\psi - \psi_0) \Big] \psi_0' \Big\|_{L^2_{-\gamma}(\mathbb{R})}.
\end{aligned} \tag{A.12}$$

Since $\alpha \in C^3(\mathbb{R})$ we have that

$$|\alpha'(z_1) - \alpha'(z_2)| \leq K_2 |z_1 - z_2|, \quad |\alpha'(z_1) - \alpha'(z_2) - \alpha''(z_2)(z_1 - z_2)| \leq K_3 |z_1 - z_2|^2, \tag{A.13}$$

for all $z_1, z_2 \in [-M-p, M+p]$, where $K_2 = \sup_{|z|\le M+p} |\alpha''(z)|$ and $K_3 = \frac{1}{2}\sup_{|z|\le M+p} |\alpha'''(z)|\}$. From (A.12) and (A.13) we obtain that

$$
\begin{aligned}
&\Big\|\Big(F_\alpha(\psi) - F_\alpha(\psi_0) - M_{\alpha'\circ\psi_0}(\psi-\psi_0)\Big)'\Big\|_{L^2_{-\gamma}(\mathbb{R})} \le \\
&\le \Big\|e^{-\frac{\gamma}{2}|\cdot|}(\alpha'\circ\psi - \alpha'\circ\psi_0)\Big\|_\infty \Big\|e^{-\frac{\gamma}{2}|\cdot|}(\psi'-\psi_0')\Big\|_2 + c\Big(\int_{\mathbb{R}} e^{-2\gamma|x|}|\psi(x)-\psi_0(x)|^4|\psi_0'(x)|^2\mathrm{d}x\Big)^{\frac{1}{2}} \\
&\le c\Big\|e^{-\frac{\gamma}{2}|\cdot|}(\psi-\psi_0)\Big\|_\infty\left(\Big\|\psi-\psi_0\Big\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})} + \Big(\int_{\mathbb{R}} e^{-\gamma|x|}|\psi(x)-\psi_0(x)|^2|\psi_0'(x)|^2\mathrm{d}x\Big)^{\frac{1}{2}}\right) \\
&\le c\Big\|\psi-\psi_0\Big\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})}\left(\Big\|\psi-\psi_0\Big\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})} + \Big(\int_{\mathbb{R}} e^{-\gamma|x|}|\psi(x)-\psi_0(x)|^2|\psi_0'(x)|^2\mathrm{d}x\Big)^{\frac{1}{2}}\right). \qquad \text{(A.14)}
\end{aligned}
$$

From (A.12) and (A.14) we conclude that

$$
\begin{aligned}
&\Big\|\Big(F_\alpha(\psi) - F_\alpha(\psi_0) - M_{\alpha'\circ\psi_0}(\psi-\psi_0)\Big)'\Big\|_{H^1_{-\gamma}(\mathbb{R})} \le \\
&\le c\Big\|\psi-\psi_0\Big\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})}\left(\Big\|\psi-\psi_0\Big\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})} + \Big(\int_{\mathbb{R}} e^{-\gamma|x|}|\psi(x)-\psi_0(x)|^2|\psi_0'(x)|^2\mathrm{d}x\Big)^{\frac{1}{2}}\right) \qquad \text{(A.15)}
\end{aligned}
$$

From (A.15) it follows that to prove that F_α is differentiable at ψ_0 and $(DF_\alpha)(\psi_0) = M_{\alpha'\circ\psi_0}$ it is enough to show that

$$
\lim_{\psi\to\psi_0, \psi\in\mathcal{A}_p} \int_{\mathbb{R}} e^{-\gamma|x|}|\psi(x)-\psi_0(x)|^2|\psi_0'(x)|^2\mathrm{d}x = 0. \qquad \text{(A.16)}
$$

To prove (A.16) we consider $\{\psi_n\}_{n\ge1}$ a sequence of functions in $\mathcal{A}_p$ such that $\psi_n \to \psi_0$ in $H^1_{-\frac{\gamma}{2}}(\mathbb{R})$ as $n\to\infty$. It follows that $e^{-\frac{\gamma}{2}|\cdot||}(\psi_n-\psi_0)\to 0$ in $L^\infty(\mathbb{R})$ as $n\to\infty$, which implies that $\lim_{n\to\infty} e^{-\gamma|x|}|\psi_n(x)-\psi_0(x)|^2|\psi_0'(x)|^2 = 0$ for all $x\in\mathbb{R}$. Moreover, since $\mathcal{A}_p \subset \{\psi \in L^\infty(\mathbb{R}) : \|\psi\|_\infty \le M+p\}$ we have that

$$
e^{-\gamma|x|}|\psi_n(x)-\psi_0(x)|^2|\psi_0'(x)|^2 \le 4(M+p)^2 e^{-\gamma|x|}|\psi_0'(x)|^2 \quad \text{for all} \quad n\ge 1, x\in\mathbb{R}.
$$

Since $\psi_0' \in L^2_{-\frac{\gamma}{2}}(\mathbb{R})$ claim (A.16) follows shortly from Lebesgue's Dominated Convergence Theorem. Thus, that F_α is differentiable on $\mathcal{A}_p$ and $(DF_\alpha)(\psi) = M_{\alpha'\circ\psi} \in \mathcal{B}(H^1_{-\frac{\gamma}{2}}(\mathbb{R}), H^1_{-\gamma}(\mathbb{R}))$. To finish the proof of lemma we have to prove that DF_α is continuous on $\mathcal{A}_p$. We fix again $\psi_0\in\mathcal{A}_p$ and let $\psi\in\mathcal{A}_p$ with $\|\psi-\psi_0\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})} \ll 1$. Since $\mathcal{A}_p \subset \{\psi\in L^\infty(\mathbb{R}) : \|\psi\|_\infty \le M+p\}$, from (A.13) and Remark A.2 we estimate

$$
\begin{aligned}
&\Big\|(DF_\alpha)(\psi) - (DF_\alpha)(\psi_0)\Big\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})\to H^1_{-\gamma}(\mathbb{R})} \le c\Big\|\alpha'\circ\psi - \alpha'\circ\psi_0\Big\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})} \\
&\le c\Big\|\alpha'\circ\psi - \alpha'\circ\psi_0\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})} + c\Big\|\big(\alpha'\circ\psi - \alpha'\circ\psi_0\big)'\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})} \\
&\le cK_2\Big\|\psi-\psi_0\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})} + c\Big\|(\alpha''\circ\psi)(\psi'-\psi_0')\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})} + c\Big\|(\alpha''\circ\psi - \alpha''\circ\psi_0)\psi_0'\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})}
\end{aligned}
$$

$$\leq c\Big\|\psi-\psi_0\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})}+c\Big\|\alpha''\circ\psi\Big\|_\infty\Big\|\psi'-\psi_0'\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})}+c\Big\|(\psi-\psi_0)\psi_0'\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})}$$

$$\leq c\Big\|\psi-\psi_0\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})}+c\Big\|\alpha''\circ\psi\Big\|_\infty\Big\|\psi'-\psi_0'\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})}+c\sup_{|z|\leq M+p}|\alpha'''(z)|\Big\|(\psi-\psi_0)\psi_0'\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})}$$

$$\leq c\Big\|\psi-\psi_0\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})}+c\Big\|\psi'-\psi_0'\Big\|_{L^2_{-\frac{\gamma}{2}}(\mathbb{R})}+c\Big(\int_{\mathbb{R}}e^{-\gamma|x|}|\psi(x)-\psi_0(x)|^2|\psi_0'(x)|^2\mathrm{d}x\Big)^{\frac{1}{2}}$$

$$\leq c\Big\|\psi-\psi_0\Big\|_{H^1_{-\frac{\gamma}{2}}(\mathbb{R})}+c\Big(\int_{\mathbb{R}}e^{-\gamma|x|}|\psi(x)-\psi_0(x)|^2|\psi_0'(x)|^2\mathrm{d}x\Big)^{\frac{1}{2}}. \tag{A.17}$$

From (A.16) and (A.17) we conclude that DF_α from $\mathcal{A}_p$ to $\mathcal{B}(H^1_{-\frac{\gamma}{2}}(\mathbb{R}),H^1_{-\gamma}(\mathbb{R}))$, proving the lemma. $\square$

Lemma A.4. *Let $I\subset\mathbb{R}^m$ be an interval and $f:\mathbb{R}\times I\to\mathbb{R}$ be a C^2 function satisfying the following properties:*

(i) $f,\partial_x f\in L^\infty(\mathbb{R}\times I)$;

(ii) For any $\theta>0$ there exits $M_\theta>0$ such that

$$|\partial_{y_j}f(x,y)|+|\partial_x\partial_{y_j}f(x,y)|\leq M_\theta\, e^{\theta|x|}\quad \textit{for all}\quad x\in\mathbb{R},y\in I,j=1,\dots,m. \tag{A.18}$$

Then, the function $K:I\to\mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R}),H^1_\eta(\mathbb{R}))$ defined by $K(y)=M_{f(\cdot,y)}$, the operator of multiplication by $f(\cdot,y)$, is of class C^1 for any $\eta,\gamma>0$.

Proof. First, we introduce the linear map $\mathcal{M}:H^1_{-\gamma}(\mathbb{R})\to\mathcal{B}(H^1_{\eta+\gamma}(\mathbb{R}),H^1_\eta(\mathbb{R}))$ defined by $\mathcal{M}h=M_h$, the operator of multiplication by h. From Remark A.2 we infer that for any $h\in H^1_{-\gamma}(\mathbb{R})$ and $\psi\in H^1_{\eta+\gamma}(\mathbb{R})$ we have that $h\psi\in H^1_\eta(\mathbb{R})$ and $\|h\psi\|_{H^1_\eta(\mathbb{R})}\leq c\|h\|_{H^1_{-\gamma}(\mathbb{R})}\,\|\psi\|_{H^1_{\eta+\gamma}(\mathbb{R})}$, which implies that the map $\mathcal{M}$ is well-defined and bounded. Moreover, from Lemma A.1 we have that the function $F:I\to H^1_{-\gamma}(\mathbb{R})$ defined by $F(y)=f(\cdot,y)$ is of class C^1. Since $K(y)=\mathcal{M}F(y)$ for all $y\in I$, the lemma follows shortly. $\square$

References

[1] G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Ration. Mech. Anal. **92** (1986), 205–245.

[2] P. Coullet, J. Lega, B. Houchmanzadeh, and J. Lajzerowicz, *Breaking chirality in nonequilibrium systems*, Phys. Rev. Lett. **65** (1990), 1352–1355.

[3] S.-I. Ei, H. Ikeda, and T. Kawana, *Dynamics of front solutions in a specific reaction-diffusion system in one dimension,* Japan J. Indust. Appl. Math. **25** (2008), 117–147.

[4] R. Goh, S. Mesuro, and A. Scheel, *Coherent structures in reaction-diffusion models for precipitation*, Special volume on "Precipitation patterns in reaction-diffusion systems", Research Signpost, 2010, 73–93.

[5] R. Goh, S. Mesuro, and A. Scheel, *Spatial wavenumber selection in recurrent precipitation,* SIAM J. Appl. Dyn. Sys. **10** (2011), 360-402.

[6] A. Hagberg and E. Meron, *Pattern formation in non-gradient reactiondiffusion systems: the effects of front bifurcations*, Nonlinearity **7** (1994) 805–835.

[7] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lect. Notes Math. **840** (1981), Springer–Verlag, New York.

[8] T. Hillen and K. Painter, *A user's guide to PDE models for chemotaxis*, J. Math. Biol. **58**, (2008), 183–217.

[9] H. Ikeda, M. Mimura, and Y. Nishiura, *Global bifurcation phenomena of traveling wave solutions for some bistable reaction-diffusion systems*, Nonlinear Anal. **13** (1989), 507–526.

[10] G. Jaramillo and A. Scheel, *Deformation of Striped Patterns by Inhomogeneities,* Math. Meth. Appl. Sci. **38** (2015), 51–65.

[11] G. Jaramillo and A. Scheel, *Pacemakers in large arrays of oscillators with nonlocal coupling,* J. Diff. Eqns., **260** (2016), 2060–2090.

[12] A. Jilkine, L. Keshet, and Y. Mori, *Wave-pinning and cell polarity from a bistable reaction-diffusion system*, Bioph. J. **94** (2008), 3684–3697.

[13] Y. Morita and T. Ogawa, *Stability and bifurcation of nonconstant solutions to a reaction-diffusion system with conservation of mass*, Nonlinearity **23** (2010), 1387–1411.

[14] S. Nasuno, N. Yoshimo, and S. Kai, *Structural transition andmotion of domain walls in liquidcrystals under a rotating magnetic field,* Phys. Rev. E **51** (1995) 1598.

[15] K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Diff. Eq. **55** (1984), 225–256.

[16] K. J. Palmer, *Exponential dichotomies and Fredholm operators*, Proc. Amer. Math. Soc. **104** (1988), 149–156.

[17] A. Pogan and A. Scheel, *Instability of Spikes in the Presence of Conservation Laws*, Z. Angew. Math. Phys. **61** (2010), 979–998.

[18] A. Pogan and A. Scheel, *Instability of radially-symmetric spikes in systems with a conserved quantity*, Fields Inst. Comm., **64** (2013), 119-140.

[19] A. Pogan and A. Scheel, *Fredholm properties of radially symmetric, second order differential operators*, Int. J. Dyn. Sys. Diff. Eqns., Int. J. Dyn. Sys. Diff. Eqns., **3** (2011), 289–327.

[20] A. Pogan and A. Scheel, *Layers in the Presence of Conservation Laws*, J. Dyn. Diff. Eq., **24** (2012), 249–287.

[21] A. Pogan, A. Scheel, and K. Zumbrun. *Quasi-gradient systems, modulational dichotomies, and stability of spatially periodic patterns* . Diff. Int. Eqns., **26** (2013), no. 3-4, 389-438.

[22] J. Rinzel and D. Terman, *Propagation phenomena in a bistable reaction-diffusion system*, SIAM J. Appl. Math. **42** (1982), 1111–1137.

[23] B. Sandstede and A. Scheel, *Relative Morse indices, Fredholm indices, and group velocities,* Discr. Cont. Dynam. Syst. **20** (2008), 139–158.