Essential instability of pulses, and bifurcations to modulated travelling waves

Björn Sandstede
Department of Mathematics
The Ohio State University
231 West 18th Avenue
Columbus, OH 43210, USA

Arnd Scheel
Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin, Germany

Abstract

Reaction-diffusion systems on the real line are considered. Localized travelling waves become unstable when the essential spectrum of the linearization about them crosses the imaginary axis. In this article, it is shown that this transition to instability is accompanied by the bifurcation of a family of large patterns that are a superposition of the primary travelling wave with steady spatially-periodic patterns of small amplitude. The bifurcating patterns can be parametrized by the wavelength of the steady patterns; they are time-periodic in a moving frame. A major difficulty in analyzing this bifurcation is its genuinely infinite-dimensional nature. In particular, finite-dimensional Lyapunov-Schmidt reductions or center-manifold theory do not seem to be applicable to pulses having their essential spectrum touching the imaginary axis.

AMS subject classification: 35B32, 35K57

Keywords: pulses, homoclinic bifurcations, essential spectrum

1 Introduction

Travelling-wave solutions of parabolic equations on the real line arise in a variety of applications. An important issue is their stability since it is expected that only stable travelling waves can be observed. Once stability has been proved over a certain range of parameters, the dynamics near a travelling wave is predictable: any solution nearby is attracted to the travelling wave or an appropriate translate of it. In that respect, interesting parameter values are those at which a transition to instability occurs. Near such transitions, other and possibly more complicated patterns may bifurcate from the primary travelling wave. A travelling wave becomes unstable if a subset of the spectrum of the linearization about it crosses the imaginary axis. One possibility is that this subset consists of isolated eigenvalues. The resulting bifurcation problem can be analyzed using standard center-manifold theory. In this article, we focus on a qualitatively different mechanism that also leads to instability, namely that of essential spectrum crossing the imaginary axis. We call this instability mechanism an *essential instability*. This route to instability is considerably more difficult to analyze since it is genuinely infinite-dimensional. In fact, to our knowledge, this transition has not been investigated previously except for small fronts [2]. Another related article is [20] where essential instabilities of fronts are studied utilizing numerical simulations and the analysis of a series of caricature problems. The waves investigated therein are of a quite different nature and we refer to Section 4 for a discussion.

We distinguish between several kinds of travelling waves. Pulses are travelling waves that converge to the same asymptotic state as the spatial variable x tends to $\pm\infty$. Fronts, on the other hand, connect different asymptotic states at $\pm\infty$. Periodic wave trains are travelling waves that are periodic in the spatial variable. In the following, we focus on pulses. Most of the results presented in this article apply also to fronts and wave trains, and we discuss these generalizations in Section 4.

Essential instabilities of pulses are caused by an instability of the asymptotic equilibrium state of the pulse. Suppose that $h(x - c_0 t)$ is a pulse that moves with speed c_0 to the right. Its essential spectrum corresponds to small-amplitude waves of the form $\mathrm{e}^{\mathrm{i}kx+\lambda t}$ that are created at the asymptotic state of the pulse, that is, at its tails. The wavenumber k and the eigenvalue λ satisfy a certain dispersion relation. At the onset of instability induced by the essential spectrum, there exist then waves of the form $\mathrm{e}^{\mathrm{i}(kx+\omega t)}$. There are four qualitatively different cases corresponding to all possible combinations of ω and k being zero or non-zero. We focus here on the case of a stationary bifurcation, that is $\omega = 0$, for non-zero k.

When both the wavenumber k and the eigenvalue ω are zero, the situation is actually considerably simpler since it corresponds to a homoclinic bifurcation in an ordinary differential equation where the equilibrium undergoes a pitchfork bifurcation. On the other hand, the case of non-zero ω is similar to the case $\omega = 0$, and we refer to Section 4 for a discussion.

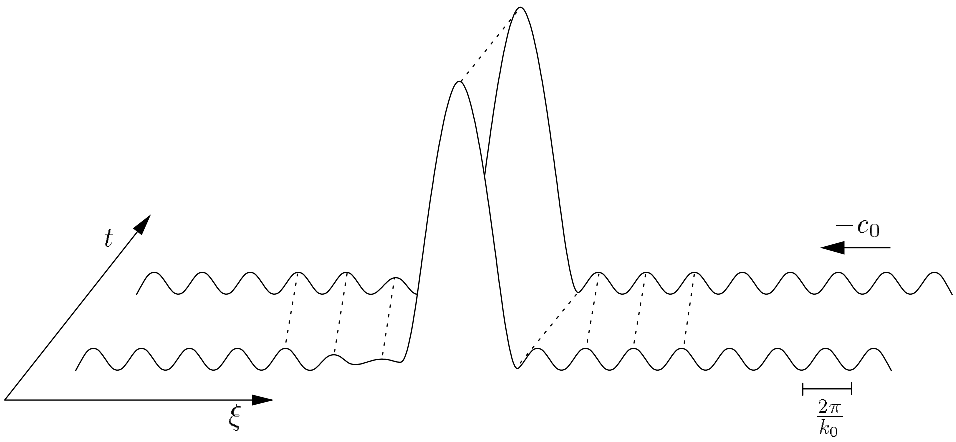

Figure 1: The superposition of the localized solitary wave h with small patterns that have a $\frac{2\pi}{k_0}$-periodic spatial structure is shown in a frame moving with speed $c_0 > 0$. The small-amplitude patterns move with speed $-c_0$ relative to the pulse h. The solution shown here has period $\frac{2\pi}{\omega_0}$ in time in the moving coordinate frame where $\omega_0 = c_0 k_0$.

From now on, we concentrate on situations where $\omega = 0$ and $k = k_0$ are non-zero, that is, we assume that the dispersion relation is satisfied by $\lambda_0 = 0$ and some $k_0 \neq 0$. The associated local bifurcation close to the equilibrium state is known as the Turing instability; see Sections 3.2 and 3.4. It generates small patterns of the form $\mathrm{e}^{\mathrm{i}k_0 x}$ that are often referred to as Turing patterns. These patterns possess a spatially oscillating structure with period $\frac{2\pi}{k_0}$. We seek time-periodic modulations of the pulse that are reminiscent of a linear superposition of these small steady patterns and the large localized pulse. In a coordinate system $\xi = x - c_0 t$ moving with the speed c_0 of the pulse, the steady patterns $\mathrm{e}^{\mathrm{i}k_0 x}$ become travelling waves $\mathrm{e}^{\mathrm{i}k_0(\xi + c_0 t)} =: \mathrm{e}^{\mathrm{i}(k_0\xi + \omega_0 t)}$ with $\omega_0 := c_0 k_0$. They move with speed $-c_0$ relative to the pulse h and have period $\frac{2\pi}{\omega_0}$ in time. In this moving frame, the modulated pulse looks roughly like

$$A\mathrm{e}^{\mathrm{i}(k_0\xi + \omega_0 t)} + h(\xi)$$

for small A; see Figure 1. We remark that Turing patterns bifurcate for any wavenumber k close to k_0. Correspondingly, we expect to find modulated pulses with asymptotic wavenumber k and temporal frequency $\omega = c_0 k$ for any k close to k_0. For the sake of clarity, we first seek modulated pulses with temporal frequency $\omega_0 = c_0 k_0$. Only at the end of the analysis, in Section 3.7, we show how modulated pulses with other temporal periods can be obtained.

To set the scene, consider

$$u_t = D u_{xx} + f(u, \mu), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $u \in \mathbb{R}^n$ and $f(0, \mu) = 0$ for all μ. Casting (1.1) in a frame $\xi = x - c_0 t$ moving with

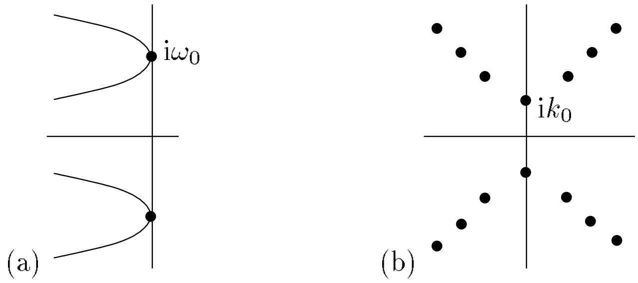

Figure 2: The essential spectrum of the parabolic reaction-diffusion system is shown in (a). Picture (b) shows the spectrum of the associated elliptic system.

speed c_0, we obtain

$$u_t = Du_{\xi\xi} + c_0 u_\xi + f(u, \mu), \qquad \xi \in \mathbb{R}, \tag{1.2}$$

which then has the equilibrium $h(\xi)$ for $\mu = 0$. We are particularly interested in localized waves satisfying $\lim_{|\xi|\to\infty} h(\xi) = 0$. The stability of h is determined by the spectrum $\mathrm{spec}(L)$ of the linearization

$$Lw := Dw_{\xi\xi} + c_0 w_\xi + \partial_u f(h(\xi), 0)w \tag{1.3}$$

of (1.2) about h. The essential spectrum of L is the complement in $\mathrm{spec}(L)$ of the set of isolated eigenvalues with finite multiplicity. It contains the spectrum of the linearization about the asymptotic state $u = 0$ that consists of all points λ in the complex plane such that

$$\det(-k^2 D + \mathrm{i}kc_0 + \partial_u f(0,0) - \lambda) = 0$$

for some $k \in \mathbb{R}$. This equation is the aforementioned dispersion relation in a coordinate frame moving with the pulse. We assume that the essential spectrum of the asymptotic state for $\mu = 0$ has the form depicted in Figure 2(a). With the transformation $\tilde{\lambda} = \lambda - \mathrm{i}kc_0$, we recover the dispersion relation

$$\det(-k^2 D + \partial_u f(0,0) - \tilde{\lambda}) = 0$$

in the original steady coordinate frame. Note that real solutions $\tilde{\lambda}$ of this dispersion relation are double zeroes corresponding to wavenumbers $\pm k$.

Since the critical eigenvalues $\lambda = \pm\mathrm{i}\omega_0$ are not isolated in the spectrum, it is difficult to reduce the dimension by applying Lyapunov-Schmidt reduction or center-manifold theory. Often, modulation equations have been used to describe the dynamics near homogeneous steady states. In the aforementioned context of bifurcations from equilibria, a partial justification of the approximation by modulation equations, that is Ginzburg-Landau equations, has been achieved in [7]; see also [3, 12] and the references therein. Modulation equations close to a pulse would have to capture both the dynamics close to the asymptotic equilibrium state of the pulse, typically described by a Ginzburg-Landau equation, and the global interaction of the modulation through the pulse. An attempt to derive such a modulation equation, at least formally, has been made in [1]. However, the resulting Ginzburg-Landau equation is still difficult to analyze.

We therefore resort to a completely different approach and cast the parabolic equation (1.2) as an elliptic equation on the space of $\frac{2\pi}{\omega_0}$-periodic functions, namely

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ D^{-1}(u_t - c_0 v - f(u,\mu)) \end{pmatrix}, \qquad t \in S^1 = \mathbb{R}/\tfrac{2\pi}{\omega_0}\mathbb{Z}. \tag{1.4}$$

In other words, we anticipate the temporal period $\frac{2\pi}{\omega_0}$ and then reverse the role of time and space by viewing (1.4) as an evolution equation in ξ. The restriction to $\frac{2\pi}{\omega_0}$-periodic functions is very efficient since most of the essential spectrum disappears for the initial-value problem (1.4). In fact, the spectrum of the linearization of (1.4) about $(u,v) = 0$ has a pair of isolated imaginary eigenvalues $\pm \mathrm{i}k_0$; see Figure 2(b). Thus, we expect a Hopf bifurcation leading to spatially periodic solutions with small amplitude. Here, we recover precisely the Turing patterns of the form $\mathrm{e}^{\mathrm{i}k_0 x} = \mathrm{e}^{i(k_0\xi + \omega_0 t)}$. On the other hand, the travelling wave $h(\xi)$ corresponds to a time-independent homoclinic solution $(h, h_\xi)(\xi)$ of (1.4). We then seek solutions close to $(h, h_\xi)(\xi)$ that are homoclinic to the aforementioned small periodic waves. If (1.4) were an ODE, we could readily investigate the existence of such connections by studying intersections of suitable global invariant manifolds associated with the periodic waves. However, the initial-value problem for (1.4) is ill-posed. Indeed, Figure 2(b) indicates that the stable and unstable eigenspaces are both infinite-dimensional, and semigroup theory fails. Therefore, it is not clear whether global invariant manifolds exist or whether dynamical-systems techniques can be used at all to investigate elliptic equations such as (1.4).

In this article, we construct global stable and unstable manifolds near the given pulse (h, h_ξ) and study their intersections upon changing the parameter μ. We use exponential dichotomies for elliptic equations to accomplish this construction. Exponential dichotomies are a well-known technique for ODEs and parabolic PDEs; see, for instance, [4, 6, 13]. For elliptic equations, however, there are major technical obstacles to their global existence that have been resolved only recently in [14].

The idea of using spatial dynamics has been introduced in [9]. Since then it has been used extensively in order to investigate bifurcations from spatially homogeneous equilibria to small steady-state or time-periodic solutions; see, for instance, [2, 7, 15]. Typically, the resulting elliptic system is reduced to a finite-dimensional equation that describes small solutions near the homogeneous steady-state. The reduced equation can then be investigated using bifurcation theory. The problem analyzed in the present article, however, involves a large pulse solution that is not close to the equilibrium state. A finite-dimensional reduction to a center manifold for the spatial dynamics is not known in this context.

This paper is organized as follows. In the next section, we present a four-dimensional model problem. The model reflects the essential features of the part of the bifurcation we are interested in, though we believe that it is inadequate for a complete description. The analysis of the infinite-dimensional problem, including all necessary hypotheses, is then carried out in Section 3. The main result, the bifurcation of modulated travelling waves asymptotic to spatially-periodic steady patterns, is stated in Section 3.7. We conclude in Section 4 with a discussion and generalizations of the result.

2 A finite-dimensional model problem

In this section, we outline the bifurcation that occurs in the elliptic problem (1.4) when the essential spectrum of (1.3) crosses the imaginary axis. For the sake of clarity, we utilize a four-dimensional model that mimics precisely the bifurcation we are interested in. Let $(u_0, u_1) \in \mathbb{R}^2 \times \mathbb{R}^2$ satisfy the differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} u_0 &= f_0(u_0, c, \mu) \\ \frac{\mathrm{d}}{\mathrm{d}\xi} u_1 &= f_1(u_0, u_1, c, \mu). \end{aligned} \tag{2.1}$$

The reader may think of $(u_0, u_1)(x)$ as the zeroth and first Fourier coefficients of the t-periodic function $(u, u_\xi)(t, \xi)$ defined in (1.4); see also equation (3.4) below.

We assume that the subspace $u_1 = 0$ is invariant for all values of the parameters, that is $f_1(u_0, 0, c, \mu) = 0$. The dynamics in the subspace $u_1 = 0$ is then governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u_0 = f_0(u_0, c, \mu). \tag{2.2}$$

Suppose that (2.2) has a homoclinic solution to the hyperbolic equilibrium $u_0 = 0$ for $(c, \mu) = (c_0, 0)$. This homoclinic orbit corresponds to the pulse solution h of equation (1.2). We assume that the homoclinic orbit of (2.2) is transversely unfolded by the parameter c, that is, stable and unstable manifold of the origin cross each other with non-zero speed upon varying c near $c = c_0$. As for the second equation in (2.1), we assume an S^1-equivariance with respect to the rotations in $\mathbb{R}^2$. This symmetry represents the time shift of non-zero solutions of (1.2). Finally, upon varying the parameter μ, suppose that the equilibrium $(u_0, u_1) = 0$ undergoes a non-degenerate Hopf bifurcation in $\mathbb{R}^4$ with critical eigenspace $\{(u_0, u_1);\ u_0 = 0\}$.

Factoring out the S^1-symmetry in the u_1-variable, we are left with a three-dimensional ODE having a homoclinic orbit to an equilibrium in a two-dimensional flow-invariant subspace. Moreover, the equilibrium experiences a pitchfork bifurcation in the direction transverse to this subspace; see Figure 3. Upon changing c, unstable and center-stable manifold of the origin cross each other with non-vanishing speed in the three-dimensional space. If the pitchfork bifurcation is supercritical, the bifurcating equilibrium has a one-dimensional

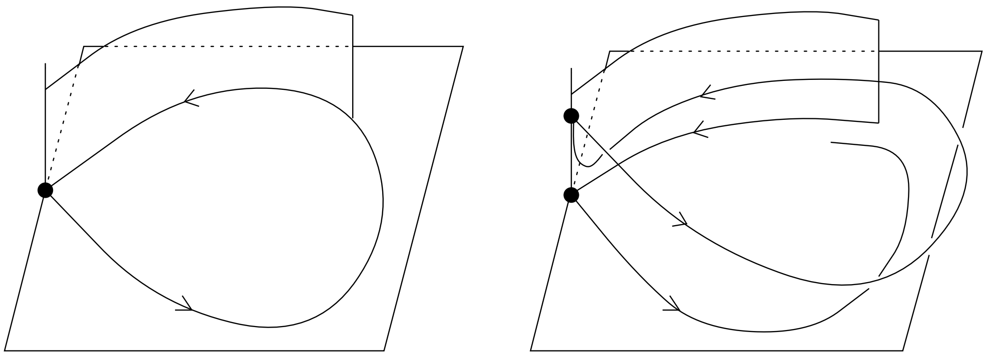

Figure 3: The picture on the left shows the homoclinic orbit (h, h_ξ) of the elliptic system at the bifurcation point. The vertical axis corresponds to the center direction in which a supercritical Hopf bifurcation takes place. The two horizontal directions coincide with the invariant subspace $u_1 = 0$. The picture on the right shows the homoclinic solution connecting the bifurcating periodic solution to itself.

unstable manifold, which is close to the (strong) unstable manifold of the origin. Therefore, upon changing c, the unstable manifold of the bifurcating equilibrium also crosses the center-stable manifold of the origin due to the persistence of transverse crossings under perturbations. The unique intersection curve corresponds to a homoclinic orbit to the bifurcating equilibrium since the origin is unstable within the center manifold; see Figure 3.

The main difference between the elliptic problem (1.4) and our model problem is that the phase space for the former equation is infinite-dimensional, and both the unstable and the center-stable manifold of the origin are infinite-dimensional. Even the existence of these manifolds far away from the equilibrium is not evident as we do not have a flow to propagate local invariant manifolds.

3 Bifurcations of time-periodic travelling waves

3.1 The parabolic and elliptic equation

We consider the semilinear parabolic equation

$$u_t = Du_{xx} + f(u, \mu), \qquad x \in \mathbb{R}, \tag{3.1}$$

where $u \in \mathbb{R}^n$, D is a diagonal matrix with positive entries, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth nonlinearity with $f(0, \mu) = 0$ for all μ.

Hypothesis (TW) *Assume that $h(x - c_0 t)$ is a travelling-wave solution of (3.1) for $\mu = 0$ and some $c_0 \neq 0$ such that $h(\xi)$ tends to zero exponentially as $|\xi|$ tends to infinity.*

Transforming (3.1) into the moving frame $(\xi, t) = (x - ct, t)$, we obtain

$$u_t = Du_{\xi\xi} + cu_\xi + f(u, \mu), \qquad \xi \in \mathbb{R}, \tag{3.2}$$

which then admits the equilibrium $h(\xi)$ for $(c, \mu) = (c_0, 0)$. Equation (3.2) is well-posed on the space $X := C^0_{\mathrm{unif}}(\mathbb{R}, \mathbb{R}^n)$ of bounded and uniformly continuous functions on $\mathbb{R}$; see, for instance, [6]. Here, we consider strong solutions $u(t)$ of (3.2) that are differentiable as functions into X, continuous with values in C^2_{unif} and satisfy (3.2) in X.

Next, we cast the parabolic equation (3.2) as an elliptic equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ D^{-1}(u_t - cv - f(u, \mu)) \end{pmatrix}, \tag{3.3}$$

reversing the role of time and space. The functions $U = (u, v)$ are contained in $Y := H^{\frac{1}{2}}_{\mathrm{per}}(0, \frac{2\pi}{\omega_0}) \times L^2_{\mathrm{per}}(0, \frac{2\pi}{\omega_0})$ for some $\omega_0 > 0$ which we specify below. The nonlinearity f maps $H^{\frac{1}{2}}_{\mathrm{per}}$ into L^2_{per} provided it has at most polynomial growth. If f has faster growth, we may consider (3.3) on the space $H^1_{\mathrm{per}} \times H^{\frac{1}{2}}_{\mathrm{per}}$. There are then no restrictions on f necessary and the analysis presented below is still valid. We say that $(u, v)(\xi)$ is a solution of (3.3) if $(u, v)(\xi)$ is differentiable in ξ as a function into Y, continuous with values in $H^1_{\mathrm{per}} \times H^{\frac{1}{2}}_{\mathrm{per}}$ and satisfies (3.3) in Y. We emphasize that the initial-value problem for (3.3) is not well-posed on Y.

On the space Y, we have the S^1-action

$$(\rho_\alpha U)(t) := U(t + \alpha)$$

with $\alpha \in \mathbb{R}/\frac{2\pi}{\omega_0}\mathbb{Z}$. Note that $(h(\xi), h_\xi(\xi))$ satisfies (3.3) for $(c, \mu) = (c_0, 0)$. We may think of this solution, which is contained in the fixed-point space $\mathrm{Fix}(S^1)$ of the S^1-action, as a homoclinic orbit to the zero equilibrium.

Throughout, we utilize the Fourier series of elements $(u, v) \in Y$ and identify (u, v) with its Fourier coefficients $(u_\ell, v_\ell)_{\ell \in \mathbb{Z}}$ where

$$(u(t), v(t)) = \Big(\sum_{\ell \in \mathbb{Z}} u_\ell \mathrm{e}^{\mathrm{i}\ell\omega_0 t}, \sum_{\ell \in \mathbb{Z}} v_\ell \mathrm{e}^{\mathrm{i}\ell\omega_0 t} \Big). \tag{3.4}$$

Note that

$$|(u, v)|^2_Y = |u|^2_{H^{\frac{1}{2}}} + |v|^2_{L^2} = \sum_{\ell \in \mathbb{Z}} \Big((1 + |\ell|)|u_\ell|^2 + |v_\ell|^2 \Big) =: \sum_{\ell \in \mathbb{Z}} |(u_\ell, v_\ell)|^2_\ell. \tag{3.5}$$

Let

$$Y_\ell = \mathrm{span}_{u_\ell, v_\ell, u_{-\ell}, v_{-\ell} \in \mathbb{R}^n} \{ (u_\ell, v_\ell)\mathrm{e}^{\mathrm{i}\ell\omega_0 t}, (u_{-\ell}, v_{-\ell})\mathrm{e}^{-\mathrm{i}\ell\omega_0 t} \}$$

equipped with the norm $|\cdot|_\ell$.

3.2 The linearization about $u = 0$

Setting $(c, \mu) = (c_0, 0)$, we linearize (3.2) about $u = 0$ and obtain the linear constant-coefficient operator

$$L_\infty w = D w_{\xi\xi} + c_0 w_\xi + \partial_u f(0,0) w.$$

First, we calculate the spectrum of L_∞ on X. Define

$$d(\lambda, \nu) := \det(\nu^2 D + \nu c_0 + \partial_u f(0,0) - \lambda). \tag{3.6}$$

Owing to [6, Theorem A.2], we have

$$\operatorname{spec}(L_\infty) = \{\lambda \in \mathbb{C};\ d(\lambda, \mathrm{i}k) = 0 \text{ for some } k \in \mathbb{R}\}, \tag{3.7}$$

since $w(\xi) = \mathrm{e}^{\mathrm{i}k\xi} w_0$ is then a bounded eigenfunction associated with the eigenvalue λ for some non-zero $w_0 \in \mathbb{C}^n$.

Hypothesis (P1) *Assume that* $\operatorname{spec}(L_\infty) \cap \mathrm{i}\mathbb{R} = \{\pm \mathrm{i}\omega_0\}$ *for some* $\omega_0 > 0$. *Furthermore, we assume that* $d(\lambda, \mathrm{i}k) = 0$ *for* λ *close to* $\mathrm{i}\omega_0$ *if, and only if,* k *is close to* $k_0 = \frac{\omega_0}{c_0}$ *and*

$$\lambda = \lambda_*(k) = \mathrm{i}\omega_0 + \mathrm{i}c_0(k - k_0) - C_\mathrm{r}(k - k_0)^2 + \mathrm{O}(|k - k_0|^3), \tag{3.8}$$

where $C_\mathrm{r} > 0$ *is real and* $c_0 \neq 0$ *denotes the wave speed. Finally, we assume that* $\partial_\nu d(\lambda, \nu)|_{(\mathrm{i}\omega_0, \mathrm{i}k_0)} \neq 0$.

Hypothesis (P1) states that the essential spectrum of L_∞ touches the imaginary axis at $\lambda = \pm \mathrm{i}\omega_0$. The corresponding eigenfunction $\mathrm{e}^{\mathrm{i}k_0\xi} w_\mathrm{H}$ is unique, up to constant multiples, and has a non-trivial spatial structure since $k_0 \neq 0$.

Note that the particular form of the dispersion relation (3.8) follows from a generic assumption on the bifurcation in the steady coordinate frame. Indeed, consider the operator

$$L^0_\infty w := D w_{\xi\xi} + \partial_u f(0,0) w. \tag{3.9}$$

Its dispersion relation for $\nu = \mathrm{i}k$ is

$$\det(-k^2 D + \partial_u f(0,0) - \lambda) = 0. \tag{3.10}$$

Eigenvalues λ^0_* of L^0_∞ transform into eigenvalues λ_* of L_∞ via $\lambda_*(k) = \lambda^0_*(k) + \mathrm{i}kc_0$. In Hypothesis (P1), we have assumed that only $k = \pm k_0$ satisfy (3.10) for $\lambda = 0$. In addition, we assumed in (P1) that the derivative of (3.10) with respect to k evaluated at $(\lambda, k) = (0, k_0)$ is not zero. Hence, there are unique solutions $\lambda^0_\pm(k)$ satisfying (3.10) for k near $\pm k_0$ with $\lambda^0_\pm(\pm k_0) = 0$. Note that (3.10) is symmetric with respect to $k \to -k$. Therefore, we conclude that $\lambda^0_\pm(k)$ are both real-valued. Summarizing, Hypothesis (P1) is satisfied by an open set of one-parameter families. Many reaction-diffusion systems that satisfy (P1) are known. One example is the Brusselator; see [5, Ch.VII, §5] or, for the first reference to Turing instabilities, [21].

Next, we compute the spectrum of the linearization

$$A_\infty = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t - \partial_u f(0,0)) & -c_0 D^{-1} \end{pmatrix}$$

of (3.3) at the equilibrium $U = 0$ considered in the space Y with ω_0 chosen as in (P1). We remark that we may consider the space Y for any frequency ω close to ω_0; see Section 3.7.

Lemma 3.1 *Suppose that (P1) is met. The operator A_∞ has then two simple eigenvalues $\pm \mathrm{i}k_0$ on the imaginary axis with eigenfunctions $\mathrm{e}^{\mathrm{i}\omega_0 t}U_\mathrm{H}$ and $\mathrm{e}^{-\mathrm{i}\omega_0 t}\overline{U_\mathrm{H}}$, respectively, for some non-zero $U_\mathrm{H} \in \mathbb{C}^{2n}$, while the rest of its spectrum is uniformly bounded away from the imaginary axis. The operator A_∞ has compact resolvent. Furthermore, there are constants $\delta \neq 0$ small and $K > 0$ such that*

$$\|(A_\infty + (\delta - \mathrm{i}k)\,\mathrm{id})^{-1}\|_{L(Y)} \leq \frac{K}{1+|k|}$$

for all $k \in \mathbb{R}$. Finally, there exist spectral projections P^u, P^c and P^s in $L(Y)$ corresponding to eigenvalues of A_∞ with positive, zero and negative real part, respectively.

Proof. Let $V = (V_1, V_2) \in Y$. We have $A_\infty V = \nu V$ if, and only if, $V_2 = \nu V_1$, and

$$(\nu^2 D + \nu c_0 + \partial_u f(0,0) - \partial_t)V_1 = 0.$$

Upon exploiting the Fourier series (3.4) of V with coefficients (a_ℓ, b_ℓ), we see that $\nu \in \mathrm{spec}(A_\infty)$ if, and only if,

$$\det(\nu^2 D + \nu c_0 + \partial_u f(0,0) - \mathrm{i}\ell\omega_0) = d(\mathrm{i}\ell\omega_0, \nu) = 0$$

for some $\ell \in \mathbb{Z}$. It follows from (P1) and (3.7) that $\nu = \pm \mathrm{i}k_0$ are the only eigenvalues of A_∞ on the imaginary axis. These eigenvalues are simple since the algebraic multiplicity of $\mathrm{i}k_0$ coincides with the order of $\mathrm{i}k_0$ as a zero of the determinant $d(\mathrm{i}\omega_0, \nu)$ with respect to ν. By Hypothesis (P1), this order is equal to one.

In particular, A_∞ is invertible on Y. It is clear that the inverse is compact since the domain $H^1_\mathrm{per} \times H^{\frac{1}{2}}_\mathrm{per}$ of A_∞ is compactly embedded into Y.

Next, we consider the eigenvalue problem for A_∞. Note that the Fourier subspaces Y_ℓ are invariant under A_∞. The associated eigenvalue problem for the Fourier coefficients (a_ℓ, b_ℓ) is given by

$$\begin{pmatrix} -\nu & \mathrm{id} \\ D^{-1}(\mathrm{i}\ell\omega_0 - \partial_u f(0,0)) & -\nu - c_0 D^{-1} \end{pmatrix} \begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix} = 0. \qquad (3.11)$$

In order to prove the remaining claims on the resolvent and spectral splittings, it suffices to investigate (3.11) for $\ell \in \mathbb{Z}$ with $|\ell|$ large. We then scale

$$a_\ell = \frac{1}{\sqrt{|\ell|}}\hat{a}_\ell, \qquad b_\ell = \hat{b}_\ell. \tag{3.12}$$

This rescaling accounts for the norm on Y_ℓ; see (3.5). In particular,

$$|(a_\ell, b_\ell)|_\ell^2 = |\ell|\,|a_\ell|^2 + |b_\ell|^2 = |\hat{a}_\ell|^2 + |\hat{b}_\ell|^2 =: |(\hat{a}_\ell, \hat{b}_\ell)|^2.$$

We also rescale the eigenvalue $\nu = \sqrt{|\ell|}\hat{\nu}$. The eigenvalue problem then reads

$$\begin{pmatrix} -\hat{\nu} & \mathrm{id} \\ D^{-1}(\mathrm{i}\omega_0 \operatorname{sign}\ell - \frac{1}{|\ell|}\partial_u f(0,0)) & -\hat{\nu} - \frac{1}{\sqrt{|\ell|}}c_0 D^{-1} \end{pmatrix} \begin{pmatrix} \hat{a}_\ell \\ \hat{b}_\ell \end{pmatrix} = 0, \tag{3.13}$$

which has a non-trivial solution if, and only if,

$$\det\left(\hat{\nu}^2 D + \frac{\hat{\nu} c_0}{\sqrt{|\ell|}} + \frac{1}{|\ell|}\partial_u f(0,0) - \mathrm{i}\omega_0 \operatorname{sign}\ell\right) = 0. \tag{3.14}$$

Taking the limit $|\ell| \to \infty$ gives

$$\begin{pmatrix} -\hat{\nu} & \mathrm{id} \\ D^{-1}\mathrm{i}\omega_0 \operatorname{sign}\ell & -\hat{\nu} \end{pmatrix} \begin{pmatrix} \hat{a}_\ell \\ \hat{b}_\ell \end{pmatrix} = 0 \tag{3.15}$$

and

$$\det(\hat{\nu}^2 D - \mathrm{i}\omega_0 \operatorname{sign}\ell) = 0,$$

respectively. The last equation has $2n$ solutions $\hat{\nu}_j$ which are not imaginary and independent of ℓ. By Rouche's Theorem, there are then $2n$ zeroes of (3.14) near the set $\{\hat{\nu}_j\}$. The rescaling $\nu = \sqrt{|\ell|}\hat{\nu}$ shows that the real parts of the corresponding eigenvalues are actually unbounded as $|\ell| \to \infty$. Similarly, the spectral projections associated with the limiting problem (3.15) perturb to spectral projections of (3.13) in Y_ℓ that are bounded uniformly in ℓ. Due to the definition of the norms on Y and the rescaling (3.12), the lemma is proved. ∎

3.3 The linearization about the travelling wave

We consider the linearizations of (3.2) and (3.3) about the pulse $h(\xi)$ for $(c, \mu) = (c_0, 0)$. For the parabolic equation, define

$$Lw = Dw_{\xi\xi} + c_0 w_\xi + \partial_u f(h(\xi), 0)w$$

for $w \in X$. The variational equation about the homoclinic solution $(h, h_\xi)(\xi)$ of the elliptic equation (3.3) is given by

$$V_\xi = A(\xi)V = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t - \partial_u f(h(\xi), 0)) & -c_0 D^{-1} \end{pmatrix} V \tag{3.16}$$

with $V \in Y$. Note that the Fourier subspaces Y_ℓ are invariant under $A(\xi)$ since $h(\xi)$ does not depend on t. In Y_ℓ, equation (3.16) reads

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\mathrm{i}\ell\omega_0 - \partial_u f(h(\xi),0)) & -c_0 D^{-1} \end{pmatrix} \begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix}. \tag{3.17}$$

The next lemma characterizes the set of bounded solutions of (3.16).

Lemma 3.2 *Assume that Hypothesis (P1) is met. We then have* $\lambda = \mathrm{i}\ell\omega_0 \in \operatorname{spec}(L)$ *for some* $\ell \in \mathbb{Z}$ *if, and only if, there exists a bounded solution* $V(\xi,t) = \mathrm{e}^{\mathrm{i}\ell\omega_0 t}V_0(\xi)$ *of (3.16) defined for* $\xi \in \mathbb{R}$.

Proof. If $V(\xi,t) = \mathrm{e}^{\mathrm{i}\ell\omega_0 t}V_0(\xi)$ is a bounded solution of (3.16) on $\mathbb{R}$, then $V_0(\xi) = (w, w_\xi)(\xi)$, and $w(\xi)$ lies in the null space of $L - i\ell\omega_0$:

$$Dw_{\xi\xi} + cw_\xi + \partial_u f(h(\xi),0)w = \mathrm{i}\ell\omega_0\, w. \tag{3.18}$$

Moreover, $w \in X$. Therefore, $\mathrm{i}\ell\omega_0 \in \operatorname{spec}(L)$.

Next, suppose that $\lambda = \mathrm{i}\ell\omega_0 \in \operatorname{spec}(L)$. If $|\ell| \neq 1$, then λ is not contained in the essential spectrum by (P1). Hence, the eigenfunction associated with $\mathrm{i}\ell\omega_0$ is localized, and therefore corresponds to a bounded solution of (3.18).

It remains to consider the case $\lambda = \pm\mathrm{i}\omega_0$. We seek bounded solutions of (3.17) with $\ell = \pm 1$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\mathrm{i}\omega_0 - \partial_u f(h(\xi),0)) & -c_0 D^{-1} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}. \tag{3.19}$$

Due to Hypothesis (P1), we have $\lambda \in \operatorname{spec}(L_\infty)$. Moreover, Lemma 3.1 shows that the spectrum of the asymptotic matrix

$$A_1 := \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\pm\mathrm{i}\omega_0 - \partial_u f(0,0)) & -c_0 D^{-1} \end{pmatrix}$$

defined on Y_1 has two simple imaginary eigenvalues $\pm ik_0$, while the other eigenvalues have non-zero real part. Since the function $h(\xi)$ converges to zero exponentially as $|\xi| \to \infty$, we can now apply ODE results on exponential dichotomies [4, 13]. Hence, there are two subspaces $E_1^{\mathrm{cs}}(0)$ and $E_1^{\mathrm{cu}}(0)$ of Y_1 such that solutions of (3.19) with initial values in $E_1^{\mathrm{cs}}(0)$ or $E_1^{\mathrm{cu}}(0)$ are bounded for $\xi \to \infty$ or $\xi \to -\infty$, respectively. Furthermore,

$$\dim E_1^{\mathrm{cs}}(0) = \#\{\nu \in \operatorname{spec}(A_1);\ \operatorname{Re}\nu \leq 0\}, \quad \dim E_1^{\mathrm{cu}}(0) = \#\{\nu \in \operatorname{spec}(A_1);\ \operatorname{Re}\nu \geq 0\},$$

counted with multiplicity; see [4]. In particular, $\dim E_1^{\mathrm{cs}}(0) + \dim E_1^{\mathrm{cu}}(0) = \dim Y_1 + 2$. Therefore, any solution of (3.19) with initial value in $E_1^{\mathrm{c}}(0) := E_1^{\mathrm{cs}}(0) \cap E_1^{\mathrm{cu}}(0)$ is bounded on $\mathbb{R}$. Moreover, $\dim E_1^{\mathrm{c}}(0) \geq 2$, and therefore $E_1^{\mathrm{c}}(0)$ contains non-trivial initial values. ∎

Note that the lemma would be wrong if the limiting matrix A_1 were containing a non-trivial Jordan block corresponding to the eigenvalue $\lambda = \mathrm{i}\omega_0$. In this situation, even though $\lambda \in \operatorname{spec}(L)$, there would in general be no bounded solution of (3.16) since solutions are expected to grow linearly in ξ.

Actually, we have proved much more. Using the notation introduced in the proof above, the set $E_1^{\mathrm{c}}(0)$ of bounded solutions of (3.17) with $|\ell| = 1$ is at least two-dimensional. If we modify the nonlinearity $f(u, 0)$ by adding a small rotation normal to the homoclinic orbit h, we can arrange that $E_1^{\mathrm{c}}(0)$ is two-dimensional. Furthermore, by the same argument, solutions associated with initial values in $E_1^{\mathrm{c}}(0)$ do generically not decay exponentially as $|\xi| \to \infty$ but oscillate. In other words, generically in $f(\cdot, 0)$, we have $E_1^{\mathrm{c}}(0) \cap E_1^{\mathrm{s}}(0) = \{0\}$ and $E_1^{\mathrm{c}}(0) \cap E_1^{\mathrm{u}}(0) = \{0\}$ where $E_1^{\mathrm{s}}(0)$ and $E_1^{\mathrm{u}}(0)$ are subspaces of Y_1 such that solutions of (3.19) with initial values in $E_1^{\mathrm{s}}(0)$ or $E_1^{\mathrm{u}}(0)$ decay exponentially for $\xi \to \infty$ or $\xi \to -\infty$, respectively.

Using similar arguments, $\lambda = \pm \mathrm{i}\ell\omega_0$ is generically not in the spectrum of L for $|\ell| > 1$. Note that $\lambda = 0 \in \operatorname{spec}(L)$ with eigenfunction h_ξ by translation invariance. This eigenvalue is typically simple. For generic nonlinearities $f(u, 0)$, the following hypothesis is therefore met.

Hypothesis (P2)

(i) $\lambda = 0 \in \operatorname{spec}(L)$ *is a simple eigenvalue.*

(ii) $(L - \mathrm{i}\omega_0)w = 0$ *has a unique, up to constant complex multiples, non-zero bounded solution* $w^{\mathrm{c}}(\xi)$, *and we have* $|w^{\mathrm{c}}(\xi) - \mathrm{e}^{\mathrm{i}k_0\xi} w_{\mathrm{H}}^{\pm}| \to 0$ *as* $\xi \to \pm\infty$ *for appropriate non-zero vectors* $w_{\mathrm{H}}^{\pm} \in \mathbb{C}^n$.

(iii) $\lambda = \pm \mathrm{i}\ell\omega_0$ *is not in* $\operatorname{spec}(L)$ *for* $\ell > 1$.

On account of Hypothesis (P2) and Lemma 3.2, the subspace of initial values in Y associated with bounded solutions of (3.16) is given by

$$E^{\mathrm{c}}(0) = \operatorname{span}\{(h, h_\xi)_\xi(0), w^{\mathrm{c}}(0)\mathrm{e}^{\mathrm{i}\omega_0 t}, \overline{w^{\mathrm{c}}(0)}\mathrm{e}^{-\mathrm{i}\omega_0 t}\}. \tag{3.20}$$

Our next goal is to solve (3.3) using the information gathered so far. Unfortunately, the initial-value problem for (3.3) is not well-posed on Y. Under certain circumstances, however, (3.3) can be solved in forward or backward ξ-direction for initial values in certain ξ-depending subspaces of Y. We say that (3.3) has an exponential dichotomy on $\mathbb{R}^+$ if there are projections $P_+(\xi)$ defined for $\xi \geq 0$ with the following property: for any $V_0 \in \mathrm{R}(P_+(0))$, there exists a unique solution $V(\xi)$ of (3.3) which is defined for $\xi \geq 0$ such that $V(0) = V_0$. Moreover, $V(\xi)$ tends to zero exponentially as $\xi \to \infty$, and $V(\xi) \in \mathrm{R}(P_+(\xi))$ for all $\xi \geq 0$. Similarly, for any V_0 in the null space of $P_+(\xi_0)$, there is a unique solution $V(\xi)$ of (3.3) which is defined for $0 \leq \xi \leq \xi_0$ such that $V(\xi_0) = V_0$; furthermore, $V(\xi)$ decays exponentially for decreasing ξ with $0 \leq \xi \leq \xi_0$. In other words, for $\xi \geq 0$, there are two complementary subspaces, $\mathrm{R}(P_+(\xi))$ and $\mathrm{R}(\mathrm{id} - P_+(\xi))$, such that we can solve the elliptic equation forward and backward in ξ for initial values in $\mathrm{R}(P_+(\xi))$ and $\mathrm{R}(\mathrm{id} - P_+(\xi))$, respectively. Exponential dichotomies on $\mathbb{R}^-$ are defined analogously; solutions in $\mathrm{R}(P_-(0))$ decay exponentially as $\xi \to -\infty$.

In the following lemma, we show that equation (3.3) has dichotomies so that we can solve it forward and backward in ξ provided the initial values are contained in appropriate subspaces. The only difference to the situation described right above is that solutions do not necessarily decay.

Lemma 3.3 *Assume that Hypothesis (P1) is met. There are bounded operators $\Phi^{\mathrm{s}}_+(\xi,\eta)$, $\Phi^{\mathrm{c}}_+(\xi,\eta)$ and $\Phi^{\mathrm{u}}_+(\xi,\eta)$ defined on Y for $0 \leq \eta \leq \xi$, $0 \leq \eta, \xi$ and $0 \leq \xi \leq \eta$, respectively, such that $\Phi^{\mathrm{s}}_+(\xi,\eta)V_0$, $\Phi^{\mathrm{c}}_+(\xi,\eta)V_0$ and $\Phi^{\mathrm{u}}_+(\xi,\eta)V_0$ satisfy (3.16) for $\xi > \eta$, any ξ and $\xi < \eta$, respectively, and are continuous in (ξ,η) for any $V_0 \in Y$. Furthermore, Φ^{s}_+ satisfies the evolution property $\Phi^{\mathrm{s}}_+(\xi,\eta)\Phi^{\mathrm{s}}_+(\eta,\zeta) = \Phi^{\mathrm{s}}_+(\xi,\zeta)$ for any $0 \leq \zeta \leq \eta \leq \xi$. Analogous properties hold for $\Phi^{\mathrm{c}}_+(\xi,\eta)$ and $\Phi^{\mathrm{u}}_+(\xi,\eta)$. Moreover,*

$$\Phi^{\mathrm{s}}_+(\xi,\xi) + \Phi^{\mathrm{c}}_+(\xi,\xi) + \Phi^{\mathrm{u}}_+(\xi,\xi) = \mathrm{id}, \quad \Phi^{i}_+(\xi,\xi)\Phi^{j}_+(\xi,\xi) = 0 \ \ \textit{for } i \neq j,$$

where $i, j \in \{\mathrm{s}, \mathrm{c}, \mathrm{u}\}$. Finally, there are constants $K > 0$ and $\kappa > 0$ such that

$$\|\Phi^{\mathrm{s}}_+(\xi,\eta)\|_{L(Y)} \leq \mathrm{e}^{-\kappa(\xi-\eta)}, \quad \|\Phi^{\mathrm{c}}_+(\xi,\eta)\|_{L(Y)} \leq K, \quad \|\Phi^{\mathrm{u}}_+(\eta,\xi)\|_{L(Y)} \leq K\mathrm{e}^{-\kappa(\xi-\eta)} \tag{3.21}$$

for any $0 \leq \eta \leq \xi$. Similar properties hold for operators $\Phi^{\mathrm{s}}_-(\xi,\eta)$, $\Phi^{\mathrm{c}}_-(\xi,\eta)$ and $\Phi^{\mathrm{u}}_-(\xi,\eta)$ defined for negative ξ and η.

Proof. The statement of the lemma follows from [14, Theorem 1]. We give another simpler proof that works for the particular case studied here. As mentioned above, the Fourier subspaces Y_ℓ are invariant under $A(\xi)$ since $h(\xi)$ does not depend on t. The Fourier coefficients (a_ℓ, b_ℓ) satisfy equation (3.17)

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\mathrm{i}\ell\omega_0 - \partial_u f(h(\xi),0)) & -c_0 D^{-1} \end{pmatrix} \begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix}.$$

We can readily solve this equation for any $\ell \in \mathbb{Z}$. Lemma 3.1 shows that the spectrum of the asymptotic operator $A_\infty|_{Y_\ell}$ is strictly hyperbolic except when $|\ell| = 1$ where it contains two simple imaginary eigenvalues. The case $|\ell| = 1$ has been discussed in Lemma 3.2. Hence, we conclude the existence of evolution operators $\Phi^{\mathrm{s}}_{+,\ell}$, $\Phi^{\mathrm{c}}_{+,\ell}$ and $\Phi^{\mathrm{u}}_{+,\ell}$ in each subspace Y_ℓ. In fact, $\Phi^{\mathrm{c}}_{+,\ell} = 0$ except when $|\ell| = 1$. Furthermore, the estimates (3.21) are true in Y_ℓ for some κ independent of ℓ due to Lemma 3.1.

It is, however, not clear whether the constant K is independent of ℓ and whether the resulting evolution operators are bounded on Y. To prove this, it suffices to estimate the norm of the evolution operators on the space Y_ℓ for large ℓ. Thus, let $|\ell|>1$. Using the scaling (3.12), that is, $a_\ell = \frac{1}{\sqrt{|\ell|}}\hat{a}_\ell$ and $b_\ell = \hat{b}_\ell$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix}\hat{a}_\ell\\ \hat{b}_\ell\end{pmatrix} = \sqrt{|\ell|}\begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\mathrm{i}\omega_0 \,\mathrm{sign}\,\ell - \frac{1}{|\ell|}\partial_u f(h(\xi),0)) & -\frac{1}{\sqrt{|\ell|}}c_0 D^{-1}\end{pmatrix}\begin{pmatrix}\hat{a}_\ell\\ \hat{b}_\ell\end{pmatrix}.$$

Rescaling the ξ-variable by $\sqrt{|\ell|}\xi = \hat{\xi}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\hat{\xi}}\begin{pmatrix}\hat{a}_\ell\\ \hat{b}_\ell\end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\mathrm{i}\omega_0 \,\mathrm{sign}\,\ell - \frac{1}{|\ell|}\partial_u f(h(\hat{\xi}/\sqrt{|\ell|}),0)) & -\frac{1}{\sqrt{|\ell|}}c_0 D^{-1}\end{pmatrix}\begin{pmatrix}\hat{a}_\ell\\ \hat{b}_\ell\end{pmatrix}. \tag{3.22}$$

Taking the limit $|\ell|\to\infty$, we obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}\hat{\xi}}\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}\mathrm{i}\omega_0 \,\mathrm{sign}\,\ell & 0\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix},$$

which is independent of $\hat{\xi}$. The matrix on the right-hand side is hyperbolic; see Lemma 3.1. A perturbation argument shows that the evolution operators $\hat{\Phi}^{\mathrm{s}}_{+,\ell}$ and $\hat{\Phi}^{\mathrm{u}}_{+,\ell}$ of (3.22) satisfy

$$\|\hat{\Phi}^{\mathrm{s}}_{+,\ell}(\hat{\xi},\hat{\eta})\| \le K\mathrm{e}^{-\kappa(\hat{\xi}-\hat{\eta})}, \qquad \|\hat{\Phi}^{\mathrm{u}}_{+,\ell}(\hat{\eta},\hat{\xi})\| \le K\mathrm{e}^{-\kappa(\hat{\xi}-\hat{\eta})}$$

for $0\le\hat{\eta}\le\hat{\xi}$, where K and κ are independent of ℓ. Due to the definition of the norms on Y and the rescaling of the ξ-variable, the lemma is proved. ∎

With Lemma 3.3 at hand, we can define the subspaces

$$\begin{aligned} E^{\mathrm{cs}}_+(0) &= \mathrm{R}(\Phi^{\mathrm{s}}_+(0,0)+\Phi^{\mathrm{c}}_+(0,0)), \qquad & E^{\mathrm{s}}_+(0) &= \mathrm{R}(\Phi^{\mathrm{s}}_+(0,0)),\\ E^{\mathrm{cu}}_-(0) &= \mathrm{R}(\Phi^{\mathrm{u}}_-(0,0)+\Phi^{\mathrm{c}}_-(0,0)), \qquad & E^{\mathrm{u}}_-(0) &= \mathrm{R}(\Phi^{\mathrm{u}}_-(0,0)). \end{aligned}$$

For any initial value in $E^{\mathrm{cs}}_+(0)$ or $E^{\mathrm{s}}_+(0)$, there exists a solution of (3.16), and it is bounded or exponentially decaying, respectively, as $\xi\to\infty$. An analogous characterization is true for $E^{\mathrm{cu}}_-(0)$ or $E^{\mathrm{u}}_-(0)$ as $\xi\to-\infty$. Note that the subspace $E^{\mathrm{c}}(0)$ defined in (3.20) is given by $E^{\mathrm{c}}(0) = E^{\mathrm{cs}}_+(0)\cap E^{\mathrm{cu}}_-(0)$.

Lemma 3.4 *Assume that Hypotheses (P1) and (P2) are true. There exists then a non-zero element $\psi_0\in Y_0$ such that*

$$(E^{\mathrm{s}}_+(0)+E^{\mathrm{u}}_-(0))\oplus\mathrm{span}\{w^{\mathrm{c}}(0)\mathrm{e}^{\mathrm{i}\omega_0 t}, \overline{w^{\mathrm{c}}(0)}\mathrm{e}^{-\mathrm{i}\omega_0 t}\}\oplus\mathrm{span}\{\psi_0\} = Y,$$

and $E^{\mathrm{s}}_+(0)\cap E^{\mathrm{u}}_-(0) = \mathrm{span}\{(h,h_\xi)_\xi(0)\}$.

Proof. It suffices to construct a complement of the stable and unstable subspaces in Y_0. Note that $Y_0 = \mathrm{Fix}(\rho)$ is invariant under the nonlinear elliptic equation (3.3). In fact, on Y_0, (3.3) coincides with the travelling-wave equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -D^{-1}(cv + f(u,\mu)) \end{pmatrix} \tag{3.23}$$

for $u \in \mathbb{R}^n$, which is satisfied by the wave $(h, h_\xi)(\xi)$. The equilibrium $u = 0$ is hyperbolic, and the intersection $T_{(h,h_\xi)(0)}W^{\mathrm{s}}(0) \cap T_{(h,h_\xi)(0)}W^{\mathrm{u}}(0)$ of tangent spaces of the stable and unstable manifolds of (3.23) is one-dimensional by Hypothesis (P2)(i). Otherwise, the geometric multiplicity of $\lambda = 0$ would be bigger than one. We may choose ψ_0 as the unit vector in the one-dimensional orthogonal complement of $T_{(h,h_\xi)(0)}W^{\mathrm{s}}(0) + T_{(h,h_\xi)(0)}W^{\mathrm{u}}(0)$. ■

3.4 Hopf bifurcations near $U = 0$ in Y

We return to the nonlinear reaction-diffusion system, first considered in the original coordinate frame

$$u_t = Du_{xx} + f(u,\mu).$$

Under the assumptions on the linearization L^0_∞ in the steady coordinate frame, see (3.9), spatially-periodic steady patterns with wavelength $\frac{2\pi}{k}$ bifurcate typically from the zero solution for k close to the critical wavelength k_0. This is usually proved using center-manifold theory or Lyapunov-Schmidt reduction in a function space of $\frac{2\pi}{k}$-periodic functions.

Next, consider the nonlinear parabolic equation (3.2)

$$u_t = Du_{\xi\xi} + cu_\xi + f(u,\mu), \qquad \xi \in \mathbb{R}.$$

In a coordinate frame moving with speed c, the aforementioned spatially-periodic steady patterns become time-periodic travelling wave-trains with frequency $\omega = ck$. We assume from now on that the wave speed c_0 of the pulse is negative, i.e. $c_0 < 0$. If $c_0 > 0$, we change $\xi \mapsto -\xi$ and obtain $c_0 < 0$ in the new spatial variable.

Since the pulse is not spatially periodic, we introduced spatial dynamics on time-periodic functions. In the next step, we rephrase the aforementioned result on bifurcation to wave trains in terms of the spatial dynamics. Consider the nonlinear elliptic problem (3.3)

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ D^{-1}(u_t - cv - f(u,\mu)) \end{pmatrix},$$

with $(u,v) \in Y$. The linearization of (3.3) at $U = 0$ is given by

$$A_\infty(c,\mu) = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t - \partial_u f(0,\mu)) & -cD^{-1} \end{pmatrix}.$$

The operator $A_\infty(c_0, 0)$ has a pair of simple eigenvalues $\pm\mathrm{i}k_0$ with eigenfunctions $\mathrm{e}^{\mathrm{i}\omega_0 t}U_\mathrm{H}$ and $\mathrm{e}^{-\mathrm{i}\omega_0 t}\overline{U_\mathrm{H}}$; see Lemma 3.1. As in (3.6), we define

$$d(\lambda, \nu, c, \mu) := \det(\nu^2 D + \nu c + \partial_u f(0, \mu) - \lambda).$$

Note that we have the relation

$$d(\lambda, \nu, c, \mu) = d(\lambda - \nu(c - c_0), \nu, c_0, \mu), \tag{3.24}$$

which follows immediately from the definition.

For a generic Hopf bifurcation, the eigenvalues $\pm\mathrm{i}k_0$ should cross the imaginary axis with non-zero velocity. We assume the following:

Hypothesis (P3) *Assume that* $C_1 = -\,\mathrm{Re}\,\dfrac{\partial_\mu d(\mathrm{i}\omega_0, \mathrm{i}k_0, c_0, 0)}{\partial_\lambda d(\mathrm{i}\omega_0, \mathrm{i}k_0, c_0, 0)} > 0.$

The reader might check, using (3.24), that the condition $C_1 \neq 0$ is equivalent to the transverse crossing of eigenvalues when considering the temporal dynamics of $\frac{2\pi}{k_0}$-periodic functions. If $C_1 < 0$, we can transform the parameter $\mu \mapsto -\mu$ to achieve $C_1 > 0$; see also Remark 3.10 below.

Note that the denominator $\partial_\lambda d(\mathrm{i}\omega_0, \mathrm{i}k_0, c_0, 0)$ is not equal to zero since $\partial_\lambda d\, \partial_\nu \lambda = -\partial_\nu d \neq 0$ by (P1). Upon differentiating (3.8) with respect to k and using $\nu = \mathrm{i}k$, it follows that $\partial_\nu \lambda = c_0$. The eigenvalue $\mathrm{i}k_0$ persists as a simple eigenvalue $\nu(\mu)$ of the operator $A_\infty(c_0, \mu)$ considered in Y. Using $-\partial_\nu d/\partial_\lambda d = \partial_\nu \lambda = c_0$, we obtain

$$\partial_\mu \nu = -\frac{\partial_\mu d}{\partial_\nu d} = -\frac{\partial_\mu d}{\partial_\lambda d}\,\frac{\partial_\lambda d}{\partial_\nu d} = \frac{1}{c_0}\frac{\partial_\mu d}{\partial_\lambda d}.$$

Hence, owing to (P3), the real part of $\nu(\mu)$ is given approximately by $-\mu C_1/c_0$. Therefore, with $c_0 < 0$ and the sign of C_1 as in (P3), the eigenvalues $\pm\mathrm{i}k_0$ cross the imaginary axis from left to right as μ becomes positive. Furthermore, exploiting (P1), (P3) and (3.24), $d(\lambda, \nu, c, \mu)$ vanishes for (λ, ν) close to $(\mathrm{i}\omega_0, \mathrm{i}k_0)$ if, and only if,

$$\begin{aligned} \lambda &= \mathrm{i}k(c - c_0) + \mathrm{i}\omega_0 + \mathrm{i}c_0(k - k_0) - C_r(k - k_0)^2 + C_1\mu + \mathrm{O}(|\mu|^2 + |k - k_0|^3) \\ &= \mathrm{i}\omega_0 + \mathrm{i}(ck - \omega_0) - C_r(k - k_0)^2 + C_1\mu + \mathrm{O}(|\mu|^2 + |k - k_0|^3). \end{aligned}$$

According to Lemma 3.1, eigenvalues of the linearization $A_\infty(c, \mu)$ are on the imaginary axis precisely when $\mathrm{Im}\,\lambda = \omega_0$ and $\mathrm{Re}\,\lambda = 0$, that is

$$\begin{aligned} 0 &= ck - \omega_0 \\ 0 &= -C_\mathrm{r}(k - k_0)^2 + C_1\mu + \mathrm{O}(|\mu|^2 + |k - k_0|^3). \end{aligned} \tag{3.25}$$

This equation can be solved for (μ, k). Thus, $A_\infty(c, \mu)$ has a pair of imaginary eigenvalues whenever

$$\begin{aligned} k - k_0 &= -\frac{k_0}{c}(c - c_0) \\ \mu &= \frac{C_\mathrm{r} k_0^2}{C_1 c_0^2}(c - c_0)^2 + \mathrm{O}(|c - c_0|^3). \end{aligned} \tag{3.26}$$

Denoting the function in the last equation by $\mu = \mu_*(c)$, we see that (3.3) has a simple pair of imaginary eigenvalues for $(c, \mu) = (c, \mu_*(c))$ for any c close to c_0. We introduce new parameters by

$$(c, \mu) = (c, \mu_*(c) + \hat{\mu}). \tag{3.27}$$

The Jacobian of this transformation is equal to the identity at $(c, \mu) = (c_0, 0)$. Also, imaginary eigenvalues occur precisely for $\hat{\mu} = 0$. Alternatively, we may solve the first equation in (3.26) with respect to k and obtain

$$(c, \mu) = (\omega_0/k, \mu_*(k) + \hat{\mu}),$$

where we again use μ_* with a slight abuse of notation.

Recall that S^1 acts on Y via $(\rho_\alpha U)(t) = U(t + \alpha)$. We say that a manifold W is invariant under equation (3.3) for $\xi \geq 0$ ($\xi \leq 0$) if, for any $U_0 \in W$, there is a solution $U(\xi)$ of (3.3) defined for $\xi \geq 0$ ($\xi \leq 0$) with $U(0) = U_0$ and $U(\xi) \in W$ for sufficiently small ξ.

Lemma 3.5 *Assume that Hypothesis (P1) is met. For any $(c, \hat{\mu})$ close to $(c_0, 0)$, there exists then a two-dimensional, smooth and S^1-invariant center-manifold $W^{\mathrm{c}}_{c,\hat{\mu}}(0) \subset Y$ that contains $U = 0$ and is tangent to* $\mathrm{span}\{\mathrm{e}^{\mathrm{i}\omega_0 t} U_{\mathrm{H}}, \mathrm{e}^{-\mathrm{i}\omega_0 t}\overline{U_{\mathrm{H}}}\}$ *at $U = 0$ for $(c, \hat{\mu}) = (c_0, 0)$. Furthermore, $W^{\mathrm{c}}_{c,\hat{\mu}}(0)$ is invariant under (3.3) and smooth in $(c, \hat{\mu})$.*

Proof. The lemma follows from results of Mielke [11]; see also [22]. The assumptions in these references are satisfied due to Lemma 3.1 and 3.3. ■

Hence, the elliptic PDE (3.3) near $U = 0$ is essentially reduced to an S^1-equivariant ODE on $W^{\mathrm{c}}_{c,\hat{\mu}}(0)$. We assume that the Hopf bifurcation is supercritical.

Hypothesis (H) *Assume that the vector field on $W^{\mathrm{c}}_{c_0,0}(0)$, projected onto the center eigenspace and in polar coordinates, is given by $r_\xi = -C_2 r^3$, $\varphi_\xi = k_0$ up to terms of fourth order for some $C_2 > 0$.*

Note that the vector field on the center eigenspace takes this particularly simple form due to the equivariance with respect to the isometric action of S^1 on the center eigenspace. We remark that the sign of C_2 is not important. The arguments given below work also in the case where $C_2 < 0$; see Remark 3.10 below.

The coefficient C_2 may be computed explicitly following standard procedures. The linear part of the vector field on the center manifold is given by the restriction of the linearization, A_∞, to the invariant center eigenspace $\mathrm{span}\{\mathrm{e}^{\mathrm{i}\omega_0 t} U_{\mathrm{H}}, \mathrm{e}^{-\mathrm{i}\omega_0 t}\overline{U_{\mathrm{H}}}\}$. By S^1-equivariance, the quadratic terms of the Taylor expansion of the vector field projected onto this subspace vanish. The computation of the cubic term requires in general the quadratic approximation of the center manifold. However, if the nonlinearity f is cubic, the computation of the cubic term simplifies greatly: in this situation, the third-order term of the vector field is obtained by simply evaluating and then projecting the nonlinearity onto the center eigenspace. A vector in the center eigenspace is of the general form $z\mathrm{e}^{\mathrm{i}\omega_0 t}U_{\mathrm{H}} + \text{c.c.}$ with $z \in \mathbb{C}$. We write $U_{\mathrm{H}} = (u_{\mathrm{H}}, v_{\mathrm{H}})^t$ where, due to the second-order structure of the equation, $v_{\mathrm{H}} = \mathrm{i}k_0 u_{\mathrm{H}}$. Evaluating f and projecting onto the subspace $\mathrm{span}\{\mathrm{e}^{\mathrm{i}\omega_0 t}U;\ U \in \mathbb{R}^{2n}\}$, we obtain

$$3\partial_u^3 f(0,0)(z\mathrm{e}^{\mathrm{i}\omega_0 t}u_{\mathrm{H}}, z\mathrm{e}^{\mathrm{i}\omega_0 t}u_{\mathrm{H}}, \overline{z\mathrm{e}^{\mathrm{i}\omega_0 t}u_{\mathrm{H}}}) + \text{c.c.}$$

up to fourth order. In order to compute the equation on the center manifold, we have to multiply with the left eigenvector $U_{\mathrm{H}}^* = (u_{\mathrm{H}}^*, v_{\mathrm{H}}^*)$ which satisfies

$$\mathrm{i}k_0 U_{\mathrm{H}}^* = U_{\mathrm{H}}^* \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\mathrm{i}\omega_0 - \partial_u f(0,0)) & -c_0 D^{-1} \end{pmatrix}.$$

For cubic nonlinearities, the cubic coefficient C_2 is therefore given by

$$C_2 = \frac{1}{2} v_{\mathrm{H}}^* (\partial_u^3 f(0,0)(u_{\mathrm{H}}, u_{\mathrm{H}}, \overline{u_{\mathrm{H}}}) + \text{c.c.}).$$

Hypotheses (P3) and (H) are related to the signs of the coefficients in the Ginzburg-Landau equation

$$A_t = A_{xx} + \beta_1 \mu A - \beta_2 |A|^2 A$$

associated with (1.1) near $u = 0$. Indeed, Hypothesis (P3) implies $\beta_1 > 0$, while (H) enforces $\beta_2 > 0$. We may now apply the S^1-equivariant Hopf bifurcation theorem and obtain the following lemma.

Lemma 3.6 *Assume that Hypotheses (P1)–(P3) and (H) are satisfied. There is then a family $\Gamma_{k,\hat{\mu}}(\xi) \in Y$ of periodic solutions of (3.3) with $(c, \mu) = (\omega_0/k, \mu_*(k) + \hat{\mu})$ defined for k close to k_0 and $\hat{\mu} \geq 0$ small. These solutions are C^2 in k uniformly in $\hat{\mu} \geq 0$. Moreover, they are relative equilibria, that is,*

$$\Gamma_{k,\hat{\mu}}(\xi, t) = \Big(\rho_{\frac{k\xi}{\omega_0}} \Gamma_{k,\hat{\mu}}(0)\Big)(t) = \Gamma_{k,\hat{\mu}}\Big(0, t + \frac{k}{\omega_0}\xi\Big).$$

In particular, $\Gamma_{k,\hat{\mu}}$ has period $\frac{2\pi}{k}$ in ξ and $\frac{2\pi}{\omega_0}$ in t. Furthermore, $\Gamma_{k,\hat{\mu}}$ is stable with respect to the dynamics on $W^c_{c,\hat{\mu}}(0)$. Finally, we have the expansion

$$\Gamma_{k,\hat{\mu}}(0,t) = A_{\mathrm{H}} \sqrt{\hat{\mu}}\, \mathrm{e}^{\mathrm{i}\omega_0 t} U_{\mathrm{H}} + \mathrm{O}(|k - k_0|\sqrt{\hat{\mu}} + |\hat{\mu}|) \tag{3.28}$$

for some $A_{\mathrm{H}} \neq 0$.

Proof. The lemma follows from the standard S^1-equivariant Hopf-bifurcation theorem. We obtain a family $\Gamma_{c,\hat{\mu}}$ of periodic solutions parametrized by $(c, \hat{\mu})$. Using the relations (3.25) and (3.27), it is easy to see that we can parametrize the periodic solutions also by the spatial wavenumber k. The relation $c = \omega_0/k$ follows since we deal with steady patterns of spatial period $\frac{2\pi}{k}$ considered in a coordinate frame moving with speed c. ■

Of course, the family of solutions $\Gamma_{k,\hat\mu}$ is precisely the family of Turing patterns that we would have obtained via standard Lyapunov-Schmidt reduction for the temporal dynamics. The periodic solutions $\Gamma_{k,\hat\mu}(\xi)$ of (3.3) correspond to solutions $\gamma_{k,\mu}(\xi,t)$ of (3.2) with $\mu = \mu_*(k)+\hat\mu$. The Turing patterns $\gamma_{k,\mu}$ have period $\frac{2\pi}{\omega_0}$ in t and $\frac{2\pi}{k}$ in ξ. Furthermore, they satisfy $\gamma_{k,\mu}(\xi,t) = \gamma_{k,\mu}(\xi-ct,0)$. In the original frame (x,t), their wave speed is zero.

For any $\tilde c$, the function

$$\tilde\Gamma_{\tilde c,k,\hat\mu}(\xi,t) := \Gamma_{k,\hat\mu}(\xi-\tilde c t,t) = \Gamma_{k,\hat\mu}\Big(0,(1-\frac{k\tilde c}{\omega_0})t+\frac{k}{\omega_0}\xi\Big)$$

satisfies (3.3) with $(c,\mu) = (\frac{\omega_0}{k}+\tilde c, \mu_*(k)+\hat\mu)$ and has frequency $\omega = \omega_0 - k\tilde c$ in t. However, $\Gamma_{k,\hat\mu,\omega}(\xi)$ is not contained in Y but in the space of $\frac{2\pi}{\omega}$-periodic functions. Solving the equation for ω, we obtain $\tilde c = \frac{\omega_0-\omega}{k}$ and we set $\Gamma_{k,\hat\mu,\omega}(\xi) = \tilde\Gamma_{\tilde c,k,\hat\mu}(\xi)$.

Remark 3.7 *The first component $\gamma_{k,\mu,\omega}(\xi,t)$ of $\Gamma_{k,\mu-\mu_*(k),\omega}$ satisfies (3.2) for $\mu\ge\mu_*(k)$ and $c=\frac{\omega}{k}$. It has period $\frac{2\pi}{\omega}$ in t and $\frac{2\pi}{k}$ in ξ.*

3.5 Existence of invariant manifolds

We state existence results for the global center-stable manifold $W^{\mathrm{cs},+}_{c,\hat\mu}(0)$ of the equilibrium $U=0$ and the local unstable manifold $W^{\mathrm{u,loc}}_{c,\hat\mu}(\Gamma_{c,\hat\mu}(0))$ of the periodic solution $\Gamma_{c,\hat\mu}(\xi)$. The key to obtain these manifolds are the exponential dichotomies derived in Lemma 3.3. In this section, we parametrize the periodic waves by $(c,\hat\mu)$ rather than using $(k,\hat\mu)$.

Proposition 1 *Assume that Hypothesis (P1) is satisfied. Equation (3.3) has then a C^2-smooth, locally invariant center-stable manifold $W^{\mathrm{cs},+}_{c,\hat\mu}(0)$ which is tangent to $E^{\mathrm{cs}}_+(0)$ at $(h,h_\xi)(0)$ for $(c,\hat\mu)=(c_0,0)$. It contains all solutions that stay close to $(h,h_\xi)(\xi)$ for all $\xi>0$. Moreover, $W^{\mathrm{cs},+}_{c,\hat\mu}(0)$ is C^2-smooth in $(c,\hat\mu)$.*

Proof. If we parametrize a neighborhood of $(h,h_\xi)(\xi)$ by $U=(h,h_\xi)+V$, we obtain the equation

$$\begin{aligned} V_\xi &= \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t - \partial_u f(h,0)) & -c_0 D^{-1}\end{pmatrix} V \\ &\quad + \begin{pmatrix} 0 \\ D^{-1}(\partial_u f(h,0)V_1 + f(h,\mu_*(c)+\hat\mu) - f(h+V_1,\mu_*(c)+\hat\mu) - (c-c_0)V_2)\end{pmatrix}\end{aligned}$$

for $V=(V_1,V_2)\in Y$. Since Y is a Hilbert space, there exists a smooth cut-off function $\chi_\epsilon(\langle V,V\rangle_Y)$. We define the modified nonlinearity

$$\begin{aligned} &G(\xi,V,c,\hat\mu) := \chi_\epsilon(\langle V,V\rangle_Y)\times \\ &\begin{pmatrix} 0 \\ D^{-1}(\partial_u f(h(\xi),0)V_1 + f(h(\xi),\mu_*(c)+\hat\mu) - f(h(\xi)+V_1,\mu_*(c)+\hat\mu) - (c-c_0)V_2)\end{pmatrix}.\end{aligned}$$

The linear equation $V_\xi = A(\xi)V$ has been solved in Lemma 3.3. For the constant κ appearing in Lemma 3.3 and any number δ with $0 < \delta < \kappa$, we define

$$Z_\delta^+ := \{V \in C^0(\mathbb{R}^+, Y);\ \sup_{\xi \geq 0} \mathrm{e}^{-\delta\xi}|V(\xi)|_Y =: |V|_\delta < \infty\}.$$

We seek the solutions in the center-stable manifold as fixed points of the equation

$$\begin{aligned} V(\xi) &= \Phi_+^{\mathrm{cs}}(\xi, 0)\, V_0^{\mathrm{cs}} + \int_0^\xi \Phi_+^{\mathrm{cs}}(\xi, \eta) G(\eta, V(\eta), c, \hat{\mu})\,\mathrm{d}\eta \\ &\quad + \int_\infty^\xi \Phi_+^{\mathrm{u}}(\xi, \eta) G(\eta, V(\eta), c, \hat{\mu})\,\mathrm{d}\eta, \end{aligned} \tag{3.29}$$

where $V_0^{\mathrm{cs}} \in E_+^{\mathrm{cs}}(0)$ and $V \in Z_\delta^+$. It follows from the estimates obtained in Lemmata 3.1 and 3.3 that the hypotheses in [22] are met. The proposition is then a consequence of the results presented in [22]. Note that any solution of the integral equation is actually a smooth solution; see [14, Lemma 3.1]. ■

Similarly, we obtain the global center-unstable manifold $W_{c,\hat{\mu}}^{\mathrm{cu},-}(0)$ of $U = 0$ that enjoys the analogous properties for $\xi \to -\infty$. Finally, we construct the local unstable manifold of $\Gamma_{c,\hat{\mu}}(0)$.

Proposition 2 *Assume that Hypotheses (P1)–(P3) and (H) are satisfied. For any $(c, \hat{\mu})$ with $|c - c_0|$ and $\hat{\mu} \geq 0$ small, equation (3.3) has a C^2-smooth, locally invariant unstable manifold $W_{c,\hat{\mu}}^{\mathrm{u,loc}}(\Gamma_{c,\hat{\mu}}(0))$ which is tangent to $R(P^{\mathrm{u}})$ at $U = 0$ for $(c, \hat{\mu}) = (c_0, 0)$. It consists precisely of those solutions U_0 that stay in a small neighborhood of $U = 0$ for $\xi \leq 0$ and satisfy*

$$|U(\xi) - \Gamma_{c,\hat{\mu}}(\xi)| \leq K\mathrm{e}^{\kappa\xi}$$

as $\xi \to -\infty$. Moreover, $W_{c,\hat{\mu}}^{\mathrm{u,loc}}(\Gamma_{c,\hat{\mu}}(0))$ is continuous in $\hat{\mu}$ in the C^2-topology and C^2-smooth in c.

Here, $\kappa > 0$ and the projection P^{u} have been defined in Lemma 3.3 and 3.1, respectively.

Proof. We use the parametrization $U(\xi) = \Gamma_{c,\hat{\mu}}(\xi) + V(\xi)$ and obtain the equation

$$\begin{aligned} V_\xi &= \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t - \partial_u f(\Gamma_{c,\hat{\mu}}, \mu_*(c) + \hat{\mu})) & -cD^{-1} \end{pmatrix} V \\ &\quad + \begin{pmatrix} 0 \\ D^{-1}(\partial_u f(\Gamma_{c,\hat{\mu}}, \mu_*(c) + \hat{\mu})V_1 + f(\Gamma_{c,\hat{\mu}}, \mu_*(c) + \hat{\mu}) - f(\Gamma_{c,\hat{\mu}} + V_1, \mu_*(c) + \hat{\mu})) \end{pmatrix} \end{aligned}$$

for $V = (V_1, V_2) \in Y$. As before, we define the modified nonlinearity

$$\begin{aligned} &G(\xi, V, c, \hat{\mu}) := \chi_\epsilon(\langle V, V\rangle_Y) \times \\ &\begin{pmatrix} 0 \\ D^{-1}(\partial_u f(\Gamma_{c,\hat{\mu}}(\xi), \mu_*(c) + \hat{\mu})V_1 + f(\Gamma_{c,\hat{\mu}}(\xi), \mu_*(c) + \hat{\mu}) - f(\Gamma_{c,\hat{\mu}}(\xi) + V_1, \mu_*(c) + \hat{\mu})) \end{pmatrix}. \end{aligned}$$

It follows from the roughness theorem for exponential dichotomies proved in [14] that the linear equation

$$V_\xi = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t - \partial_u f(\Gamma_{c,\hat\mu}(\xi), \mu_*(c) + \hat\mu)) & -cD^{-1} \end{pmatrix} V$$

has evolution operators $\Phi^{\mathrm{cs}}_{c,\hat\mu}(\xi,\eta)$ and $\Phi^{\mathrm{u}}_{c,\hat\mu}(\eta,\xi)$ defined for $\eta \le \xi \le 0$ provided $|c - c_0|$ and $\hat\mu > 0$ are small. The evolution operators satisfy the estimates

$$\|\Phi^{\mathrm{cs}}_{c,\hat\mu}(\xi,\eta)\|_{L(Y)} \le K, \qquad \|\Phi^{\mathrm{u}}_{c,\hat\mu}(\eta,\xi)\|_{L(Y)} \le K\mathrm{e}^{-\kappa(\xi-\eta)}$$

and

$$\|\Phi^{\mathrm{cs}}_{c,\hat\mu}(\xi,\eta) - \Phi^{\mathrm{cs}}_{c_0,0}(\xi,\eta)\|_{L(Y)} + \|\Phi^{\mathrm{u}}_{c,\hat\mu}(\eta,\xi) - \Phi^{\mathrm{u}}_{c_0,0}(\eta,\xi)\|_{L(Y)} \le K(|c - c_0| + \sqrt{\hat\mu})$$

for $\eta \le \xi \le 0$; see [14]. For any δ with $0 < \delta < \kappa$, we define

$$Z^-_\delta := \{V \in C^0(\mathbb{R}^-, Y);\ \sup_{\xi\le 0} \mathrm{e}^{-\delta\xi}|V(\xi)|_Y =: |V|_\delta < \infty\}.$$

We seek the unstable manifold as a fixed point of the equation

$$\begin{aligned} V(\xi) &= \Phi^{\mathrm{u}}_{c,\hat\mu}(\xi,0)\, V^{\mathrm{u}}_0 + \int_0^\xi \Phi^{\mathrm{u}}_{c,\hat\mu}(\xi,\eta) G(\eta, V(\eta), c, \hat\mu)\,\mathrm{d}\eta \\ &\quad + \int_{-\infty}^\xi \Phi^{\mathrm{cs}}_{c,\hat\mu}(\xi,\eta) G(\eta, V(\eta), c, \hat\mu)\,\mathrm{d}\eta, \end{aligned} \tag{3.30}$$

where $V^{\mathrm{u}}_0 \in E^{\mathrm{u}}_*$ and $V \in Z^-_\delta$. Since $G(\xi, V, c, \hat\mu) = \mathrm{O}(|V|^2_Y)$ uniformly in c and $\hat\mu$, the nonlinearity G is C^2 as a map from Z^-_δ into itself. By the uniform-contraction theorem, there exists a unique fixed point of (3.30) that depends smoothly on V^{u}_0 and c, and continuously on $\hat\mu$. This proves the proposition. ∎

By the above proof and equation (3.28), $W^{\mathrm{u,loc}}_{c,\hat\mu}(\Gamma_{c,\hat\mu}(0))$ is given by

$$\begin{aligned} U &= \Gamma_{c,\hat\mu}(0) + \Phi^{\mathrm{u}}_{c,\hat\mu}(0,0)\, V^{\mathrm{u}}_0 + \mathrm{O}(|V^{\mathrm{u}}_0|^2_Y) \\ &= A_{\mathrm{H}}\sqrt{\hat\mu}\, \mathrm{e}^{\mathrm{i}\omega_0 t} U_{\mathrm{H}} + V^{\mathrm{u}}_0 + \mathrm{O}(|\hat\mu| + \sqrt{\hat\mu}(|c - c_0| + |V^{\mathrm{u}}_0|_Y) + |V^{\mathrm{u}}_0|^2_Y), \end{aligned} \tag{3.31}$$

where $V^{\mathrm{u}}_0 \in E^{\mathrm{cu}}_*$.

3.6 Transversality

We seek solutions of (3.3) connecting the bifurcating periodic solution $\Gamma_{c,\hat\mu}$ with itself. Therefore, we are interested in intersections of the local unstable manifold $W^{\mathrm{u,loc}}_{c,\hat\mu}(\Gamma_{c,\hat\mu}(0))$ with the global center-stable manifold $W^{\mathrm{cs},+}_{c,\hat\mu}(0)$. For $(c, \hat\mu) = (c_0, 0)$, the former manifold coincides with $W^{\mathrm{u,loc}}_{c_0,0}(0)$. We may then shift the variable ξ such that $(h, h_\xi)(0)$ is contained in the local unstable manifold $W^{\mathrm{u,loc}}_{c_0,0}(0)$. In particular, $W^{\mathrm{u,loc}}_{c_0,0}(0)$ and $W^{\mathrm{cs},+}_{c_0,0}(0)$ intersect along the homoclinic solution $U(\xi) = (h, h_\xi)(\xi)$. In order to find intersections for $\hat{\mu} \neq 0$, we consider the suspended manifolds

$$\begin{aligned}\tilde{W}^{\mathrm{cs},+}_{\hat{\mu}} &:= \{(U,c);\ |c-c_0|<\delta,\ U \in W^{\mathrm{cs},+}_{c,\hat{\mu}}(0)\}\\ \tilde{W}^{\mathrm{u},-}_{\hat{\mu}} &:= \{(U,c);\ |c-c_0|<\delta,\ U \in W^{\mathrm{u,loc}}_{c,\hat{\mu}}(\Gamma_{c,\hat{\mu}}(0))\}\end{aligned}$$

as manifolds in $Y \times \mathbb{R}$. Note that they are indeed C^2 due to the propositions proved above. For $\hat{\mu} = 0$, these manifolds intersect along $(U,c) = ((h,h_\xi), c_0)$.

Lemma 3.8 *For $\hat{\mu} = 0$, we have*

$$\begin{aligned}T_{((h,h_\xi)(0),c_0)}\tilde{W}^{\mathrm{cs},+}_0 \cap T_{((h,h_\xi)(0),c_0)}\tilde{W}^{\mathrm{u},-}_0 &= \operatorname{span}\{((h,h_\xi)_\xi(0),0)\},\\ T_{((h,h_\xi)(0),c_0)}\tilde{W}^{\mathrm{cs},+}_0 + T_{((h,h_\xi)(0),c_0)}\tilde{W}^{\mathrm{u},-}_0 &= Y \times \mathbb{R}.\end{aligned}$$

In other words, the suspended manifolds intersect transversely in the extended phase space $Y \times \mathbb{R}$.

Proof. We observe that $\Gamma_{c_0,0}(\xi) = 0$ vanishes identically for all ξ. The tangent spaces of $\tilde{W}^{\mathrm{cs},+}_0$ and $\tilde{W}^{\mathrm{u},-}_0$ are given by

$$\begin{aligned}T_{((h,h_\xi)(0),c_0)}\tilde{W}^{\mathrm{cs},+}_0 &= (E^{\mathrm{s}}_+(0)\times\{0\}) + (E^{\mathrm{c}}(0)\times\{0\}) + \operatorname{span}\{(\tilde{V}^{\mathrm{cs},+}_{c_0,0}(0),1)\},\\ T_{((h,h_\xi)(0),c_0)}\tilde{W}^{\mathrm{u},-}_0 &= (E^{\mathrm{u}}_-(0)\times\{0\}) + \operatorname{span}\{((h,h_\xi)_\xi(0),0)\} + \operatorname{span}\{(\tilde{V}^{\mathrm{u},-}_{c_0,0}(0),1)\}.\end{aligned}$$

The tangent vector $\tilde{V}^{\mathrm{cs},+}_{c_0,0}(\xi)$ of the center-stable manifold in the c-direction can be calculated by taking the derivative of (3.29) with respect to c at $V^{\mathrm{cs}}_0 = 0$. Similarly, $\tilde{V}^{\mathrm{u},-}_{c_0,0}(\xi)$ is the derivative of (3.30) with respect to c at $V^{\mathrm{u}}_0 = (h,h_\xi)(0)$. Computing these derivatives, we obtain the expressions

$$\begin{aligned}\tilde{V}^{\mathrm{u},-}_{c_0,0}(0) &= \int_{-\infty}^0 \Phi^{\mathrm{u}}_-(0,\eta)\begin{pmatrix}0\\ -D^{-1}h_\xi(\eta)\end{pmatrix}\mathrm{d}\eta,\\ \tilde{V}^{\mathrm{cs},+}_{c_0,0}(0) &= \int_{\infty}^0 \Phi^{\mathrm{cs}}_+(0,\eta)\begin{pmatrix}0\\ -D^{-1}h_\xi(\eta)\end{pmatrix}\mathrm{d}\eta.\end{aligned}$$

On account of Lemma 3.4, it suffices to prove that

$$\Big\langle \psi_0, \tilde{V}^{\mathrm{cs},+}_{c_0,0}(0)\Big\rangle \neq \Big\langle \psi_0, \tilde{V}^{\mathrm{u},-}_{c_0,0}(0)\Big\rangle,$$

that is,

$$\Big\langle \psi_0, \int_{-\infty}^0 \Phi^{\mathrm{u}}_-(0,\eta)(0,-D^{-1}h_\xi(\eta))^t\,\mathrm{d}\eta - \int_{\infty}^0 \Phi^{\mathrm{cs}}_+(0,\eta)(0,-D^{-1}h_\xi(\eta))^t\,\mathrm{d}\eta\Big\rangle \neq 0. \tag{3.32}$$

Note that the integrands are actually contained in Y_0. In particular, the term on the left-hand side in (3.32) is given by

$$M := \int_{-\infty}^{\infty} \langle \psi(\eta), (0, -D^{-1}h_\xi(\eta))^t\rangle\,\mathrm{d}\eta \tag{3.33}$$

where $\psi(\xi)$ is the unique, up to constant multiples, bounded solution of the adjoint variational equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} 0 & \mathrm{id} \\ -D^{-1}\partial_u f(h(\xi),0) & -c_0 D^{-1} \end{pmatrix}^t \begin{pmatrix} u \\ v \end{pmatrix}$$

for $(u,v) \in \mathbb{R}^n \times \mathbb{R}^n$. Since zero is a simple eigenvalue by (P2)(i), we can conclude that M, defined in (3.33), is non-zero; see [16, Lemma 5.5]. A similar argument, and more details, can be found in [17, Section 5]. ∎

Therefore, for any $\hat\mu > 0$, the manifolds $\tilde W^{\mathrm{cs},+}_{\hat\mu}$ and $\tilde W^{\mathrm{u},-}_{\hat\mu}$ intersect along a unique line $(U_{\hat\mu}(\xi), c(\hat\mu))$ that depends on $\hat\mu$. The associated solution $U_{\hat\mu}(\xi)$ of (3.3) with $c = c(\hat\mu)$ converges exponentially to $\Gamma_{c(\hat\mu),\hat\mu}$ as $\xi \to -\infty$ by definition. It is also contained in the center-stable manifold $W^{\mathrm{cs},+}_{c(\hat\mu),\hat\mu}(0)$.

Lemma 3.9 *We have the estimate* $|c(\hat\mu) - c_0| \le K|\hat\mu|$.

Proof. We consider the suspended local center-unstable manifold

$$\tilde W^{\mathrm{cu},-}_{\hat\mu} := \{(U,c);\ |c-c_0| < \delta,\ U \in W^{\mathrm{cu,loc}}_{c,\hat\mu}(0)\};$$

see the comment after Proposition 1. Since the manifolds $\tilde W^{\mathrm{cu},-}_{\hat\mu}$ and $\tilde W^{\mathrm{cs},+}_{\hat\mu}$ are smooth in $\hat\mu$, we can parametrize them locally near $((h,h_\xi)(0), c_0)$ according to

$$\begin{aligned}
\tilde W^{\mathrm{cu},-}_{\hat\mu} &= ((h,h_\xi)(0),c_0) + (V^{\mathrm{cu}}_-,0) + (c-c_0)(\tilde V^{\mathrm{cu}}_{c_0,0}(0),1) + \mathrm{O}(|c-c_0|^2 + |V^{\mathrm{cu}}_-|^2_Y + |\hat\mu|) \\
\tilde W^{\mathrm{cs},+}_{\hat\mu} &= ((h,h_\xi)(0),c_0) + (V^{\mathrm{cs}}_+,0) + (c-c_0)(\tilde V^{\mathrm{cs}}_{c_0,0}(0),1) + \mathrm{O}(|c-c_0|^2 + |V^{\mathrm{cs}}_+|^2_Y + |\hat\mu|),
\end{aligned}$$

where $V^{\mathrm{cu}}_- \in E^{\mathrm{cu}}_-(0)$ and $V^{\mathrm{cs}}_+ \in E^{\mathrm{cs}}_+(0)$. Projecting the difference of elements (U^{cu}_-, c) and (U^{cs}_+, c) in $\tilde W^{\mathrm{cu},-}_{\hat\mu}$ and $\tilde W^{\mathrm{cs},+}_{\hat\mu}$, respectively, onto $\mathrm{span}\{(\psi_0,0)\}$, we obtain

$$\langle(\psi_0,0),(U^{\mathrm{cu}}_-,c) - (U^{\mathrm{cs}}_+,c)\rangle = (c-c_0)M + \mathrm{O}(|c-c_0|^2 + |V^{\mathrm{cu}}_-|^2_Y + |V^{\mathrm{cs}}_+|^2_Y + |\hat\mu|).$$

Upon inserting the intersection point, the left-hand side vanishes. On the other hand, the distance between the intersection point and $(h,h_\xi)(0)$ is of the order $\sqrt{\hat\mu}$. This proves the lemma. ∎

It remains to show that $U_{\hat\mu}(\xi)$ cannot converge to zero as $\xi \to \infty$ but approaches the periodic solution $\Gamma_{c(\hat\mu),\hat\mu}$. Since it already converges to $\Gamma_{c(\hat\mu),\hat\mu}$ for $\xi \to -\infty$ by construction, it is then a homoclinic orbit to $\Gamma_{c(\hat\mu),\hat\mu}$. On account of Lemma 3.6, it suffices to show that $U_{\hat\mu}(0)$ is not contained in the stable manifold of the origin $W^{\mathrm{s},+}_{c(\hat\mu),\hat\mu}(0)$. Firstly, we shift time such that $(h,h_\xi)(0)$ is contained in the local unstable manifold of zero and has distance $r > 0$ small from zero. We shall now estimate the distance between $W^{\mathrm{u,loc}}_{c(\hat\mu),\hat\mu}(\Gamma_{c,\hat\mu}(0))$ and

$W^{\mathrm{u,loc}}_{c(\hat\mu),\hat\mu}(0)$, measured near $(h,h_\xi)(0)$. Using the expansion (3.31) and Lemma 3.9, we can estimate this distance from below by

$$\sqrt{\hat\mu}\,|A_{\mathrm{H}}|\,|U_{\mathrm{H}}| - K\sqrt{\hat\mu}(|V_0^{\mathrm{u}}|_Y + |c(\hat\mu)-c_0|) \geq \sqrt{\hat\mu}\,|A_{\mathrm{H}}|\,|U_{\mathrm{H}}| - K\sqrt{\hat\mu}(r+\hat\mu).$$

Note that we do not have to account for the quadratic terms $\mathrm{O}(|V_0^{\mathrm{u}}|_Y)$ in (3.31) since they correspond to the local unstable manifold for $\hat\mu = 0$. Hence, they disappear when computing the distance. After choosing r sufficiently small, we conclude from the above estimate that the aforementioned distance between $W^{\mathrm{u,loc}}_{c(\hat\mu),\hat\mu}(\Gamma_{c,\hat\mu}(0))$ and $W^{\mathrm{u,loc}}_{c(\hat\mu),\hat\mu}(0)$ is bigger than $\delta\sqrt{\hat\mu}$ for some $\delta > 0$. On the other hand, the distance between $W^{\mathrm{s},+}_{c(\hat\mu),\hat\mu}(0)$ and $W^{\mathrm{u,loc}}_{c(\hat\mu),\hat\mu}(0)$ is of the order $\hat\mu$ since both are smooth in $\hat\mu$. Therefore, $W^{\mathrm{s},+}_{c(\hat\mu),\hat\mu}(0)$ and $W^{\mathrm{u,loc}}_{c(\hat\mu),\hat\mu}(\Gamma_{c,\hat\mu}(0))$ cannot intersect near $(h,h_\xi)(0)$.

3.7 The homoclinic bifurcation

We summarize our findings in the following existence theorem.

Theorem 1 *Assume that Hypotheses (H), (P1)-(P3) and (TW) are satisfied. There is then a smooth function $\mu_*(\omega) \geq 0$ with $\mu_*(\omega_0) = \mu_*'(\omega_0) = 0$ and $\mu_*''(\omega_0) > 0$ such that, for any ω close to ω_0 and any small $\mu > \mu_*(\omega)$, the following is true. For a unique wave speed $c = c_*(\mu,\omega)$ close to c_0, equation (3.2) has a solution $h_{\mu,\omega}(\xi,t)$ with the following properties.*

(i) *$h_{\mu,\omega}(\xi,t)$ is periodic in t with period $\frac{2\pi}{\omega}$. In other words, the bifurcating pulse is time-periodic in an appropriate moving frame. The family $h_{\mu,\omega}(\cdot,\cdot)$ is continuous in (μ,ω) with values in $C^0(\mathbb{R}^2,\mathbb{R}^n)$ provided with the local topology.*

(ii) *We have $c_*(0,\omega_0) = c_0$ and $h_{0,\omega_0}(\xi,t) = h(\xi)$.*

(iii) *There exists a constant $\delta > 0$ such that, for $c_0 < 0$,*

$$\begin{aligned} |h_{\mu,\omega}(\xi,t) - \gamma_{k_*(\mu,\omega),\mu,\omega}(\xi+\varphi_+,t)| &\leq K\mathrm{e}^{-\delta\mu\xi} & \qquad \xi &\to \infty \\ |h_{\mu,\omega}(\xi,t) - \gamma_{k_*(\mu,\omega),\mu,\omega}(\xi+\varphi_-,t)| &\leq K\mathrm{e}^{-\kappa|\xi|} & \qquad \xi &\to -\infty \end{aligned}$$

for some $\varphi_\pm = \varphi_\pm(\mu,\omega)$ independent of ξ and t, where $k_*(\mu_*(\omega),\omega) = \frac{\omega}{c_*(\mu,\omega)}$. If $c_0 > 0$, replace ξ by $-\xi$ in the above expressions.

(iv) *The functions $\gamma_{k_*(\mu,\omega),\mu,\omega}(\xi,t)$ have amplitude of the order $\sqrt{\mu-\mu_*(\omega)}$, spatial period $\frac{2\pi}{k_*(\mu_*(\omega),\omega)}$ in ξ and temporal period $\frac{2\pi}{\omega}$; see Remark 3.7.*

Proof. For $\omega = \omega_0$, the claims in the theorem have been proved in the previous sections. It is straightforward to see that these proofs remain valid for any fixed ω close to ω_0. Indeed, all hypotheses are open conditions. In order to show that the estimates and existence domains are uniform in ω, we change the time and space variables in (3.1) according to

$$(\tau,\eta) = \Big(\frac{\omega_0}{\omega}t, \sqrt{\frac{\omega_0}{\omega}}\xi\Big).$$

Equation (3.1) then reads

$$u_\tau = Du_{\eta\eta} + \frac{\omega}{\omega_0} f(u,\mu). \tag{3.34}$$

In particular, the nonlinearity depends smoothly on the parameter ω. Any solution of (3.34) with period $\frac{2\pi}{\omega_0}$ in t corresponds to a solution of (3.1) with period $\frac{2\pi}{\omega}$ in t. We may now cast (3.34) in a moving frame as an elliptic problem on the space Y and apply the analysis of the previous sections to this problem with the additional parameter ω. The crucial point is that only the nonlinearity depends on the parameter ω, and in fact smoothly. A pure time-rescaling would result in an operator $\frac{\omega_0}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}$ in the elliptic problem; the dependence on ω is then more delicate. ■

The solutions $h_{\mu,\omega}$ are relative periodic orbits with respect to the group of translations acting on the function space $X = C^0_{\mathrm{unif}}(\mathbb{R},\mathbb{R}^n)$. At the tails, they have small spatial oscillations of period $\frac{2\pi}{k_0}$ that move with speed of about $-c_0 = -\frac{\omega_0}{k_0}$ relative to the resting large pulse profile; see Figure 1. Their temporal period is close to $\frac{2\pi}{\omega_0}$.

Remark 3.10 *In Hypotheses (P3) and (H), we assumed that $C_1 > 0$ and $C_2 > 0$ are both positive. As mentioned earlier, these conditions are not really necessary. In fact, Theorem 1 holds provided $C_1 \neq 0$ and $C_2 \neq 0$. We expect, however, that the modulated pulses are stable only if the Hopf bifurcation, which generates the Turing patterns, is supercritical; see Section 4.1 below.*

3.8 Other modulated waves

There are many more qualitatively different modulated waves that bifurcate near an essential instability. For instance, it is straightforward to prove the following. Suppose that the assumptions of Theorem 1 are met. Also, without loss of generality, assume that $c_0 < 0$. Then, for any ω near ω_0 and any fixed $\mu > \mu_*(\omega)$, there exists a one-parameter family of modulated waves that connect the trivial asymptotic state $u = 0$ at $\xi \to -\infty$ to a Turing pattern at $\xi \to \infty$. These waves can be parametrized by their distance, measured at $\xi = -\xi_0$ for some large ξ_0, from the invariant subspace of time-independent functions. The proof is very similar to the one given above and we shall omit it.

4 Discussion

The approach presented here is a natural generalization of deriving ODEs describing travelling waves. It allows us to investigate the local Turing bifurcation to patterns with small amplitude and the global bifurcation involving the pulse separately. The method is applicable to a variety of other instability phenomena. We restricted ourselves to the case of an instability from a primary pulse only in order to make our strategy more transparent. The most general framework would be a bifurcation from a heteroclinic orbit for the spatial dynamics, that is, a travelling-wave solution of a parabolic equation posed on an infinite cylinder that approaches stationary states at both ends of the cylinder.

4.1 Modulated waves bifurcating from pulses

In this section, we discuss some other issues related to bifurcations from pulses as considered in Section 3.

Modulated waves connecting different Turing patterns

The modulated waves we described in Theorem 1 converge to the same Turing pattern as $\xi \to \pm\infty$. One might expect that, near an essential instability, generalized modulated waves arise that connect two Turing patterns with different wavenumbers. These generalized modulated waves would be quasi-periodic in time with two frequencies that are associated with the temporal periods of the asymptotic Turing patterns. The approach pursued here does not work when investigating waves that are quasi-periodic in time. It is natural and tempting to consider the elliptic equation on the space of quasi-periodic functions. Unfortunately, this procedure leads to small-divisor problems that are difficult to resolve even on the linear level.

Shape of the modulated pulses

An interesting aspect of our analysis are the rates of convergence of the pulse towards the oscillatory patterns. A computation shows that in the moving coordinate frame, for $\mu > 0$, the origin is unstable in the center manifold for the ξ-dynamics when $c < 0$ and stable if $c > 0$. Suppose that $c < 0$. This is the situation we discussed in Section 3.7; the other case is obtained by reversing $\xi \to -\xi$. For $\xi \to -\infty$, the time-periodic pulse that bifurcates from the original pulse $h(\xi)$ converges to the periodic pattern $\Gamma_{k,\mu}$ exponentially with rate $\kappa = \mathrm{O}(1)$ with respect to $\hat{\mu}$ as it lies in the strong unstable manifold of the periodic orbit. For $\xi \to \infty$, however, it approaches first the center manifold with exponential rate $\mathrm{O}(1)$ at a point with distance $\mathrm{O}(\mu)$ to the periodic pattern $\Gamma_{k,\mu}$. It then approaches the periodic pattern, which is of amplitude $\mathrm{O}(\sqrt{\mu})$, with exponential rate $\mathrm{O}(\mu)$.

In physical space, in a steady coordinate frame, we can interpret this as follows. In front of the pulse, we see the possibly stable periodic pattern with amplitude $\mathrm{O}(\sqrt{\mu})$. We then observe the pulse that passes by and moves exponentially fast away from the oscillatory pattern. Behind the pulse, there is some kind of recovery zone where the amplitude of the oscillations grows or decays exponentially towards the same value as ahead of the wave, but on the large spatial scale $\xi \sim 1/\mu$.

Stability

Having established the existence of modulated pulses, an important issue is their stability. We say that a time-periodic solution is spectrally stable if the spectrum of the linearization of the time-period map about the wave is contained strictly inside the unit circle with the exception of the point $\lambda = 1$. Note that $\lambda = 1$ is always contained in the essential spectrum of modulated pulses on account of translation and time invariance.

The bifurcating time-periodic pulses are spectrally stable in the moving coordinate frame if the small-amplitude periodic patterns are spectrally stable and the point spectrum of the primary pulse is strictly contained in the left half-plane with the exception of a simple eigenvalue at zero. This statement is proved in the second part [18] of this work. Assume that the Hopf bifurcation leading to Turing patterns is supercritical. It then follows that, for $\mu > 0$ fixed, there is an open interval of wavenumbers k and an open set of temporal frequencies ω such that the modulated pulses with asymptotic wavenumber k and temporal period $\frac{2\pi}{\omega}$ described in Theorem 1 are spectrally stable; see [18].

Genuine Hopf bifurcations

Similar dynamical problems arise when genuine Hopf bifurcations are considered. We again restrict ourselves to pulse solutions that decay to the zero equilibrium at both ends of the real axis. Suppose that this equilibrium becomes unstable in the non-moving coordinate frame with essential spectrum crossing the imaginary axis at $\pm \mathrm{i}\omega_0$ for some non-zero ω_0. The associated critical wave vector k may again be zero or non-zero. If $k = 0$, spatially homogeneous oscillations $\mathrm{e}^{\mathrm{i}\omega_0 t}$ are created. Reversing space and time, these correspond to equilibria of the elliptic system (1.4). Due to the time-shift symmetry, there is a whole group orbit of equilibria bifurcating from zero in a two-dimensional center-manifold for (1.4). Arguing again as in Section 3.7, we obtain the following result: there exists a homoclinic trajectory to this circle of 'equilibria'. It corresponds to a pulse solution where both tails experience a spatially homogeneous time-periodic oscillation.

The case of a Hopf bifurcation with non-zero wavenumber involves non-resonance conditions. Small-amplitude waves of the form $\mathrm{e}^{\mathrm{i}(\omega_0 t \pm k_0 x)}$ correspond to waves moving with speed $\pm\frac{\omega_0}{k_0} - c$ relative to the pulse. In addition, we expect the creation of standing waves

$e^{i(\omega_0 t + k_0 x)} - e^{i(\omega_0 t - k_0 x)}$. However, in a moving coordinate frame and fixing the temporal frequency $\omega_0 + ck_0$ (or $\omega_0 - ck_0$), only one of the linear waves $e^{i(\omega_0 t \pm k_0 x)}$ yields a bounded solution of the linearized elliptic operator A_∞ defined in Section 3.2 – provided c is non-resonant, that is, $c \neq -\frac{n-1}{n+1} \cdot \frac{\omega_0}{k_0}$ (or $c \neq \frac{n-1}{n+1} \cdot \frac{\omega_0}{k_0}$). The center manifold is again two-dimensional with the time shift acting as rotational symmetry, and we recover precisely the same setting as in the stationary bifurcation. We obtain time-periodic modulated pulses converging at both tails to travelling-wave patterns moving with speed $\pm \frac{\omega_0}{k_0} - c$ relative to the pulse.

Standing pulses

If the wave speed vanishes so that $c_0 = 0$, then the bifurcation problem reduces to an ODE since the Turing patterns and the primary pulse are stationary. Note, however, that the center manifold is four-dimensional due to the reversibility; see [8] or [18]. It is then not obvious whether standing pulses bifurcate that converge to one of the stationary Turing patterns as $x \to \pm\infty$.

Numerical computation of the time-periodic pulses

We mention that the analysis presented here also indicates how the bifurcating periodic pulses might be computed numerically. Indeed, we sought and found them as homoclinic orbits towards a periodic orbit for the elliptic equation (3.3). In particular, a Galerkin approximation of (3.3) in time and the subsequent computation of a homoclinic connection to the small-amplitude patterns should provide a robust method of computing modulated pulses; we refer to [10] for details and more references of such methods for elliptic equations. The only difficulty here is the start-off near the bifurcation point. Also, an additional phase condition has to be incorporated to factor the S^1-symmetry induced by the shift in time.

4.2 Modulated waves bifurcating from fronts or wave trains

Heteroclinic connections and fronts

In this section, we consider travelling waves $u(\xi)$ of (1.1) with wave speed c_0 that converge to two different equilibria $p_\pm$ as $\xi \to \pm\infty$. The asymptotic states $p_\pm$ could be either stable or unstable.

First, suppose that the travelling wave connects a stable with an unstable equilibrium or vice versa. Such waves are often called fronts. Fronts exist typically for a continuum of wave speeds and move towards the unstable state. The stability of a front is recovered upon choosing a suitable weighted norm in the relevant function space. We may investigate the situation when the stable asymptotic state becomes unstable. Whether this route to instability produces modulated waves depends crucially on certain Fredholm properties of the linearization about the primary front. If a certain Fredholm index becomes positive, the approach introduced in Section 3 is applicable under conditions as in Hypothesis (P1). The resulting pattern is a time-periodic front that connects the spatially homogeneous unstable state to a spatially oscillatory pattern; see [2] for an example of such a phenomenon. We refer to [19] for a more thorough discussion. Sherratt [20] investigated a series of certain caricature problems numerically and analytically for fronts with negative Fredholm index. He observed several modulated waves of different wave speed that are glued together. Following the approach presented in this paper, we can show that modulated fronts cannot bifurcate in this situation.

The second possibility is that the travelling wave connects two stable equilibria. In this case, one of the stable equilibria may destabilize. Whether or not modulated waves bifurcate depends again on the sign of a certain Fredholm index. The details can be found in [19].

Periodic wave trains

Another interesting example are periodic wave trains that destabilize due to essential spectrum that crosses the imaginary axis. The difference to the previous cases is that there are no asymptotic states involved. Suppose that the linearized operator about the wave train has two bounded eigenfunctions associated with the essential instability. Investigating the spatial dynamics in an appropriate moving frame, we obtain a trichotomy that characterizes solutions that decay for either forward or backward ξ-direction, or are bounded for $\xi \in \mathbb{R}$. The bifurcation problem can then be reduced to equations that live on the two-dimensional space of bounded solutions. The resulting bifurcation is similar to a Hopf bifurcation to an invariant circle for the Poincare map associated with periodic orbits in ODEs. The bifurcation direction depends upon higher-order terms of the equation restricted to the space of bounded functions at the bifurcation point. Generically, modulated wave trains bifurcate that are periodic in time. Their spatial structure is given by a superposition of the bounded eigenfunctions and the primary wave train.

Cylindrical domains

Generalizations to travelling-wave problems posed on cylinders $\mathbb{R} \times \Omega$ require some more technical preparation. Consider, for example,

$$u_t = D\Delta_{x,y} u + f(u), \qquad (x, y) \in \mathbb{R} \times \Omega,$$

with, say, Dirichlet or Neumann boundary conditions on $\mathbb{R} \times \partial\Omega$. Here, Ω is a bounded domain in $\mathbb{R}^m$. Travelling waves are of the form $u = u(x - ct, y)$ and satisfy the elliptic system of partial differential equations

$$cu_\xi = D\Delta_{\xi,y} u + f(u, \mu), \qquad (x, y) \in \mathbb{R} \times \Omega. \tag{4.1}$$

Again, there is a dynamical interpretation for solutions of this elliptic equation induced by the shift of solutions along the cylinder. If $u(\xi, y) \to p_\pm(y)$ as $\xi \to \pm\infty$, then $D\Delta_y p_\pm + f(p_\pm) = 0$ and $p_\pm(y)$ are 'equilibria' of (4.1). In this sense, a travelling wave u can be interpreted as a heteroclinic orbit of the dynamical system associated with the elliptic system (4.1). A generic choice for the phase space would then be $(u, u_\xi) \in H_0^1(\Omega)^n \times L^2(\Omega)^n$ if, for example, Dirichlet boundary conditions were chosen. The bifurcation analysis can now be carried out as in Section 3 making extensive use of the results in [14]. In particular, the existence of global invariant manifolds and a Fredholm property for the resulting bifurcation equation has been proved there.

4.3 Applications

Nishiura [personal communication] has recently observed modulated fronts of the form described in this paper in numerical simulations of reaction-diffusion systems. Modulated wave trains have been observed numerically by Ogawa [personal communication] in dissipatively perturbed generalized KdV equation. The appearance of these wave trains can probably be explained rigorously by essential instabilities of periodic wave trains; this is work in progress. As mentioned earlier, Sherratt [20] investigated, numerically and otherwise, fronts that destabilize in an essential instability. The reaction-diffusion systems he used are predator-prey models. The patterns he observed, however, are not modulated fronts but are composed of several modulated waves with different wave speeds.

Acknowledgement The main part of this work was done while we were visiting the Institute of Mathematics and its Applications (Minneapolis, USA). We thank the IMA for providing a stimulating atmosphere and for the financial support with funds provided by the National Science Foundation.

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