DYNAMICAL BEHAVIOR OF PATTERNS WITH SYMMETRY

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Abstract. Recent results on the dynamical behavior of patterns in two and three spatial dimensions are reviewed. Based upon spatio-temporal symmetries of patterns, it is shown that transitions to other patterns can be explained by analyzing low-dimensional model equations. Examples include the dynamics of periodically forced twisted scroll waves and transitions from rigidly-rotating one spiral waves to meandering or drifting spirals.

Key words. Spiral waves, twisted scroll waves, Euclidean symmetry, meandering, drifting.

AMS(MOS) subject classifications. 34C40, 35B32, 35K57.

1. Introduction. Recent efforts [1, 3, 11, 14, 23, 31, 32, 33] to understand the dynamical behavior of patterns are reviewed. Given a certain shape of the pattern, we would like to predict its dynamics and investigate transitions where the temporal behavior of patterns changes qualitatively. Patterns may evolve in time by changing their shape in a certain fashion or by translating and rotating a fixed shape in space. The latter mechanism is facilitated by the homogeneity of the underlying medium. The translations and rotations which arise in this fashion constitute the spatio-temporal symmetries of the pattern. Both mechanisms may occur simultaneously: the pattern may change its shape only to look like the original pattern translated and rotated by a certain amount at a specific later point in time. Examples of patterns are spirals and scroll waves. Patterns having a spiral-like structure emanating from their center or tip are referred to as spiral waves; see Figure 1. Scroll waves consist of one or more filaments together with planar spiral waves whose tips or cores are aligned along the filaments.

Spiral waves have been observed in various different biological, chemical, and physical systems. They occur, for instance, in the Belousov-Zhabotinsky reaction [8, 18, 22, 35, 43] and the catalysis on platinum surfaces [26]. Spiral waves have also been found in convection in cylindrical shells [29] and, quite recently, in vertically vibrating granular layers [37]. Another interesting problem, where spiral waves play an important role, are fibrillations in cardiac tissue. There are indications that meandering spiral waves are related to certain heart failures. See the recent focus issue

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[9] and [17] for more details; results of in vitro experiments can be found, for instance, in [20]. We refer to [25, 38] for references to other systems in which spiral waves or other related patterns have been observed.

Understanding the formation of patterns involves many more issues other than their spatio-temporal symmetries; it is important to understand when patterns arise in experiments or numerical simulations and what their shape might be. Defects, which may occur in homogeneous media, may have to be taken into account. The generation of spiral waves in mathematical models such as reaction-diffusion systems, Ginzburg-Landau equations or $\lambda - \omega$ systems is often analyzed using the dynamics of their interfaces and utilizing slow-fast structures; see [36] for a review.

The organization of this paper is as follows. In Section 2, we focus on spiral wave patterns in two-dimensional media. Section 3 contains results on waves in three dimensions. We conclude with a brief summary in Section 4.

2. Patterns in two dimensions: spiral waves. The simplest possible motions of spiral waves in the plane are rigid rotations. A *rigidly-rotating* spiral wave is periodic in time; in the laboratory frame, the spiral tip moves on a circle with uniform angular velocity while the spiral wave rotates about its tip with the same velocity; see Figure 1. Therefore, the pattern is stationary in an appropriate rotating coordinate frame. *Meandering* or *drifting* spiral waves are slightly more complicated. The motion of a meandering wave is quasi-periodic in the laboratory and time-periodic in a co-rotating frame. Its tip traces out a flower pattern with inward or outward petals, see Figure 2, while the spiral rotates quasi-periodically about the tip. Drifting spiral waves arise if the petality of the flower pattern changes from inward to outward. At such a transition point, the radius of the circle traced out by the tip tends to infinity and the spiral-wave tip drifts along a line while oscillating about it. A drifting spiral wave is time-periodic in a suitable moving frame. Meandering and drifting patterns are also often referred to as modulated rotating or travelling waves. We are interested in the transition from rigidly-rotating to either meandering or drifting spiral waves. Note that this transition seemingly does not involve changes in the shape of the pattern but is seen in the qualitatively different nature of the tip dynamics.

The transition outlined above generates quasi-periodic behavior of the spiral tip which involves two frequencies. There is the possibility of other secondary transitions which lead to modulated waves with three or more frequencies. This route to more complicated temporal behavior is facilitated by adding periodic forcing to the system.

2.1. The Euclidean symmetry of the plane. We consider patterns arising in a homogeneous two-dimensional medium described by the

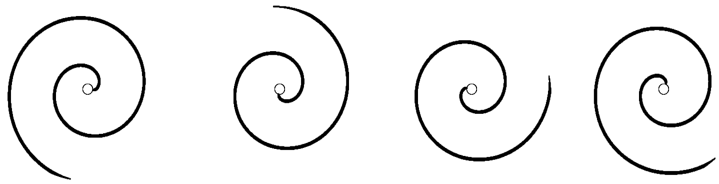

FIG. 1. *Temporal evolution of rigidly-rotating spiral waves. Plotted are those values of the spatial variable x for which one of the components of the spiral-wave solution $u(x,t)$ is larger than a certain constant. The small circles are not part of the spiral wave; they correspond to the motion of the tip, i.e. the center or core of the spiral wave. The spiral waves and their tip paths plotted here and in Figure 2 were computed numerically using the package* EZ-SPIRAL *written by Barkley [5].*

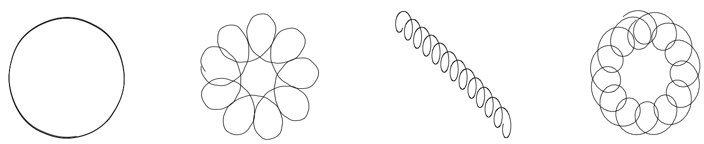

FIG. 2. *The patterns traced out by the tips of rigidly-rotating, meandering (outward petals), drifting, and meandering (inward petals) spiral waves (from left to right).*

reaction-diffusion equation

$$u_t = D\Delta u + f(u,\mu), \qquad x \in \mathbb{R}^2,\ u \in \mathbb{R}^N, \tag{2.1}$$

where the diffusion matrix D is diagonal with positive entries. Also, $\mu \in \mathbb{R}^p$ represents parameters present in the system. Each solution to (2.1) is given by a function $u(x,t)$ where the vector $u(x,t) \in \mathbb{R}^N$ contains the values of certain chemical or physical quantities such as concentrations or membrane potentials at the point $x \in \mathbb{R}^2$ in the medium and at time t. Note that (2.1) may be bistable, excitable or oscillatory; even though the mechanisms creating spiral waves may depend upon the type of the equation, the approach presented here to explain their dynamical properties does not, as we shall see below. Throughout, we identify the real plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

Homogeneity implies that any solution behaves in the same fashion if we move it to a different location in the medium and rotate it about its center. Rotating the pattern $u(x,t)$ by the angle $\varphi \in S^1$ about $x=0$, and subsequently translating it by the vector $a \in \mathbb{R}^2$, which moves a point x to $x+a$, results in a new solution to (2.1) given by

$$[(\varphi,a)u](x,t) := \tilde{u}(x,t) = u(\mathrm{e}^{-\mathrm{i}\varphi}(x-a),t); \tag{2.2}$$

see Figure 3. The set of all rotations and translations $(\varphi,a) \in S^1 \times \mathbb{R}^2$ constitutes the Euclidean symmetry group SE(2) of the plane. The combined effect of translating and rotating a solution first by (φ,a) and then

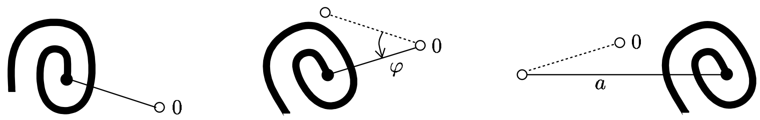

FIG. 3. *The effect of an element (φ, a) on a pattern.*

by $(\tilde{\varphi}, \tilde{a})$ is expressed by the group multiplication

$$(\tilde{\varphi}, \tilde{a})(\varphi, a) = (\tilde{\varphi} + \varphi, \tilde{a} + \mathrm{e}^{\mathrm{i}\tilde{\varphi}} a) \tag{2.3}$$

on SE(2), which can be derived by applying (2.2) twice.

2.2. Rigidly-rotating spiral waves. We are interested in patterns $u(x,t)$ whose time evolution is given by rotating and translating a fixed function $u_*(x)$ by an angle $\varphi(t)$ and a vector $a(t)$, respectively, so that

$$u(x,t) = [(\varphi(t), a(t))u_*](x) = u_*(\mathrm{e}^{-\mathrm{i}\varphi(t)}(x - a(t))) \tag{2.4}$$

for all x and t. It follows that $(\varphi(t), a(t))$ satisfies a certain ordinary differential equation

$$(\dot{\varphi}(t), \dot{a}(t)) = G(\varphi(t), a(t)). \tag{2.5}$$

Homogeneity of the medium imposes considerable restrictions on the time-dependence of $(\varphi(t), a(t))$. Indeed, we had seen that $[(\tilde{\varphi}, \tilde{a})u](x,t)$ satisfies (2.1) for any fixed $(\tilde{\varphi}, \tilde{a})$ whenever $u(x,t)$ does. Hence, rotating and translating the pattern u appearing in (2.4) by $(\tilde{\varphi}, \tilde{a})$ and exploiting (2.5), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\varphi}, \tilde{a})(\varphi(t), a(t)) = G((\tilde{\varphi}, \tilde{a})(\varphi(t), a(t)))$$

for any fixed $(\tilde{\varphi}, \tilde{a}) \in \mathrm{SE}(2)$. Using the group multiplication (2.3) in SE(2), we can evaluate the left and right-hand sides of the above equation and obtain

$$(\dot{\varphi}(t), \mathrm{e}^{\mathrm{i}\tilde{\varphi}}\dot{a}(t)) = G(\tilde{\varphi} + \varphi, \tilde{a} + \mathrm{e}^{\mathrm{i}\tilde{\varphi}} a).$$

Since $(\tilde{\varphi}, \tilde{a})$ is arbitrary, it can be shown that there exist fixed numbers $\omega_* \geq 0$ and $a_* \in \mathbb{C}$ such that

$$G(\varphi, a) = (\omega_*, \mathrm{e}^{\mathrm{i}\varphi} a_*).$$

Therefore, the equation for (φ, a) is given by

$$\dot{\varphi} = \omega_*, \qquad \dot{a} = \mathrm{e}^{\mathrm{i}\varphi} a_* \tag{2.6}$$

which has the solution

$$(\varphi(t), a(t)) = (\omega_* t, (1 - \mathrm{e}^{\mathrm{i}\omega_* t})\mathrm{i}a_*/\omega_*).$$

Note that $u(x,0) = u_*(x) = [(\varphi(0), a(0))u_*](x)$ so that the initial condition is $(\varphi(0), a(0)) = 0$. We remark that the actual value of ω_* is determined by the specific reaction-diffusion equation at hand. For $\omega_* = 0$, we obtain a travelling wave

$$u(x,t) = u_*(x - a_* t)$$

since $(\varphi(t), a(t)) = (0, a_* t)$. If $\omega_* > 0$, then the associated pattern is given by

$$u(x,t) = u_*(\mathrm{e}^{-\mathrm{i}\omega_* t}(x - (1 - \mathrm{e}^{\mathrm{i}\omega_* t})\mathrm{i}a_*/\omega_*)).$$

The core of the spiral centered at $x = 0$ for $t = 0$ moves along a circle with center $\mathrm{i}a_*/\omega_*$. Hence, upon moving the origin of the coordinate system to $\mathrm{i}a_*/\omega_*$, we obtain

$$u(x,t) = [(\omega_* t, 0)u_*](x) = u_*(\mathrm{e}^{-\mathrm{i}\omega_* t}x).$$

This pattern is rigidly-rotating with temporal period equal to $2\pi/\omega_*$.

We reformulate the results obtained above in a slightly more abstract way. The differential equation (2.6) lives on the group SE(2). Any equivariant equation on the group is given by right multiplication with a fixed element in the Lie algebra se(2) of SE(2). The solution $(\varphi(t), a(t))$ reflects the temporal evolution $u(x,t)$ of the pattern $u_*(x)$; the solution $u(x,t)$ is contained in the group orbit $\{[(\varphi, a)u_*];\ (\varphi, a) \in \mathrm{SE}(2)\}$ of u_*.

2.3. Transitions to meandering or drifting patterns. Barkley [2, 3] proposed, and verified numerically, that Hopf bifurcations from rigidly-rotating spiral waves cause transitions to meandering or drifting spirals. Such transitions have been observed experimentally, for instance, in [22, 35].

Suppose that $u(x,t) = [(\omega_* t, 0)u_*](x)$ is a rigidly-rotating spiral wave with temporal period $2\pi/\omega_*$. It is convenient to cast (2.1) in a coordinate frame which rotates with frequency ω_* so that, in the new coordinates, (2.1) is given by

$$u_t = D\Delta u + \omega_* \partial_\varphi u + f(u, \mu). \tag{2.7}$$

Note that the rigidly-rotating spiral wave u_* is then an equilibrium of (2.7). We linearize (2.7) about this pattern at $\mu = 0$ and obtain the operator

$$L_* = D\Delta + \omega_* \partial_\varphi + \mathrm{D}_u f(u_*(x), 0).$$

HYPOTHESIS 1. *The spectrum of L_* considered in the space L^2 has $n+3$ isolated eigenvalues on the imaginary axis, counted with multiplicity, for some $n \geq 0$, and the rest of the spectrum is contained strictly in the left half-plane.*

We emphasize that $\lambda = 0$ and $\lambda = \pm\mathrm{i}\omega_*$ are always eigenvalues of L_* due to the Euclidean symmetry group. These eigenvalues correspond to the

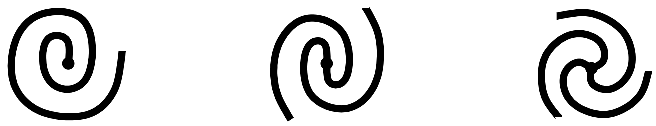

FIG. 4. *Schematic pictures of one-armed, two-armed, and three-armed spiral waves. The spiral wave to the left does not have any internal symmetry; any rotation except the identity changes the pattern. The two-armed spiral in the center is transformed into itself by the rotation by π. Similarly, the three-armed spiral to the right has the internal symmetries consisting of rotations by $2\pi/3$ and $4\pi/3$. Hence, the isotropy subgroups are $\{\mathrm{id}\}$, $\mathbb{Z}_2$, and $\mathbb{Z}_3$, respectively.*

derivatives of $(\varphi, a)u_*$ with respect to φ and a. Throughout, we denote the generalized eigenspace associated with the remaining n eigenvalues which are not related to the symmetry by V_*; note that $V_* \cong \mathbb{R}^n$. Transitions to meandering and drifting spirals arise due to the interaction of the n eigenvalues in V_* with the Euclidean symmetry group SE(2), as we shall see below.

It is important to take symmetries of the pattern u_* into account. Upon rotating the pattern u_* by an angle $\varphi \neq 0$ and translating it by $a \in \mathbb{R}^2$, the pattern may not change; see Figure 4. The set of all such pairs (φ, a) is called the *isotropy subgroup* Σ_* of u_*; it is defined by

$$\Sigma_* = \{(\varphi, a) \in \mathrm{SE}(2);\ u_*(\mathrm{e}^{-\mathrm{i}\varphi}(x-a)) = u_*(x) \text{ for all } x \in \mathbb{R}^2\} \subset \mathrm{SE}(2).$$

It has been shown [31] that the isotropy subgroup does not contain any translations and is not equal to S^1 provided u_* is not a stationary state and satisfies Hypothesis 1; in other words, the isotropy consists of finitely many rotations:

$$\Sigma_* = \{(2\pi k/\ell, 0);\ k = 0, \dots, \ell - 1\} = \mathbb{Z}_\ell$$

for some $\ell \in \mathbb{N}$. A spiral wave with isotropy $\mathbb{Z}_\ell$ is called ℓ-armed; see Figure 4.

We are interested in patterns which are, for any fixed time, close to $(\varphi, a)u_*$ for some (φ, a).

THEOREM 2.1 ([11, 32, 33]). *Assume that the rigidly-rotating spiral wave u_* satisfies Hypothesis 1. Any pattern $u(x,t)$ of (2.1) for μ close to zero which is close to u_*, or a rotated and translated version of it, is of the form*

$$\begin{aligned} u(x,t) &= [(\varphi(t), a(t))(u_* + v(t) + \mathrm{O}(|v(t)|^2 + |\mu|))](x) \\ &= u_*(\mathrm{e}^{-\mathrm{i}\varphi(t)}(x - a(t))) + v(\mathrm{e}^{-\mathrm{i}\varphi(t)}(x - a(t)), t) \\ &\quad + \mathrm{O}(v(\mathrm{e}^{-\mathrm{i}\varphi(t)}(x - a(t)), t)|^2 + |\mu|) \end{aligned} \tag{2.8}$$

where $v(t) \in V_*$ for all t. Moreover, $(\varphi, a, v)(t)$ satisfies the equation

$$\dot\varphi = \omega_* + G_1(v, \mu), \qquad \dot a = \mathrm{e}^{\mathrm{i}\varphi} G_2(v, \mu), \qquad \dot v = G_3(v, \mu) \tag{2.9}$$

for some nonlinearity $(G_1, G_2, G_3)(v, \mu)$ *such that* $(G_1, G_2, G_3)(0, \mu) = 0$ *and*

$$(G_1, G_2, G_3)((\hat{\varphi}, 0)v, \mu) = (G_1, \mathrm{e}^{\mathrm{i}\hat{\varphi}} G_2, (\hat{\varphi}, 0) G_3)(v, \mu) \tag{2.10}$$

for all $(\hat{\varphi}, 0) \in \Sigma_*$.

In other words, the finite-dimensional model (2.9) characterizes the temporal behavior of any pattern which, for any given time t, is close to the rigidly-rotating spiral u_*. The model lives on the space $\mathrm{SE}(2) \times V_*$. The first two equations for (φ, a) determine the location of the spiral tip and the angle of rotation about it, while the last equation for $v \in V_* \cong \mathbb{R}^2$ determines the change of the shape of the spiral wave. An important feature of (2.9) is its *skew-product* nature, meaning that the last equation for v decouples from the first two equations on the group. This decomposition allows us to investigate the dynamics of v separately and to apply standard results [16] for systems with symmetries to the equation for v. The dynamics of patterns near u_* can be extracted upon substituting the solutions $(\varphi(t), a(t))$ and $v(x, t)$ of (2.9) into (2.8). Note that the model reflects the symmetries Σ_* of the rigidly-rotating wave u_* which act upon the eigenfunctions in the eigenspace V_*; see (2.10).

We expect that the change of the shape due to the solution $v(x, t)$ does not affect the dynamical behavior; this is certainly true if the spiral wave has sharp transition fronts. We refer to [15] for a more thorough discussion of this issue. Thus, the important factor is the time-behavior of the drift $(\varphi(t), a(t))$ along the group $\mathrm{SE}(2)$ which results in rotating and shifting the pattern in the plane in a certain fashion.

We return to the issue of meandering versus drifting. As outlined above, Barkley proposed that these transitions arise near Hopf bifurcations of the original pattern u_*. Hence, we apply Theorem 2.1 with $n = 2$ so that $V_* \cong \mathbb{R}^2$, denoting the Hopf eigenvalues of L_* on the imaginary axis by $\lambda_H = \pm \mathrm{i}\omega_H$ for some $\omega_H > 0$. Any bifurcating pattern is then described by equation (2.9).

First, we suppose that the rigidly-rotating spiral wave u_* at $\mu = 0$ is one-armed so that $\Sigma_* = \{\mathrm{id}\}$. The nonlinearity $G(v, \mu)$ has then no additional structure, that is, (2.10) does not give any information. We assume that the equation for v

$$\dot{v} = G_3(v, \mu)$$

describes a generic Hopf bifurcation. Hence, it has a small periodic solution $v(t)$ with frequency close to ω_H. Substituting this solution into the equation for φ, we get

$$\dot{\varphi} = \omega_* + G_1(v(t), \mu),$$

which we can integrate. This yields $\varphi(t) = \omega_* t + \tilde{v}(t)$ where $\tilde{v}(t)$ is periodic with frequency $\omega_H(\mu)$ close to ω_H. Finally, substituting $\varphi(t)$ and $v(t)$ into the second equation for a and integrating, we obtain

$$a(t) = a_0 + \int_0^t \mathrm{e}^{\mathrm{i}\omega_* t}\mathrm{e}^{\mathrm{i}\tilde{v}(t)} G_2(v(t), \mu)\, \mathrm{d}t.$$

Expanding the term $\mathrm{e}^{\mathrm{i}\tilde{v}(t)} G_2(v(t), \mu)$ into a Fourier series, we get

$$a(t) = a_0 + \sum_{k=-\infty}^{\infty} B_k \frac{\mathrm{e}^{\mathrm{i}(\omega_* + k\omega_H(\mu))t} - 1}{\mathrm{i}(\omega_* + k\omega_H(\mu))}.$$

Therefore, the translation $a(t)$ is bounded, and in fact quasi-periodic in t, as long as $\omega_* + k\omega_H(\mu) \neq 0$ is bounded away from zero uniformly in $k \in \mathbb{Z}$. The resulting pattern is meandering. If, however,

$$\omega_* + \tilde{k}\omega_H(\mu) = 0 \tag{2.11}$$

for some $\tilde{k} \in \mathbb{Z}$, then we have

$$a(t) = a_0 + B_{\tilde{k}} t + \sum_{k \neq \tilde{k}} B_k \frac{\mathrm{e}^{\mathrm{i}(\omega_* + k\omega_H(\mu))t} - 1}{\mathrm{i}(\omega_* + k\omega_H(\mu))}.$$

The tip of the associated spiral wave moves in an oscillatory fashion along the direction $B_{\tilde{k}}$ towards infinity. Hence, the spiral wave is drifting. These calculations show that drifting arises as a consequence of a resonance between the Hopf frequency ω_H and the frequency ω_* of the rigidly-rotating spiral wave.

For multi-armed spiral waves, meandering or drifting need not occur [14, 11]. To see this, note that from (2.10), the nonlinearity G_2 appearing in the equation

$$\dot{a} = \mathrm{e}^{\mathrm{i}\varphi} G_2(v, \mu)$$

for the translation satisfies

$$G_2((\hat{\varphi}, 0)v, \mu) = \mathrm{e}^{\mathrm{i}\hat{\varphi}} G_2(v, \mu) \tag{2.12}$$

for any element $(\hat{\varphi}, 0) \in \Sigma_*$. Recall that $v \in V_*$ corresponds to an element in the generalized eigenspace of the linearization about the spiral wave. Suppose that there exists a rotation $(\hat{\varphi}, 0) \in \Sigma_*$ in the isotropy subgroup of u_* with $\hat{\varphi} \neq 0$ such that

$$[(\hat{\varphi}, 0)v](x) = v(\mathrm{e}^{-\mathrm{i}\hat{\varphi}} x) = v(x) \tag{2.13}$$

for any $v \in V_* \cong \mathbb{R}^2$. Due to (2.12), we then have

$$G_2(v, \mu) = \mathrm{e}^{\mathrm{i}\hat{\varphi}} G_2(v, \mu)$$

and therefore $G_2(v,\mu) = 0$. Hence, $\dot{a} = 0$, and neither meandering nor drifting can occur. In other words, if the symmetries of the eigenfunctions v contain symmetries of the underlying spiral-wave pattern u_*, then meandering or drifting cannot occur. For instance, for two-armed spirals, drifting does not occur if the Hopf eigenfunctions are even.

The results discussed above can also be interpreted as follows. We had seen that rigidly-rotating spirals are described by a differential equation on the group SE(2). At a Hopf bifurcation, this differential equation is no longer autonomous but periodically forced. The associated Poincare map is given by left multiplication with some fixed group element. It can be seen from (2.3) that iterated multiplications of an element in SE(2) yield bounded translation components except when its initial rotation component is the identity.

2.4. Periodic forcing of meandering spiral waves. Finally, we investigate periodic forcing of meandering spiral waves in order to illustrate the route to more complicated tip motions; see [23, 33]. The reaction-diffusion equation with small periodic forcing is given by

$$u_t = D\Delta u + f(u) + \mu\, g(t), \qquad x \in \mathbb{R}^2,\ u \in \mathbb{R}^N, \tag{2.14}$$

where $g(t)$ is periodic with frequency Ω_*. Let $u(x,t)$ with $u(x,0) = u_*(x)$ be a meandering spiral wave of (2.14) with $\mu = 0$ so that

$$u(x,T) = [(\varphi_*, 0)u_*](x) = u_*(\mathrm{e}^{-\mathrm{i}\varphi_*}x)$$

for appropriate numbers φ_* and $T > 0$ possibly after changing the origin of the coordinate system. Let $\omega_* = \varphi_*/T$ denote the non-zero rotation frequency of the meandering spiral. The following result characterizes patterns near u_* under periodic forcing.

THEOREM 2.2 ([33]). *Assume that the meandering spiral wave u_* is stable. For μ close to zero, any pattern $u(x,t)$ of (2.14) which is close to u_*, or a rotated and translated version of it, can then be parametrized by (φ, a, θ) where (φ, a) correspond to the position of the spiral tip and the rotation angle of the spiral, respectively, and the variable $\theta \in [0,T]$ measures changes of the shape due to the time-dependence of the meandering spiral. Moreover, $(\varphi, a, \theta)(t)$ satisfies the equation*

$$\dot{\varphi} = \omega_* + G_1(t,\theta,\mu), \quad \dot{a} = \mathrm{e}^{\mathrm{i}\varphi} G_2(t,\theta,\mu), \quad \dot{\theta} = 1 + G_3(t,\theta,\mu) \tag{2.15}$$

for some nonlinearity $(G_1, G_2, G_3)(t,\theta,\mu)$ which has frequency Ω_* in t and obeys $(G_1, G_2, G_3)(t,\theta,0) = 0$ at $\mu = 0$.

Note that the linearization about a meandering spiral always has four neutral directions corresponding to derivatives with respect to φ, a and t. The hypothesis in the above theorem is then understood as stability in directions transverse to these neutral eigendirections.

We observe that the equation for θ in (2.15) decouples. Since this equation is periodically forced with frequency Ω_*, the solution $\theta(t)$ is quasi-periodic with frequencies Ω_* and $\omega_T := 2\pi/T$. We substitute this to solve the equation for (φ, a) and denote the solution by $(\varphi(t), a(t))$. The function $\varphi(t)$ is quasi-periodic, and its three frequencies are equal to ω_*, ω_T, and Ω_*. For one-armed spirals, we expect that, in analogy to the results presented in Section 2.3, $a(t)$ grows linearly provided

$$\omega_* = k_1\omega_T + k_2\Omega_*$$

for some $k_1, k_2 \in \mathbb{Z}$; see [23, 33] for the details. If $a(t)$ is unbounded, the spiral is drifting. Such spirals are sometimes called *generalized* drifting solutions since they are not periodic but quasi-periodic in an appropriate moving frame. They have been found in experiments [34, 43].

We remark that Hopf bifurcations of meandering spiral waves, which have been observed in numerical simulations [30], also lead to modulated waves with three frequencies [30, 33].

2.5. Discussion. As mentioned before, Barkley [3] was the first who observed the importance of the Euclidean symmetry group for meandering and drifting of spiral waves. He also proposed the five-dimensional model (2.9) for one-armed spirals and calculated the resonance condition (2.11). Wulff [42] derived the resonance condition rigorously using Lyapunov-Schmidt reduction. A rigorous reduction to a finite dimensional dynamical system was achieved in [31, 32]. Simultaneously, a formal center-manifold reduction was proposed in [6]. The normal form (2.9) for multi-armed spiral waves was uncovered in [11] and investigated in [14]. Refined normal-form equations with Euclidean symmetry were studied in [12]. Finally, reductions near meandering spirals were investigated in [33].

The major difficulty in obtaining rigorous results in this area is that the Euclidean symmetry group is non-compact and may act discontinuously on functions. Krupa [21] obtained similar reduction results for compact groups with smooth representations.

We remark that the reduction outlined in Theorem 2.1 applies also to inhomogeneous or symmetry-breaking perturbations of spiral waves. The only necessary adjustment in this case occurs because the equivariance properties (2.10) are no longer obeyed. Instead, the finite-dimensional model corresponding to (2.9) is given by

$$\dot{\varphi} = \omega_* + \mu G_1(\varphi, a), \qquad \dot{a} = \mu\, \mathrm{e}^{\mathrm{i}\varphi} G_2(\varphi, a), \tag{2.16}$$

where μ measures the strength of the perturbation. Analogous results hold near meandering spiral waves.

3. Patterns in three dimensions: scrolls. In this section, we investigate the dynamics of patterns in three dimensions under homogeneous periodic forcing governed by the reaction-diffusion system

$$u_t = D\Delta u + f(u) + \mu\, g(t), \qquad x \in \mathbb{R}^3, \tag{3.1}$$

where $g(t)$ is periodic in t with frequency Ω_*.

Scrolls consist of one or more filaments together with planar spiral waves whose tips or cores are attached to the filaments. They have been studied, for instance, in [40, 41]; see also [38]. In order to explain their dynamical behavior, dynamics for their filaments [7, 19, 27, 39] have been postulated and investigated. Instead, following the reasoning in the last section, we shall exploit their spatio-temporal symmetries.

3.1. Symmetries in three dimensions. Since the medium and the forcing modeled by (3.1) are homogeneous, the dynamics of waves do not depend upon their location. Similar to the situation in two dimensions, we may rotate and translate patterns arbitrarily in space without changing their dynamical behavior. Mathematically, this is expressed by the fact that

$$[(R,S)u](x,t) := \tilde{u}(x,t) = u(R^{-1}(x-S),t)$$

satisfies (3.1) whenever $u(x,t)$ does; see also (2.2). Here, R is an arbitrary rotation in $\mathbb{R}^3$, and $S \in \mathbb{R}^3$ is a translation. The set of all rotations and translations in the three-dimensional space is called the Euclidean symmetry group SE(3).

In contrast to the situation in the plane, however, it is not always the full group SE(3) but only a smaller set of symmetries which affects the dynamics of waves in three dimensions. To illustrate this fact, consider a fixed pattern $u(x,t)$ with $u(x,0) = u_*(x)$. It is reasonable to assume that translations act continuously upon $u_*(x)$. This means that the values $u_*(x)$ and $u_*(x-S)$ are close together *uniformly* in x for any fixed sufficiently small translation S. Next, consider rotations R by the angle φ about an axis $r \in \mathbb{R}^3$. We say that the axis r is *admissible* if the values $u_*(x)$ and $u_*(R^{-1}x)$ are close together *uniformly* in x for any sufficiently small, fixed angle φ. The *effective symmetry group* Λ_* of u_* consists then of all translations and all rotations about admissible axes. The latter are also referred to as admissible rotations. We emphasize that this set may not contain all rotations, and examples are given below.

Next, consider the temporal evolution $u(x,t)$ of the pattern $u_*(x)$. For any (R,S) which is not an effective symmetry, the patterns $(R,S)u_*$ are not close to the initial value u_* even if the translation S and the angle associated with the rotation R are arbitrarily small. Therefore, the solution $u(x,t)$ is also far away from $(R,S)u_*$ for small times t. In other words, the solution $u(x,t)$ can only reach the patterns $(R,S)u_*$ where (R,S) is admissible.

3.2. Twisted scroll waves. Scroll waves have spiral waves in each horizontal plane such that the tips of the spirals are aligned along the vertical axis. The spiral waves in the horizontal planes may differ by a phase, that is, they are rotated against each other. A twisted scroll wave has the following internal symmetry: shifting the scroll wave along the vertical axis and rotating at the same time in the horizontal plane does not change the pattern. In other words, its isotropy group is $\Sigma_* = S^1$.

We expect that rotations about the vertical filament are admissible; they act upon the spiral waves in the horizontal plane in the same fashion as the rotations in Section 2. Rotations about any other axis, however, are most likely not admissible, as first pointed out in [11, 33]. Indeed, any such rotation moves the tips of those spirals which are located far away from zero arbitrarily far away from the vertical filament. Hence, the effective symmetry group Λ_* of a twisted scroll wave consists of the rotations about the vertical axis and all translations, i.e.

$$\Lambda_* = \mathrm{SE}(2) \times \mathbb{R} \subset \mathrm{SE}(3).$$

Each element of Λ_* can be written as (φ, a, ψ) where $(\varphi, a) \in \mathrm{SE}(2)$ denote rotations and translations in the horizontal planes and $\psi \in \mathbb{R}$ denotes vertical translations.

Without periodic forcing, twisted scroll waves $u(x,t)$ rotate about the vertical filament with constant frequency ω_*; the spiral waves in the horizontal plane are rigidly-rotating [1]. Therefore, more formally, we have

$$u(x,t) = u_*(R_{\mathrm{vert}}(-\omega_* t)x),$$

where $R_{\mathrm{vert}}(\varphi)$ denotes the rotation about the vertical (admissible) axis by the angle φ.

The finite-dimensional reduction [33] under periodic forcing is given by

$$\dot{\varphi} = \omega_* + G_1(t,\mu), \qquad \dot{a} = \mathrm{e}^{\mathrm{i}\varphi} G_2(t,\mu), \qquad \dot{\psi} = G_3(t,\mu).$$

However, the isotropy group S^1 of the twisted scroll wave implies that $G_2(t,\mu) = 0$. Hence, $\dot{a} = 0$, and the spiral waves in the horizontal planes can neither meander nor drift. Instead, their motion is a non-uniform rotation by a quasi-periodic angle $\varphi(t)$ with frequencies ω_* and Ω_* about the vertical axis. In addition, the pattern may oscillate periodically with frequency Ω_* along the vertical axis.

3.3. Twisted scroll rings. Twisted scroll rings are closely related to the scroll waves introduced in the previous section. The difference is that scroll rings have their cores aligned along a circle rather than along a vertical filament. Thus, the spatial pattern consists of a core circle together with a spiral wave in each plane normal to the circle so that the tips of the spirals lie on the core circle. In addition, the spiral waves may have a phase difference along the core circle. If we move along the circle by an angle φ and transport a spiral with us, then we have to rotate the spiral wave inside the normal plane by an angle $\ell\varphi$ for a certain fixed integer ℓ in order to match it with the spiral attached to the new core point; see Figure 5. Such patterns are called *ℓ-armed twisted scroll rings.* Simply-twisted scroll rings

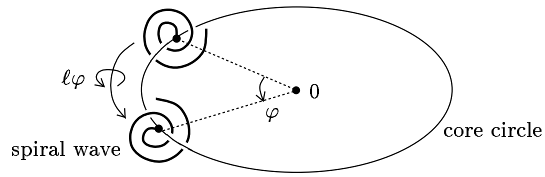

FIG. 5. *Schematic picture of an ℓ-armed twisted scroll wave.*

correspond to the case $l = 0$. We may assume that the circle along which the cores are aligned is contained in the horizontal plane.

Based upon the results in [1], we expect that simply-twisted scroll rings typically drift along a direction other than the vertical axis *provided* the effective symmetry group of the scroll ring is the full Euclidean symmetry group SE(3). The same behavior is then bound to occur under periodic forcing [33]. The direction of drift varies arbitrarily in μ regardless of resonances between the rotation frequency ω_* of the scroll ring and the forcing frequency Ω_* of the periodic forcing.

However, we have argued in [33] that the effective symmetry should be equal to the smaller group $\mathrm{SE}(2) \times \mathbb{R}$. In this situation, only the rotations about the vertical axis are admissible. This has been confirmed recently by numerical simulations [24]. If rotations about axes different from the vertical axis act discontinuously, then the scroll ring drifts along the vertical axis. In fact, without periodic forcing, twisted scroll rings rotate with frequency ω_* about the vertical axis, say, while drifting with constant velocity c_* along the vertical axis. Therefore,

$$u(x,t) = u_*(R_{\mathrm{vert}}(-\omega_* t)x + c_* t\, \mathrm{e}_{\mathrm{vert}})$$

where $R_{\mathrm{vert}}(\varphi)$ denotes the rotation about the vertical axis by the angle φ and $\mathrm{e}_{\mathrm{vert}}$ is the unit vector in the vertical direction. These patterns have been observed in numerical simulations of reaction-diffusion systems in three dimensions [10, 28].

Next, we assume $\mu \neq 0$ so that the system is periodically forced. The spirals in the vertical planes then start to meander. In the case of ℓ-twisted scroll rings with $\ell > 1$, drift is only possible along the vertical symmetry axis of the scroll ring [33]. Simply-twisted scroll rings may exhibit additional slow horizontal drift at resonances where $\omega_* = k\Omega_*$; see [33].

Finally, we remark that similar phenomena occur if a twisted scroll wave destabilizes due to a Hopf bifurcation. We refer to [11, Section 6] for the details.

4. Summary. Recent results [1, 3, 6, 11, 14, 23, 31, 32, 33] offer an alternative approach to analyzing the dynamical behavior of patterns in the plane or in three-dimensional space. Similar results for compact groups were derived much earlier [13, 21]. The approach is based entirely on the homogeneity of the underlying equation and the resulting spatio-temporal symmetry properties of solutions. With only this knowledge at hand, patterns and their transitions to various qualitatively different forms of motion can be investigated by analyzing finite-dimensional model equations which were derived rigorously. Transition points can be calculated, at least numerically, by computing eigenvalues and the associated eigenfunctions of the linearized equation about the pattern. The internal symmetries of the pattern and the eigenfunctions determine which transition occurs. An important observation is that we have to restrict the symmetries associated with a pattern to a smaller group of effective symmetries. This contains only those translations and rotations which are admissible, i.e. whose associated one-parameter family of group elements acts continuously upon the pattern. This group determines the dynamics of the pattern to a large extent. The discontinuity of the action of the Euclidean symmetry group upon functions therefore seems not to be a mathematical artefact.

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