Math 8501, Fall 2009

Theory of Ordinary Differential Equations

Arnd Scheel, VinH 251, phone 625-4065, scheel@math.umn.edu

— HA 4, due Friday, November 20, 10:00 a.m. —

1) Consider a differential equation $\dot{x} = f(x)$ with scaling symmetry of order p, $f(\lambda x) = \lambda^p f(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$. Write $0 \neq x = r \cdot u$, with $r > 0$ and $u \in S^{n-1}$ and derive an equation for r and u. Show that after introducing a suitable Euler-multiplier, the equation for u is independent of r. What do equilibria in the u-equation correspond to? Use these results to find examples of cubic vector fiels in $\mathbb{R}^2$ (p=3) so that

   (i) all trajectories converge to the origin;

   (ii) all trajectories converge to infinity;

   (iii) an open set of trajectories converges to infinity and an open set of trajectories converges to the origin.

   Draw the phase portrait in the third example. What happens if you perturb $\dot{x} = f(x) + g(x)$, and $g(x) = \mathrm{o}(|x|^p)$?

2) Suppose that the two flows Φ_t and Ψ_t with vector fields f and g, respectively, are C^1-conjugate near the equilibrium $x = 0$: there is a local C^1-diffeomorphism T, $T(0) = 0$, such that $\Phi_t \circ T = T \circ \Psi_t$ for all $t \in \mathbb{R}$. Show that the spectra of $Df(0)$ und $Dg(0)$ coincide.

3) Consider the nonlinear damped pendulum

$$x' = y, \qquad y' = -\mu y + x - x^3,$$

   for damping parameter values $\mu > 0$,small. Show that almost all solutions (in the sense of Lebesgue measure) inside a ball $|(x,y)| \leq R$ converge to either $(1,0)$ or $(-1,0)$. Show that the boundary of the basin of attraction for those two equilibria is given by the stable manifold of the origin. Compute this stable manifold numerically by integrating a local approximation backwards. Describe the shape of this manifold for small $\mu = 0.2, 0.1, 0.05, 0.02, 0.01$.

4) Consider the linear equation $\dot{x} = Ax$, $A = \mathrm{diag}\,(\lambda_j)$, $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. Find all equilibria and heteroclinic orbit of the induced flow on the unit sphere, that is, write $x = u \cdot |x|$ and derive an equation for $u \in S^{n-1}$. Then find all equilibria and all heteroclinic orbits in the equation for u.

   2 bonus points: Can you describe equilibria and heteroclinic orbits for the (non-self-adjoint) $A = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}$ for all $\mu \in \mathbb{R}$?

5) Consider again the linear equation $\dot{x} = Ax$, with $A = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, and $\lambda_1 > \ldots > \lambda_n$. Let $x_1, \ldots, x^k$ be k linearly independent vectors in $\mathbb{R}^n$ and denote by E the linear subspace spanned by those vectors.

(i) Show that the linear equation generates a flow on the set of k-dimensional subspaces of $\mathbb{R}^n$.

(ii) Show that the space of subspaces is a smooth manifold (the Grassmannian) by locally writing subspaces as graphs: any subspace F "near" E can be written as a graph of a linear map $h(F) : E \mapsto E^\perp$.

(iii) Show that the flow on subspaces is a smooth flow.

(iv) Determine all equilibria of this flow. Find the (unique) stable equilibrium.

All exercises 4 points!