Math 8502, Spring 2003

Dynamical Systems and Differential Equations

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— HA 2, due Friday, February 21, 11 a.m. —

1) Let $q(t) \to 0$ be a solution in the stable manifold of the hyperbolic equilibrium $x = 0$ of $\dot{x} = f(x)$. Assume that $\lambda^{\mathrm{s}} < 0$ is an algebraically simple eigenvalue of $Df(0)$ and for all other eigenvalues λ we have either $\mathrm{Re}\,\lambda > 0$ of $\mathrm{Re}\,\lambda < \lambda^{\mathrm{s}}$. Show that there are constants q_1 and $\delta > 0$ such that

$$q(t) = q_1 \mathrm{e}^{\lambda^{\mathrm{s}} t} + \mathrm{O}(\mathrm{e}^{(\lambda^{\mathrm{s}}-\delta)t}) \text{ for } t \to \infty.$$

   (i) To prove this, restrict the flow to the stable manifold and project on the stable subspace: the stable manifold is given as a graph $x_{\mathrm{u}} = h(x_{\mathrm{s}})$, and we find an equation for x_{s}

   $$\dot{x}_{\mathrm{s}} = P^{\mathrm{s}} f(x_{\mathrm{s}} + h(x_{\mathrm{s}}).$$

   Show that $x_{\mathrm{s}} = 0$ is a hyperbolic, asymptotically stable equilibrium for this flow. Consider the equation for x_{s} from now on and omit superscripts.

   (ii) Consider $\dot{q}(t)$, the solution to the linearized equation

   $$\dot{x} = A(t)x, \quad A(t) = Df(q(t)) = Df(0) + \mathrm{O}(\mathrm{e}^{-\delta t}) \text{ for } t \to \infty,$$

   with some $\delta > 0$. Choose a constant η such that $\eta + \lambda^{\mathrm{s}} > 0$, but $\eta + \lambda < 0$ for all other $\lambda \in \mathrm{spec}\, Df(0)$ with $\mathrm{Re}\,\lambda < 0$. Relate solutions $x(t)$ to solutions $y(t)$ of

   $$\dot{y} = A(t)y + \eta y.$$

   Show next that there exists an exponential dichotomy $\Phi^{\mathrm{u}}(t,\tau)$ and $\Phi^{\mathrm{s}}(t,\tau)$ for the y-equation, with $\mathrm{Rg}\Phi^{\mathrm{u}}(t,t)$ one-dimensional.

   (iii) Show that the projection y^{u} of solutions y in the unstable subspace onto the kernel of $Df(0) - \lambda^{\mathrm{s}}$ solves a scalar differential equation

   $$\dot{y}_{\mathrm{u}} = a(t) y_{\mathrm{u}},$$

   with $a(t) = \lambda^{\mathrm{s}} + \eta + \mathrm{O}(\mathrm{e}^{-\delta t})$, for some $\delta > 0$.

   (iv) Compute the asymptotic of solutions $y_{\mathrm{u}}(t)$ then of solutions in the unstable subspace, and then also for general solutions in the y-equation. Use (iii) to finde the expansions for solutions of the linearized x-equation. Now integrate and prove the expansion for q.

2) Consider the map $\Phi : r \mapsto \mu + dr^{1+\alpha}$, arising in a resonant homoclinic bifurcation, with $\alpha, \mu, r \geq 0$, small.

(i) For $d > 1$, show that there exists a curve $\mu = \mu(\alpha)$ such that for $\mu < \mu(\alpha)$ there are two fixed points for Φ, and for $\mu > \mu(\alpha)$, there are no fixed points.

(ii) If $d < -1$, show that there exists a unique fixed point for all values of $\mu, \alpha > 0$. Furthermore, show that in this case there are two curves $\mu_{\mathrm{pd}}(\alpha) < \mu_{\mathrm{hh}}(\alpha)$ such that for $\mu_{\mathrm{pd}}(\alpha) < \mu < \mu_{\mathrm{hh}}(\alpha)$ there exists a fixed point of $\Phi \circ \Phi$ which does not coincide with the fixed point of Φ. What happens on the curves $\mu_{\mathrm{pd}}(\alpha)$ and $\mu_{\mathrm{hh}}(\alpha)$?

(iii) Try to interpret the consequences for the resonant homoclinic bifurcation.