

Dynamical Systems and Differential Equations

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— HA 1, due Monday, February 23, 11 a.m. —

1. A continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called expansive if there exists $\kappa < 1$ so that for all $x, y \in \mathbb{R}^n$

$$\kappa|f(x) - f(y)| \geq |x - y|.$$

Is f necessarily onto?

2. Compute the kneading invariant of an infinitely renormalizable unimodal map (see Ex. 3, §1.18, in [Devaney] for detailed steps).
3. For $f : I \rightarrow I$ unimodal, assume that the critical point is periodic, $f^\ell(c) = c$ for some $\ell > 0$. Show that f is not structurally stable.
4. Let $f : I \rightarrow I$ be unimodal, continuous, and continuously differentiable in $I \setminus \{c\}$. Suppose $|f'| > 1$ in $I \setminus \{c\}$. Show that the itinerary map is one-one.
5. (C^0 -Closing Lemma) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a homeomorphism and $x \in \mathbb{R}^n$ a point so that $x \in \omega(x)$. Show that for any $\epsilon > 0$ there is a continuous g and $N > 0$ so that

$$\sup_x |g(x) - f(x)| < \epsilon, \quad \text{and} \quad g^N(x) = x.$$

6. Consider the rigid rotation $f_0 : S^1 \rightarrow S^1$, $x \mapsto x + \alpha \pmod{1}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $J = [1 - \alpha, 1)$. We define the itinerary map $\underline{i}(x)$:

$$(\underline{i}(x))_j = \begin{cases} 0, & \text{if } f_0^j(x) \in J \\ 1, & \text{if } f_0^j(x) \in S^1 \setminus J. \end{cases}$$

Show that for a suitable $k_0 \geq 0$, the symbol 0 is followed by wither $k_0 - 1$ or k_0 times the symbol 1. Consider now the return map

$$f_1 : J \rightarrow J, \quad x \mapsto f^{k(x)}(x),$$

where $k(x) \geq 1$ is minimal, so that $f^{k(x)}(x) \in J$. Show that f_1 can be viewed again as a rigid rotation of a (renormalized) circle. What is the new rotation number α_1 ?

Can you iterate the procedure? What is the sequence of rotation numbers and k_0 's? What happens when $\alpha = (\sqrt{5} - 1)/2$?

7. Plot devil's staircase numerically for $x \mapsto x + \rho + \epsilon \sin(2\pi x)$, for $\rho \in [0, 1)$ and $\epsilon = 0.01, 0.02, 0.05, 0.1, 0.15, 0.2$. Also, study the width of the period-3 window systematically as $\epsilon \rightarrow 0$.

Four correct answers required for maximal score!