Solution to HA 1, Ex. 6

Let $R_\alpha : x \mapsto x + \alpha \mod 1$, $J := [1-\alpha, 1)$. Also, define $k := [1/\alpha]$, so that $k\alpha < 1 < (k+1)\alpha$.

First note that there is $x_* \in J$ such that

$$R_\alpha^{k+1}([1-\alpha, x_*)) \subset J.$$

Moreover, x_* is readily computed as $x_* = 2 - (k+1)\alpha$.

Similarly,

$$R_\alpha^k([x_*, 1)) \subset J.$$

The first-return map to J therefore is

$$g : x \in J \mapsto \begin{cases} x + k\alpha, & x > x_* \\ x + (k+1)\alpha, & x < x_* \end{cases}$$

Next, define the renormalization operator

$$L : J \longrightarrow (0,1], \qquad x \mapsto \frac{1-x}{\alpha},$$

and the renormalized map $f_1 := L \circ g \circ L^{-1} : S^1 \mapsto S^1$. The map f_1 then clearly is a rigid rotation. A short computation gives that

$$f_1 = R_\beta, \text{ with } \beta = G(\alpha) := \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right].$$

Repeating the renormalization procedure gives a sequence of renormalized maps f_ℓ with rotation numbers α_k, so that

$$\alpha_{k+1} = G(\alpha_k).$$

The process can be repeated indefinitely if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. If $\alpha \in \mathbb{Q}$, it ends with $\alpha_k = 0$ for some k.

The map G is called the Gauss map. It possesses countably many fixed points. The numbers $k_\ell = \left[\frac{1}{\alpha}\right]$ are the denominators in a continued-fractions expansion of α. The golden mean $\alpha = \frac{\sqrt{5}-1}{2}$ is a fixed point with $k_\ell \equiv 1$.