

Dynamical Systems and Differential Equations

Arnd Scheel, VinH 251, phone 625-4065, scheel@math.umn.edu

— HA 2, due Monday, March 30, 11 a.m. —

1. Consider the piecewise linear map T on $(x, y) \in \mathbb{R}^2$, defined as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

for $x \leq 1$, and

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

for $\xi = x - 3 > -2$, and $\eta = y + 1$.

- Show that $p_- = (0, 0)$ and $p_+ = (3, -1)$ are hyperbolic fixed points.
 - Show that $q = (1, 0)$ is a heteroclinic point, that is, $T^n(q) \rightarrow p_{\pm}$ for $n \rightarrow \pm\infty$.
 - Show that $\Lambda := \overline{\bigcup_{n \in \mathbb{Z}} T^n(q)}$ is a compact hyperbolic set.
 - Analyze the differentiability of the stable subspace at accumulation points of Λ .
2. Consider the map

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda^s & 0 \\ 0 & \lambda^{ss} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

for $0 < \lambda^{ss} < \lambda^s < 1$. Show that it possesses infinitely many 'weak stable' manifolds, that is, invariant manifolds that are tangent to the eigenspace corresponding to λ^s . Assume now that $\lambda^s/\lambda^{ss} \notin \mathbb{Z}$. Determine which of those manifolds are analytic and find the differentiability properties of all other manifolds. Show that even for $\lambda^s/\lambda^{ss} \notin \mathbb{Z}$, there are infinitely many manifolds which are not C^∞ .

- Assume that the linear map T on the Banach space X satisfies $|T^n|_X \leq C\lambda^n$, for some $\lambda < 1$ and some $C > 0$. Show that for any $\sigma > \lambda$ there exists an equivalent norm $\|\cdot\|$ on X such that $\|T\| \leq \sigma < 1$.
- Find a horseshoe in the cat map!
- Find stable and unstable manifolds in the solenoid!
- Consider the operator $T : (x)_{n \in \mathbb{Z}} \mapsto (x_{n+1} - Ax_n)_{n \in \mathbb{Z}}$ on ℓ^∞ , with $x \in \mathbb{R}^N$, $A \in M^{N \times N}$. Show that T is bounded invertible if and only if A is hyperbolic. Show the same statement when the operator is considered on ℓ^2 .

7. Formulate a nontrivial finite-time shadowing lemma! You do not need to prove the result.

Four correct answers required for maximal score!