THE STRUCTURE OF TOTALLY RAMIFIED EXTENSIONS OF P-ADIC FIELDS

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Abstract. Using the field-of-norms functors, we prove a result analogous to a classical Noether's theorem about the Galois module structure of the ring of p-adic integers. We start with a tower of discrete valuation rings that is defined via the Frobenius map.

1. Introduction

The aim of this paper is to study finite Galois extensions of complete discrete valuation fields of characteristic zero and characterize a certain kind of tamely ramified extensions in terms of the vanishing of some Galois cohomology modules. By a p-adic field, we mean a complete discrete valuation field of characteristic zero with perfect residue field of characteristic $p > 0$. As a classical result, it is known that the Galois group of a finite unramified extension of p-adic fields with finite residue field is isomorphic to the cyclic group, generated by the Frobenius map on the residue fields. Fontaine and Wintenberger [3], [5] established this Galois correspondence between p-adic fields and complete discrete valuation fields of positive characteristic via the method of field-of-norms functors. Recently, Andreatta [1] generalized this Galois correspondence to higher dimensional Noetherian rings, which also extends the classical theory of (φ, Γ)-modules.

In Fontaine and Wintenberger's theory, one requires a fine study of the behavior of the ramification of extensions of p-adic fields to construct complete discrete valuation rings in positive characteristic in a functorial way. We call the resulting valuation ring a *small Fontaine ring*, which is valid only in this paper. Let L/K be a finite Galois extension of p-adic fields. Then the Galois group $G_{L/K}$ acts on the ring of integers of L. We fix a *basic sequence* $K_\bullet := (K_n \mid n \in \mathbb{N})$ (see the definition below), and let $L_\bullet := (L_n \mid n \in \mathbb{N})$ be the sequence defined by $L_n := K_n L$. Then one can define the small Fontaine ring $\mathbf{E}^+_{K_n}$ (resp. $\mathbf{E}^+_{L_n}$) for $n \geq n_0$ with some $n_0 \in \mathbb{N}$, which depends on the extension L/K. The basic properties of this ring with its naturality are contained in Proposition 2.5. Consider the Galois cohomology module:

$$H^i(G_{L_n/K_n}, \mathbf{E}^+_{L_n})$$

for $n \geq n_0$ with some $n_0 \in \mathbb{N}$. Then we make use of a fact that the Galois cohomology is annihilated by the image of the trace map $\mathrm{Tr} : \mathbf{E}^+_{L_n} \to \mathbf{E}^+_{K_n}$, which is non-zero as the extension is Galois (Proposition 2.5, (3)). The following theorem is well-known:

Theorem 1.1 (E. Noether)**.** *The extension L/K is tamely ramified if and only if the $\mathfrak{o}_K[G]$-module $\mathfrak{o}_L$ is projective.*

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Recall that a finite extension L/K of p-adic fields is called tamely ramified if $p \nmid [L : K]$. In the situation of the above theorem, all the higher Galois cohomology groups of the ring of integers $\mathfrak{o}_L$ are zero. Our next result is regarded as a variant of Noether's theorem.

Theorem 1.2. *Suppose L/K is a totally ramified cyclic extension. If $K_\bullet$ is a basic sequence, then the following conditions are equivalent:*

(1) *$H^i(G_n, \mathbf{E}^+_{L_n})$ is zero for all $i > 0$ and $n \geq n_0$.*
(2) *$H^i(G_n, \mathbf{E}^+_{L_n})$ is zero for some $i > 0$ and $n \geq n_0$.*
(3) *$p \nmid [L_n : K_n]$ for $n \geq n_0$.*

Note that the extension L_∞/L_n is totally ramified for sufficiently large $n > 0$. If L/K is totally ramified, then so is K_nL/K_n for every $n \geq 0$, which is easily verified by taking the maximal unramified subextension K^{ur}/K for K_n/K.

2. Totally ramified infinite extension

In this section, we start with a valuation field K_∞ obtained as an infinite Galois extension of K. Then we attach to K_∞ a sequence $K_\bullet = (K_n \mid n \in \mathbb{N})$ such that every ring of integers $\mathfrak{o}_{K_n}$ is a complete discrete valuation ring for all $n \geq 0$ whose residue field is perfect. Let L be a finite extension of K and denote by L_n the composite field K_nL and by $\mathfrak{o}_{L_n}$ the ring of integers for L_n. Then we have a diagram of finite extensions of complete discrete valuation rings:

$$\begin{array}{ccccccccc}
\mathfrak{o}_K & \longrightarrow & \mathfrak{o}_{K_1} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{o}_{K_n} & \longrightarrow & \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{o}_L & \longrightarrow & \mathfrak{o}_{L_1} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{o}_{L_n} & \longrightarrow &
\end{array}$$

Since the fields K_∞ and L_n will be linearly disjoint over K_n for all $n \geq n_1$ for some $n_1 \in \mathbb{N}$, we have $G_{L_n/K_n} = \cdots = G_{L_\infty/K_\infty}$ for $n \geq n_1$. Following Tate, we make the following definition.

Definition 2.1 (the basic field K_∞)**.** Let K be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic $p > 0$. Let K_∞/K be an infinite Galois extension such that:

(1) $\mathrm{Gal}(K_\infty/K)$ contains $\mathbb{Z}_p$ as a finite index subgroup, and
(2) the extension $K_n \to K_\infty$ is totally ramified for large $n > 0$, where $K_n := K_\infty^{p^{n+1}\mathbb{Z}_p}$.

The following theorem was first proved by Tate in the case of a finite extension $K/\mathbb{Q}_p$, and then Faltings extended it to the case of an arbitrary p-adic field. Let $\mathfrak{D}_{L_n/K_n}$ denote the different ideal for the extension L_n/K_n.

Theorem 2.2 (Tate, Faltings)**.** *Let $K_\bullet$ be the same as above, and let $L_\bullet$ be the induced sequence. Let v_p be some fixed p-adic valuation on L_∞. Then $\lim_{n\to\infty} v_p(\mathfrak{D}_{L_n/K_n}) = 0$.*

Proof. The proof may be found in ([1], Proposition 3.6). □

Using the above theorem, we prove the following proposition. In fact, this is proved in a more general form in [1]. For the reader's convenience, we insert a short proof below.

Proposition 2.3. *Under the hypotheses of Theorem 2.2, there exist $n_0 > 0$ and ϵ, $0 < \epsilon \leq 1$ such that the Frobenius map on $\mathfrak{o}_{L_{n+1}}/p^\epsilon \mathfrak{o}_{L_{n+1}}$ induces a surjection:*

$$\mathfrak{o}_{L_{n+1}}/p^\epsilon \mathfrak{o}_{L_{n+1}} \twoheadrightarrow \mathfrak{o}_{L_n}/p^\epsilon \mathfrak{o}_{L_n}$$

for every $n \geq n_0$.

The following proof is expanded from [1]. First of all, let δ_{L_n/K_n} denote the p-adic valuation of the different ideal $\mathfrak{D}_{L_n/K_n}$. Let also $\epsilon_n := 1 - p\delta_{L_n/K_n}$. Then by Theorem 2.2, we have $\epsilon_n \to 1$ as $n \to \infty$ and the well-defined Frobenius map:

$$\mathfrak{o}_{L_{n+1}}/p^{\epsilon_n}\mathfrak{o}_{L_{n+1}} \to \mathfrak{o}_{L_n}/p^{\epsilon_n}\mathfrak{o}_{L_n} \quad \cdots (1).$$

It suffices to prove that there exists sufficiently large $n_0 \in \mathbb{N}$ such that the Frobenius map (1) yields a surjection:

$$\mathfrak{o}_{L_{n+1}}/p^{\epsilon}\mathfrak{o}_{L_{n+1}} \twoheadrightarrow \mathfrak{o}_{L_n}/p^{\epsilon}\mathfrak{o}_{L_n}$$

for every $n \geq n_0$ and some fixed $0 < \epsilon \leq 1$. In the case when L/K is unramified, there is nothing to prove since the extension L_n/K_n is unramified, in which case one can take $\epsilon = 1$. So assume the extension L/K is ramified. Then we must prove that the map:

$$\mathfrak{o}_{L_{n+1}}/p^{\epsilon}\mathfrak{o}_{L_{n+1}} \to \mathfrak{o}_{L_n}/p^{\epsilon}\mathfrak{o}_{L_n}.$$

is surjective by choosing small enough $\epsilon > 0$. By choosing n large, we may assume that $[L_n : K_n]$ is constant. Then by Nakayama's lemma, we deduce that $\mathfrak{o}_{L_n}/\mathfrak{m}_n\mathfrak{o}_{L_n}$ is a vector space over $\mathfrak{o}_{K_n}/\mathfrak{m}_n$ such that:

$$[\mathfrak{o}_{L_n}/\mathfrak{m}_n\mathfrak{o}_{L_n} : \mathfrak{o}_{K_n}/\mathfrak{m}_n] = [L_n : K_n] = \text{const.}$$

for $n \gg 0$, where $\mathfrak{m}_n$ is the maximal ideal of $\mathfrak{o}_{K_n}$. Furthermore, since $\epsilon_n \to 1$, we may assume that $p^{\epsilon_n} \in \mathfrak{m}_n$ and let $\mathfrak{m}_n = (p^{\epsilon})$ for some $\epsilon > 0$. Then one has:

$$[\mathfrak{o}_{L_{n+1}}/\mathfrak{m}_{n+1}\mathfrak{o}_{L_{n+1}} : \mathfrak{o}_{K_{n+1}}/\mathfrak{m}_{n+1}] = [\mathfrak{o}_{L_n}/\mathfrak{m}_n\mathfrak{o}_{L_n} : \mathfrak{o}_{K_n}/\mathfrak{m}_n]$$

and it follows that the ring homomorphism

$$\mathfrak{o}_{L_{n+1}}/p^{\epsilon p^{-1}}\mathfrak{o}_{L_{n+1}} \to \mathfrak{o}_{L_n}/p^{\epsilon}\mathfrak{o}_{L_n}$$

induced by the Frobenius map is well-defined and is an isomorphism for every $n \geq n_0$ for some n_0. Hence the map

$$\mathfrak{o}_{L_{n+1}}/p^{\epsilon}\mathfrak{o}_{L_{n+1}} \to \mathfrak{o}_{L_n}/p^{\epsilon}\mathfrak{o}_{L_n}$$

is surjective. Inductively, we conclude that the map

$$\mathfrak{o}_{L_{n+l+1}}/p^{\epsilon p^{-l}}\mathfrak{o}_{L_{n+l+1}} \to \mathfrak{o}_{L_{n+l}}/p^{\epsilon p^{-l}}\mathfrak{o}_{L_{n+l}}$$

is surjective for every $l \in \mathbb{N}$. We set $A := \mathfrak{o}_{L_{n+l+1}}$, $B := \mathfrak{o}_{L_{n+l}}$, and $b := p^{\epsilon p^{-(l+1)}}$. Now let $\mathrm{Fil}^{\bullet}(A/p^{\epsilon}A)$ (resp. $\mathrm{Fil}^{\bullet}(B/p^{\epsilon}B)$) be a filtration induced by

$$(b^m A \mid 0 \leq m \leq p^l) \text{ (resp. } (b^{pm}B \mid 0 \leq m \leq p^l)).$$

Then the Frobenius map induces a filtered homomorphism $\mathrm{Fil}^{\bullet}(A/p^{\epsilon}A) \to \mathrm{Fil}^{\bullet}(B/p^{\epsilon}B)$, and the associated map of graded abelian groups $\Phi^{\bullet} : \mathrm{Gr}^{\bullet}(A/p^{\epsilon}A) \to \mathrm{Gr}^{\bullet}(B/p^{\epsilon}B)$ turns out to be surjective, due to the following commutative square:

$$\begin{array}{ccc} A/bA & \xrightarrow{\text{Frob}} & B/b^pB \\ {\scriptstyle b^l}\downarrow & & {\scriptstyle b^{pl}}\downarrow \\ b^lA/b^{l+1}A & \xrightarrow{\Phi^l} & b^{pl}B/b^{p(l+1)}B \end{array}$$

for all $l \in \mathbb{N}$ in which Φ^l is induced by the Frobenius map. Then it follows from ([2], Ch.III, §2.8, Corollary 2) that the the Frobenius map $A/p^{\epsilon}A \to B/p^{\epsilon}B$ is surjective, which is the desired claim. □

In the situation of Proposition 2.3, we say that the sequence $L_\bullet$ is *deeply ramified* for $n \geq n_0$. Now we begin with such a sequence to construct a tower of discrete valuation rings in a functorial way, due to Fontaine:

Definition 2.4 (small Fontaine ring)**.** Let the sequence $L_\bullet$ be as above and let $n \geq n_0$. Then define

$$\mathbf{E}^+_{L_n} := \varprojlim \Big(\cdots \xrightarrow{\text{Frob}} \mathfrak{o}_{L_{n+1}}/p^\epsilon \mathfrak{o}_{L_{n+1}} \xrightarrow{\text{Frob}} \mathfrak{o}_{L_n}/p^\epsilon \mathfrak{o}_{L_n} \Big)$$

where each transitive map is surjective and is induced by the Frobenius map. We also set

$$\mathbf{E}^{\text{rad}}_{L_\bullet} := \bigcup_{n \geq n_0} \mathbf{E}^+_{L_n}.$$

An element of $\mathbf{E}^+_{L_n}$ is of the form $(a_n \mid n \in \mathbb{N})$, and $a_n \in \mathfrak{o}_{L_n}/p^\epsilon \mathfrak{o}_{L_n}$ such that $a^p_{n+1} = a_n$. Denote by $\langle p \rangle := (p, p^{1/p}, p^{1/p^2}, \ldots)$ in the following. Then $\mathbf{E}^{\text{rad}}_{L_\bullet}$ is a perfect ring of characteristic $p > 0$. We define a natural projection map:

$$\Phi_{L_n} : \mathbf{E}^+_{L_n} \to \mathfrak{o}_{L_n}/p^\epsilon \mathfrak{o}_{L_n},\ (a_0, a_1, a_2, \ldots) \mapsto a_0,$$

which is surjective. Let $\mathbf{E}_{K_n}$ (resp. $\mathbf{E}_{L_n}$) denote the field of fractions of $\mathbf{E}^+_{K_n}$ (resp. $\mathbf{E}^+_{L_n}$). The construction of small Fontaine rings is functorial.

Proposition 2.5 (Andreatta)**.** *Let the notations be the same as above.*

1. *The extension $\mathbf{E}^+_{K_n} \to \mathbf{E}^+_{L_n}$ is module-finite of complete discrete valuation rings.*
2. *The ring $\mathbf{E}^{\text{rad}}_{L_\bullet}$ is the perfect closure of $\mathbf{E}^+_{L_n}$ that surjects onto $\mathfrak{o}_{L_\infty}/p^\epsilon \mathfrak{o}_{L_\infty}$.*
3. *The extension L_n/K_n is Galois if and only if the extension $\mathbf{E}_{L_n}/\mathbf{E}_{K_n}$ is Galois for all $n \geq n_0$. Furthermore,*

$$\text{Gal}(L_n/K_n) \simeq \text{Gal}(\mathbf{E}_{L_n}/\mathbf{E}_{K_n})$$

for all $n \geq n_0$. In particular, the ring $\mathbf{E}^+_{L_n}$ admits a semilinear action of the Galois group.

Proof. See ([1], Proposition 4.5, Corollary 4.6, Corollary 5.4, and Corollary 6.4). ☐

From the proposition together with the fact that the extension $\mathbf{E}_{L_{n+1}}/\mathbf{E}_{L_n}$ is purely inseparable, it is always the case that $n_1 \leq n_0$, where n_0 and n_1 are defined previously.

3. Galois cohomology of small Fontaine rings

We recall from [4] the definition of the Galois cohomology for finite groups. Let G be a finite group, and let M be a discrete G-module and let Γ denote the functor that sends a module M to M^G, the submodule of M of G-invariants. Then let $H^i(G, M) := R^i\Gamma(M)$ for $i \geq 0$. Henceforth, we denote by $\mathfrak{o}_K$ the ring of integers of a complete discrete valuation field K of characteristic zero with perfect residue field k_K of characteristic $p > 0$. Let L/K be a finite Galois extension with Galois group $G = \text{Gal}(L/K)$. Let $K_\bullet$ be a basic sequence and let $L_\bullet$ be its induced sequence. If $G_n := \text{Gal}(L_n/K_n)$, then the group G_n acts on the small Fontaine ring $\mathbf{E}^+_{L_n}$ by Proposition 2.5, and we take its Galois cohomology:

$$H^i(G_n, \mathbf{E}^+_{L_n}), i \geq 0.$$

In the following, we assume that L/K is a finite Galois extension. Consider the following commutative diagram of natural inclusions:

$$\begin{array}{ccc} \mathbf{E}^+_{K_n} & \longrightarrow & \mathbf{E}^+_{K_{n+k}} \\ \downarrow & & \downarrow \\ \mathbf{E}^+_{L_n} & \longrightarrow & \mathbf{E}^+_{L_{n+k}} \end{array}$$

Then the above diagram induces a homomorphism of abelian groups:

$$H^i(G_n, \mathbf{E}^+_{L_n}) \to H^i(G_{n+k}, \mathbf{E}^+_{L_{n+k}})$$

for all $k \geq 0$. In what follows, we choose $n_0 \in \mathbb{N}$ such that:

(1) The sequence $L_\bullet$ defines a deeply ramified tower for all $n \geq n_0$.
(2) $G_n = G_{n+1} = \cdots = G_\infty$ for all $n \geq n_0$.

Then the inclusion $\mathbf{E}^+_{L_n} \to \mathbf{E}^+_{L_{n+k}}$ admits a splitting as an $\mathbf{E}^+_{K_n}$-module map with G_n-action, and the above Galois cohomology map is an injective homomorphism of $\mathbf{E}^+_{K_n}$-modules. Let $\mathfrak{m}_{\mathbf{E}^+_{K_n}}$ (resp. $\mathfrak{m}_{\mathbf{E}^+_{L_n}}$) be the maximal ideal of $\mathbf{E}^+_{K_n}$ (resp. $\mathbf{E}^+_{L_n}$).

Lemma 3.1. *Let the notations be as above with $n \geq n_0$. Then the following hold:*

(1) $\mathfrak{m}_{\mathbf{E}^+_{K_n}} \subset \mathrm{Tr}_{\mathbf{E}_{L_n}/\mathbf{E}_{K_n}}(\mathbf{E}^+_{L_n})$.
(2) $H^i(G_{L_n/K_n}, \mathbf{E}^+_{L_n})$ *is annihilated by* $\mathfrak{m}_{\mathbf{E}^+_{K_n}}$ *for all* $i > 0$.

Proof. For (1), we notice that $\mathbf{E}^{\mathrm{rad}}_{L_\bullet}$ is a perfect ring and the extension $\mathbf{E}^{\mathrm{rad}}_{K_\bullet} \to \mathbf{E}^{\mathrm{rad}}_{L_\bullet}$ is Galois, so the trace map is non-zero. Hence we get the following commutative square:

$$\begin{array}{ccc} \mathbf{E}^{\mathrm{rad}}_{L_\bullet} & \xrightarrow[\sim]{\mathrm{Frob}} & \mathbf{E}^{\mathrm{rad}}_{L_\bullet} \\ {\scriptstyle \mathrm{Tr}}\downarrow & & {\scriptstyle \mathrm{Tr}}\downarrow \\ \mathbf{E}^{\mathrm{rad}}_{K_\bullet} & \xrightarrow[\sim]{\mathrm{Frob}} & \mathbf{E}^{\mathrm{rad}}_{K_\bullet}, \end{array}$$

and it follows easily that $\mathfrak{m}_{\mathbf{E}^{\mathrm{rad}}_{K_\bullet}} \subset \mathrm{Tr}(\mathbf{E}^{\mathrm{rad}}_{L_\bullet})$. Then this implies that

$$\mathrm{Tr}(\mathbf{E}^+_{L_n}) = \mathrm{Tr}(\mathbf{E}^{\mathrm{rad}}_{L_\bullet}) \cap \mathbf{E}^+_{K_n},$$

which obviously contains the maximal ideal $\mathfrak{m}_{\mathbf{E}^+_{K_n}}$.

For (2), this follows immediately from (1) by taking an injective resolution $\mathbf{E}^+_{L_n} \to I^\bullet$ by $\mathbf{E}^+_{K_n}$-modules, which is compatible with the G_n-action. □

We prove the main theorem now.

Theorem 3.2. *Suppose L/K is a totally ramified cyclic extension. If $K_\bullet$ is a basic sequence, then the following conditions are equivalent:*

(1) $H^i(G_n, \mathbf{E}^+_{L_n})$ *is zero for all* $i > 0$ *and* $n \geq n_0$.
(2) $H^i(G_n, \mathbf{E}^+_{L_n})$ *is zero for some* $i > 0$ *and* $n \geq n_0$.
(3) $p \nmid [L_n : K_n]$ *for* $n \geq n_0$.

Proof. (1) ⇒ (2): This is trivial.

(2) ⇒ (3): We note that the maximal ideal $\mathfrak{m}_{\mathbf{E}^{\mathrm{rad}}_{L_\bullet}}$ is the direct limit of principal ideals. Let b be the generator of $\mathfrak{m}_{\mathbf{E}^+_{K_n}}$. Then b^{1/p^k} is the generator of the maximal ideal $\mathfrak{m}_{\mathbf{E}^+_{K_{n+k}}}$, and it is easy

to verify that this maximal ideal is the same as the limit of the direct system: $(A_m \mid m \in \mathbb{N})$, where $A_m := \mathbf{E}^{\mathrm{rad}}_{K_\bullet}$ and the transition map is given by

$$b^{p^{-m}(p-1)} : A_m \to A_{m+1}.$$

Then it follows from Lemma 3.1 (2) that the cohomology $H^i(G_\infty, \mathfrak{m}_{\mathbf{E}^{\mathrm{rad}}_{K_\bullet}} \mathbf{E}^{\mathrm{rad}}_{L_\bullet})$ is zero for $i > 0$. Since the valuation of $\mathbf{E}^{\mathrm{rad}}_{K_\bullet}$ (resp. $\mathbf{E}^{\mathrm{rad}}_{L_\bullet}$) is not discrete of rank one, it follows that $\mathfrak{m}_{\mathbf{E}^{\mathrm{rad}}_{L_\bullet}} = \mathfrak{m}_{\mathbf{E}^{\mathrm{rad}}_{K_\bullet}} \mathbf{E}^{\mathrm{rad}}_{L_\bullet}$. Using the fact that the small Fontaine ring has perfect residue field and the extension $\mathbf{E}^{+}_{L_n} \hookrightarrow \mathbf{E}^{+}_{L_{n+k}}$ is purely inseparable, we deduce that $H^i(G_{n+k}, \mathbf{E}^{+}_{L_{n+k}})$ is zero for all $k \geq 0$ by assumption. Since

$$H^i(G_\infty, \mathbf{E}^{\mathrm{rad}}_{L_\bullet}) = \varinjlim_{k \geq 0} H^i(G_{n+k}, \mathbf{E}^{+}_{L_{n+k}}),$$

we have $H^i(G_\infty, \mathbf{E}^{\mathrm{rad}}_{L_\bullet}) = 0$. Taking the cohomology exact sequence associated to the short exact sequence:

$$0 \longrightarrow \mathfrak{m}_{\mathbf{E}^{\mathrm{rad}}_{K_\bullet}} \mathbf{E}^{\mathrm{rad}}_{L_\bullet} \longrightarrow \mathbf{E}^{\mathrm{rad}}_{L_\bullet} \longrightarrow k_{L_\infty} \longrightarrow 0,$$

where k_{L_∞} is the residue field of L_∞, we have isomorphisms:

$$0 \simeq H^i(G_\infty, \mathbf{E}^{\mathrm{rad}}_{L_\bullet}) \simeq H^i(G_\infty, k_{L_\infty}),\ i > 0.$$

Since L/K is totally ramified by assumption, so is L_∞/K_∞. Hence $k_{K_\infty} = k_{L_\infty}$. Now assume that $\#G_\infty = [L_\infty : K_\infty]$ is divisible by p. Then since G_∞ is a cyclic group acting trivially on k_{L_∞}, $H^i(G_\infty, k_{L_\infty}) \simeq k_{L_\infty}$ by ([4], P. 133), which is a contradiction and thus $p \nmid [L_\infty : K_\infty]$.

(3) $\Rightarrow$ (1): Since the Galois cohomology in question is killed by both $[L_n : K_n]$ and p, this implication is obvious. □

The next corollary is immediate from the theorem.

Corollary 3.3. *In addition to the hypotheses of the theorem, if $G_{L/K}$ is a p-group and $H^i(G_n, \mathbf{E}^{+}_{L_n})$ is zero for some $i > 0$ and $n \geq n_0$, then $L \subset K_n$ for all $n \geq n_0$.*

Proof. By Theorem 3.2, we must have $K_n = L_n$ for $n \geq n_0$, which implies that L is the subfield of K_n. □

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