CONDITIONS FOR SUPERCONVERGENCE OF HDG METHODS FOR SECOND-ORDER ELLIPTIC PROBLEMS

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Abstract. We provide a projection-based analysis of a large class of finite element methods for second order elliptic problems. They include the hybridized version of well known mixed methods as well as old and new hybridizable discontinuous Galerkin methods. The main feature of this unifying approach is that it reduces the main difficulty of the analysis to the verification of some properties of an auxiliary, locally defined projection and of the local spaces defining the methods. Sufficient conditions for the optimal convergence of the approximate flux and the superconvergence of an element-by-element postprocessing of the scalar variable are obtained. New hybridizable discontinuous Galerkin methods with these properties are devised which are defined on squares and cubes.

1. Introduction. In this paper, we propose a projection-based a priori error analysis of finite element methods for second-order elliptic problems. The analysis is *unifying* because it applies to a large class of methods including the hybridized version of most well known mixed methods as well as several hybridizable discontinuous Galerkin (HDG) methods. The novelty of the approach is that it reduces the whole error analysis to the element-by-element construction of an auxiliary projection satisfying certain orthogonality and approximation properties, and to the verification of very simple inclusion properties of the local spaces defining the methods. For the sake of simplicity, we present our approach in the framework of the following diffusion problem:

$$\boldsymbol{q} + \nabla u = 0 \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla \cdot \boldsymbol{q} = f \qquad \text{in } \Omega \tag{1.1b}$$

$$u = g \qquad \text{on } \partial\Omega. \tag{1.1c}$$

Here $\Omega \in \mathbb{R}^n$ $(n = 2, 3)$ is a bounded polyhedral domain, $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$.

Two ideas led to this approach. The first is that many mixed methods, including the method of Raviart-Thomas (RT), [15, 11, 1], Brezzi-Douglas-Marini, [5], and Brezzi-Douglas-Fortin-Marini, [4], were analyzed by using suitably defined auxiliary projections; see also [6]. The second is that both mixed and HDG methods can be seen as particular cases of a single, general numerical method uncovered in [9]. This suggested the possibility of using a similar projection-based approach to analyze HDG methods. Recently, this was actually achieved, first for a particular case of the so-called local discontinuous Galerkin hybridizable (LDG-H) methods (defined on simplexes) in [8], and then for the whole family of those methods in [10]. In this paper, we continue this effort and show that a single error analysis of many of the methods fitting in the general framework proposed in [9] can be realized.

To better describe our results, let us begin by introducing the general form of the methods under consideration; we follow [9]. Let $\Omega_h := \{K\}$ denote a conforming triangulation of Ω, where K is a polyhedral element. Let $\mathcal{E}_h$ denote the set of all faces

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F of all the element $K \in \Omega_h$. The methods we are interested in seek an approximation to $(u, \boldsymbol{q}, u|_{\mathcal{E}_h})$, $(u_h, \boldsymbol{q}_h, \widehat{u}_h)$, in the finite element space $W_h \times \boldsymbol{V}_h \times M_h$, where

$$\begin{aligned}
\boldsymbol{V}_h &:= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_h) : \boldsymbol{v}|_K \in \boldsymbol{V}(K),\ \ K \in \Omega_h\},\\
W_h &:= \{w \in L^2(\Omega_h) : w|_K \in W(K), K \in \Omega_h\},\\
M_h &:= \{\mu \in L^2(\mathcal{E}_h) :\ \mu|_F \in W(F),\ \ F \in \mathcal{E}_h\},
\end{aligned}$$

and determine it as the only solution of the following weak formulation:

$$-(u_h\, , \nabla\cdot\boldsymbol{v})_{\Omega_h} + (\boldsymbol{q}_h\, , \boldsymbol{v})_{\Omega_h} + \langle \widehat{u}_h\, , \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0, \tag{1.2a}$$

$$-(\boldsymbol{q}_h\, , \nabla w)_{\Omega_h} + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\, , w\rangle_{\partial\Omega_h} = (f\, , w)_{\Omega_h}, \tag{1.2b}$$

$$\langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h\setminus\partial\Omega} = 0, \tag{1.2c}$$

$$\langle \widehat{u}_h, \mu\rangle_{\partial\Omega} = \langle g, \mu\rangle_{\partial\Omega}, \tag{1.2d}$$

for all $(w, \boldsymbol{v}, \mu) \in W_h \times \boldsymbol{V}_h \times M_h$. Here we write $(\eta\, , \zeta)_{\Omega_h} := \sum_{K\in\Omega_h} (\eta, \zeta)_K$, where $(\eta, \zeta)_D$ denotes the integral of $\eta\zeta$ over the domain $D \subset \mathbb{R}^n$. We also write $(\eta\, , \zeta)_{\Omega_h} := \sum_{K\in\Omega_h} \langle\eta\, , \zeta\rangle_{\partial K}$, where $\langle\eta\, , \zeta\rangle_D$ denotes the integral of $\eta\zeta$ over the domain $D \subset \mathbb{R}^{n-1}$ and $\partial\Omega_h := \{\partial K : K \subset \Omega_h\}$. The definition of the method is completed with the definition of the normal component of the numerical trace:

$$\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = \boldsymbol{q}_h\cdot\boldsymbol{n} + \boldsymbol{\alpha}((u_h - \widehat{u}_h)\boldsymbol{n})\cdot\boldsymbol{n} \quad \text{on} \quad \partial\Omega_h. \tag{1.3}$$

By taking particular choices of the local spaces $\boldsymbol{V}(K)$, $W(K)$ and $M(F)$, and the *linear local stabilization* operator $\boldsymbol{\alpha}$, the different mixed and HDG methods are obtained.

Our main result is to show that if we can construct, in an element-by-element fashion, an auxiliary projection $\Pi_u(\boldsymbol{q}, u) := (\boldsymbol{\Pi}_V\boldsymbol{q}, \Pi_W u)$ satisfying certain orthogonality and approximation conditions, and if the local spaces $\boldsymbol{V}(K)$, $W(K)$ and $M(F)$, for all the faces F of the element K, satisfy some inclusion properties, then the method is well defined and we have the estimates

$$\begin{aligned}
\|\boldsymbol{q} - \boldsymbol{q}_h\|_{\Omega_h} &\le 2\,\|\boldsymbol{q} - \boldsymbol{\Pi}_V\boldsymbol{q}\|_{\Omega_h},\\
\|\Pi_W u - u_h\|_{\Omega_h} &\le C\,h\|\boldsymbol{q} - \boldsymbol{\Pi}_V\boldsymbol{q}\|_{\Omega_h},
\end{aligned}$$

where $\|\cdot\|_{\Omega_h}$ denotes the $L^2(\Omega_h)$-norm.

Note that if the error $\Pi_W u - u_h$ converges to zero *faster* than the error $u - u_h$, this *superconvergence* property can be advantageously exploited; see [1, 5, 16, 12, 17]. Indeed, following [16, 12, 17], we define a new approximation to u, u_h^*, in the space

$$W_h^* := \{w \in L^2(\Omega_h) : w|_K \in W^*(K), K \in \Omega_h\}, \tag{1.4a}$$

as follows. On each element $K \in \Omega_h$, the postprocessing u_h^* is the element of $W^*(K)$ such that

$$(\nabla u_h^*, \nabla\omega)_K = -\,(\boldsymbol{q}_h, \nabla\omega)_K \qquad \forall\, \omega \in W^*(K) :\ (\omega, 1)_k = 0, \tag{1.4b}$$

$$(u_h^*, 1)_K = (u_h, 1)_K. \tag{1.4c}$$

It is not difficult to prove that the postprocessing u_h^* is well defined and that we have

$$\|u - u_h^*\|_{\Omega_h} \le \|\Pi_W u - u_h\|_{\Omega_h} + C\,h\,(\|\boldsymbol{q} - \boldsymbol{q}_h\|_{\Omega_h} + \inf_{\omega\in W_h^*} \|\nabla(u - \omega)\|_{\Omega_h}),$$

which means that it is possible to define u_h^* converging to u as fast as $\Pi_W u - u_h$ converges to zero.

Moreover, we do provide a *single* template for the construction of the auxiliary projection Π_h with which we can analyze *all* the methods, old and new, that we give as examples fitting our general framework. The old ones are the (hybridized versions of) the main mixed methods, the LDG-H method for simplexes, and the so-called BMMPR-H method; see [9]. The last method, which had never been analyzed before, is proven to superconverge even though it uses a local stabilization operator $\boldsymbol{\alpha}$ which is different from those of the previous examples. The new methods are the superconvergent LDG-H methods for squares and cubes. The definition of these methods had remained elusive in the last few years and it is thanks to the our projection-based approach that it became clear. It is important to emphasize that, although it is very easy to devise HDG methods that are well defined, it is far from obvious to devise them so that they display the above-mentioned superconvergence property. The technique we propose here is a *new and effective* tool to achieve this goal.

The rest of the paper is organized as follows. In Section 2, we describe the conditions on the auxiliary projection Π_h and the local spaces associated with our finite element methods and present our a priori error estimates. In section 3, we give various particular examples of hybridized mixed and HDG methods with superconvergent properties. We then end with some extensions and concluding remarks in section 4.

2. Main results.

In this section we show how an a priori error analysis of the HDG methods can be *reduced* to the verification of a few conditions on the local spaces and on some properties of an associated, auxiliary projection Π_h defined in an element-by-element fashion.

2.1. A priori error estimates.

The main idea of our error analysis is to estimate the projection of the errors $\Pi_h(\boldsymbol{q} - \boldsymbol{q}_h, u - u_h)$ and then deduce bounds of the $L^2(\Omega)$-norm of the errors $\boldsymbol{q} - \boldsymbol{q}_h$, $u - u_h$ and $u - u_h^*$.

2.1.1. Estimate of $\boldsymbol{q} - \boldsymbol{q}_h$.

Our first result gives an estimate of the projection of the error $\boldsymbol{\Pi}_V \boldsymbol{q} - \boldsymbol{q}_h$ solely in terms of the approximation error of the projection $\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}$. To state it, we need to describe our assumptions on the projection Π_h and on the local finite element spaces $\boldsymbol{V}(K)$, $W(K)$ and $M(F)$.

Assumptions A:

- *Orthogonality properties of* Π_h. On each element K, there exist a projection $\Pi_h(\boldsymbol{q}, u) = (\boldsymbol{\Pi}_V \boldsymbol{q}, \Pi_W u) \in \boldsymbol{V}(K) \times W(K)$ satisfying the following properties:

(A.1) $(\boldsymbol{\Pi}_V \boldsymbol{q}, \boldsymbol{v})_K = (\boldsymbol{q}, \boldsymbol{v})_K$ for all $\boldsymbol{v} \in \nabla W(K)$,
(A.2) $(\Pi_W u, w)_K = (u, w)_K$ for all $w \in \nabla \cdot \boldsymbol{V}(K)$,
(A.3) For all faces F of the element K,

$$\langle \boldsymbol{\Pi}_V \boldsymbol{q} \cdot \boldsymbol{n} + \boldsymbol{\alpha}(\Pi_W u\, \boldsymbol{n}) \cdot \boldsymbol{n}, \mu \rangle_F = \langle \boldsymbol{q} \cdot \boldsymbol{n} + \boldsymbol{\alpha}(P_M u\, \boldsymbol{n}) \cdot \boldsymbol{n}, \mu \rangle_F \quad \text{for all } \mu \in M(F).$$

We also need to assume suitable relations between the traces on the faces F of the local spaces $\boldsymbol{V}(K)$ and $W(K)$ with the local space $M(F)$.

- *Properties of the traces of the local spaces.* For each element K, and for any of its faces F,

(A.4) $\boldsymbol{V}(K) \cdot \boldsymbol{n}|_F \subset M(F)$,

(A.5) $W(K)|_F \subset M(F)$.

Here, $\boldsymbol{V}(K)\cdot\boldsymbol{n}|_F$ denotes the space of the *traces* of normal components of functions of $\boldsymbol{V}(K)$ on the face F of K. Similarly, $W(K)|_F$ denotes the space of *traces* of functions of $W(K)$ on the face F.

Finally, we need a simple assumption reflecting the stabilizing role of the linear operator $\boldsymbol{\alpha}$.

- *The semi-positivity property of* $\boldsymbol{\alpha}$. For each element K and any of its faces F,

(A.6) $\langle \boldsymbol{\alpha}(\mu\,\boldsymbol{n}),\, \mu\,\boldsymbol{n}\rangle_F \geq 0$ for all $\mu \in M(F)$.

We are now ready to state our first result. In what follows, we use $\|\cdot\|_{k,D}$, $|\cdot|_{k,D}$ to denote the standard norm and seminorm on any Sobolev space $H^k(D)$, respectively. For simplicity, we use $\|\cdot\|_D$ to denote the $L^2(D)-$norm on any D.

THEOREM 2.1. *Suppose that the Assumptions A are satisfied. Then we have*

$$\|\boldsymbol{\Pi}_V \boldsymbol{q} - \boldsymbol{q}_h\|_{\Omega_h} \leq \|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\Omega_h}.$$

Note that, since this implies that

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{\Omega_h} \leq 2\|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\Omega_h},$$

the quality of the approximation $\boldsymbol{q}_h$ depends on the approximation properties of the first component of the projection *only.*

2.1.2. Estimate of $u - u_h$. Our next result shows that $\Pi_W u - u_h$ can *also* be controlled solely in terms of the approximation error of the projection $\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}$.

It is valid under a typical elliptic regularity property we state next. We assume that, for any given $\eta \in L^2(\Omega)$, we have

$$\|\phi\|_{2,\Omega} + \|\boldsymbol{\theta}\|_{1,\Omega} \leq C\|\eta\|_\Omega, \tag{2.1}$$

where C only depends on the domain Ω, and $(\boldsymbol{\theta}, \phi)$ is the solution of the *dual* problem:

$$\boldsymbol{\theta} + \nabla\phi = 0 \quad \text{in } \Omega, \tag{2.2a}$$
$$\nabla\cdot\boldsymbol{\theta} = \eta \quad \text{in } \Omega, \tag{2.2b}$$
$$\phi = 0 \quad \text{on } \partial\Omega. \tag{2.2c}$$

We also need a couple of additional assumptions.

Assumptions B:

The first is an approximation property of a projection $\Pi_h^*(\boldsymbol{q}, u) = (\boldsymbol{\Pi}_V^* \boldsymbol{q}, \Pi_W^* u)$ which satisfies the assumptions *(A.1)*, *(A.2)* and *(A.3)* where the local stabilization operator $\boldsymbol{\alpha}(\cdot)$ is *replaced* by its dual $\boldsymbol{\alpha}^*(\cdot)$, that is, by the linear function defined by

$$\langle \boldsymbol{\alpha}(\boldsymbol{p})\,, \boldsymbol{q}\rangle_{\partial K} = \langle \boldsymbol{p}\,, \boldsymbol{\alpha}^*(\boldsymbol{q})\rangle_{\partial K} \quad \text{for all} \quad \boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{L}^2(\partial K).$$

- *The approximation property of the projection* $\boldsymbol{\alpha}^*$. For each element K and any $(\boldsymbol{q}, u) \in \boldsymbol{H}^1(K) \times H^2(K)$, *(B.1)* $\|\boldsymbol{\Pi}_V^* \boldsymbol{q} - \boldsymbol{q}\|_K \le C_{app}^* \, h_K (|u|_{2,K} + |\boldsymbol{q}|_{1,K})$.

The second assumption is a condition on the local space $W(K)$.

- *The local space $W(K)$ is not too small.* For each element K, we have that

(B.2) $\boldsymbol{P}^0(K) \subset \nabla W(K)$,

Here $\boldsymbol{P}^0(K) := [P^0(K)]^n$ and $P^0(K)$ is the space of constants defined on K.

We are now ready to state our second result.

THEOREM 2.2. *Suppose that the Assumptions A and B are satisfied. Also, suppose that the elliptic regularity property* (2.1) *holds. Then we have*

$$\|\Pi_W u - u_h\|_{\Omega_h} \le C\, h \|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\Omega_h},$$

for some constant C depending on C_{app}^* but independent of h and the exact solution.

From this result, we immediately get that

$$\|u - u_h\|_{\Omega_h} \le \|u - \Pi_W u\|_{\Omega_h} + C\, h\, \|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\Omega_h},$$

and we see that the quality of the approximation u_h only depends on the approximation error of the projection.

2.1.3. Estimate of $u - u_h^*$. Note that if the second term of the above right-hand side converges faster than the first, the convergence of u_h to $\Pi_W u$ is *faster* than that of u_h to u. As mentioned before, we can take advantage of this *superconverge* result to show that the postprocessing u_h^* defined by (1.4) converges to u as fast as u_h superconverges to $\Pi_W u$. For that purpose, we need the following assumption.

Assumption C:

- *The local space $\boldsymbol{V}(K)$ is not too small.* For each element K,

(C.1) $P^0(K) \subset \nabla \cdot \boldsymbol{V}(K)$.

We can now state our third and last result.

THEOREM 2.3. *Suppose that the Assumptions A, B, and C are satisfied. Then, we have*

$$\|u - u_h^*\|_{\Omega_h} \le \|\Pi_W u - u_h\|_{\Omega_h} + C\, h\, (\|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\Omega_h} + \inf_{\omega \in W_h^*} \|\nabla (u - \omega)\|_{\Omega_h}).$$

3. Applications: Examples of superconvergent methods. In this section, we give several examples of methods fitting in our framework. They include (the hybridized version of) well known mixed methods as well as HDG methods. To do that, we show how to construct the auxiliary projection. We then study the main three examples of local stabilization operators. Finally, we give examples of superconvergent methods using simplexes and then consider those using squares and cubes.

3.1. A template for the auxiliary projection Π_h. For all our examples, the associated projection $\Pi_h(\boldsymbol{q}, u) := (\boldsymbol{\Pi}_V \boldsymbol{q}, \Pi_W u)$ has a single form devised in such a way that the first three properties of *Assumption A are satisfied.* On the element K, $\Pi_h(\boldsymbol{q}, u)$ is the element of $\boldsymbol{V}(K) \times W(K)$ satisfying the equations

$$(\boldsymbol{\Pi}_V \boldsymbol{q}, \boldsymbol{v})_K = (\boldsymbol{q}, \boldsymbol{v})_K \qquad \forall\, \boldsymbol{v} \in \widetilde{\boldsymbol{V}}(K), \tag{3.1a}$$

$$(\Pi_W u, w)_K = (u, w)_K \qquad \forall\, w \in \widetilde{W}(K), \tag{3.1b}$$

$$\langle \boldsymbol{\Pi}_V \boldsymbol{q} \cdot \boldsymbol{n} + \boldsymbol{\alpha}(\Pi_W u\, \boldsymbol{n}) \cdot \boldsymbol{n}, \mu \rangle_F = \langle \boldsymbol{q} \cdot \boldsymbol{n} + \boldsymbol{\alpha}(P_M u\, \boldsymbol{n}) \cdot \boldsymbol{n}, \mu \rangle_F \;\forall\, \mu \in M(F), \tag{3.1c}$$

for all faces F of the element K. The auxiliary spaces $\widetilde{\boldsymbol{V}}(K) \times \widetilde{W}(K)$ are chosen in such a way that

$$\nabla W(K) \times \nabla \cdot \boldsymbol{V}(K) \;\subset\; \widetilde{\boldsymbol{V}}(K) \times \widetilde{W}(K) \;\subset\; \boldsymbol{V}(K) \times W(K), \tag{3.1d}$$

In all of our examples, the local stabilization operator $\boldsymbol{\alpha}$ is self-adjoint so that we have $\Pi_h^* = \Pi_h$.

3.2. The local stabilization operators $\boldsymbol{\alpha}$. There are three main examples of local stabilization operators $\boldsymbol{\alpha}$ satisfying Assumption (*A.6*).

The first example is the trivial choice $\boldsymbol{\alpha} := \boldsymbol{0}$ which is used in all the mixed methods.

The second example is $\boldsymbol{\alpha} := \tau\, \boldsymbol{Id}$. If τ is taken to be non-negative on each face on each of the elements $K \in \mathcal{T}_h$, it is very easy to see that this operator satisfies Assumption (*A.6*) and that it is self-adjoint.

The third and last example is the local stabilization operator used by the so-called BMMPR-H methods; see [9] and the references therein. It is is given by $\boldsymbol{\alpha} := \tau \boldsymbol{r}$, where $\boldsymbol{r}$ is a suitably defined *lifting* operator. In [3] and [7], see also [2], such operator was introduced in the case in which K is a simplex and $\boldsymbol{V}(K) \times W(K) := \boldsymbol{P}^k(K) \times P^k(K)$ and $M(F) := P^k(F)$. It is not difficult to extend its definition as follows. Given an element K, for any $\mu \in M(F)$, we define $\boldsymbol{r}(\mu\, \boldsymbol{n})$ on K as the element of $\boldsymbol{V}(K)$ satisfying

$$\frac{1}{|K|}(\boldsymbol{r}(\mu\, \boldsymbol{n}), \boldsymbol{v})_K = \frac{1}{|F|}\langle \mu, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_F, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(K). \tag{3.2}$$

We now see that Assumption (*A.6*) is satisfied, and that $\boldsymbol{\alpha}$ is self-adjoint. Indeed, taking $\boldsymbol{v} := \boldsymbol{\alpha}(\eta\, \boldsymbol{n})$ in the definition of the lifting operator $\boldsymbol{r}$,(3.2), we get

$$\langle \boldsymbol{\alpha}(\eta\, \boldsymbol{n}),\, \mu\, \boldsymbol{n} \rangle_F = \tau_F\, \langle \boldsymbol{r}(\eta\, \boldsymbol{n}),\, \mu\, \boldsymbol{n} \rangle_F = \tau_F\, \frac{|F|}{|K|}(\boldsymbol{r}(\eta\, \boldsymbol{n}), \boldsymbol{r}(\mu\, \boldsymbol{n}))_K. \tag{3.3}$$

3.3. Methods using simplexes. We begin by considering methods for which the element K is a simplex. To describe them we use the following notation: $P^k(D)$ denotes the space of polynomials of total degree k defined on D, $\boldsymbol{P}^k(F)$ denotes the space $[P^k(D)]^n$, $\mathcal{R}^k(\partial K)$ denotes the functions whose restriction to each face F of K belong to $P^k(F)$, and $\boldsymbol{\Phi}_k(K)$ denotes the space of functions in $\boldsymbol{P}^k(K)$ which are divergence-free and whose normal component on ∂K is zero.

In Table 1, we give methods that satisfy conditions *A, B* and *C*, and in Table 2, we show their orders of convergence predicted by our theoretical results.

The methods include the well known mixed methods of Raviart-Thomas, $\mathbf{RT}_k$, of Brezzi-Douglas-Marini, $\mathbf{BDM}_k$, and of Brezzi-Douglas-Fortin-Marini, $\mathbf{BDFM}_{k+1}$.

Table 1

Methods satisfying assumptions A, B, and C for which $M(F) = P^k(F), k \geq 1$, and K is a simplex.

method	$\boldsymbol{V}(K)$	$W(K)$	$\widetilde{\boldsymbol{V}}(K)$	$\widetilde{W}(K)$
$\mathbf{BDFM}_{k+1}$	$\{\boldsymbol{q} \in \boldsymbol{P}^{k+1}(K) :$ $\boldsymbol{q} \cdot \boldsymbol{n}\|_{\partial K} \in \mathcal{R}^k(\partial K)\}$	$P^k(K)$	$\nabla P^k(K) \oplus \boldsymbol{\Phi}_{k+1}(K)$	$P^k(K)$
$\mathbf{RT}_k$	$\boldsymbol{P}^k(K) \oplus \boldsymbol{x}\widetilde{P}^k(K)$	$P^k(K)$	$\boldsymbol{P}^{k-1}(K)$	$P^k(K)$
$\mathbf{HDG}_k$	$\boldsymbol{P}^k(K)$	$P^k(K)$	$\boldsymbol{P}^{k-1}(K)$	$P^{k-1}(K)$
$\underset{k\geq 2}{\mathbf{BDM}_k}$	$\boldsymbol{P}^k(K)$	$P^{k-1}(K)$	$\nabla P^{k-1}(K) \oplus \boldsymbol{\Phi}_k(K)$	$P^{k-1}(K)$

Table 2

Orders of convergence for methods for which $M(F) = P^k(F), k \geq 1$, and K is a simplex.

Methods	τ	$\\|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\\|_{\Omega_h}$	$\\|u - \Pi_W u\\|_{\Omega_h}$
$\mathbf{BDFM}_{k+1}$	0	$k+1$	$k+1$
$\mathbf{RT}_k$	0	$k+1$	$k+1$
$\mathbf{HDG}_k$	$> 0, \mathcal{O}(1)$	$k+1$	$k+1$
$\underset{k\geq 2}{\mathbf{BDM}_k}$	0	$k+1$	k

Methods	$\\|\boldsymbol{q} - \boldsymbol{q}_h\\|_{\Omega_h}$	$\\|\Pi_W u - u_h\\|_{\Omega_h}$	$\\|u - u_h^\star\\|_{\Omega_h}$
$\mathbf{BDFM}_{k+1}$	$k+1$	$k+2$	$k+2$
$\mathbf{RT}_k$	$k+1$	$k+2$	$k+2$
$\mathbf{HDG}_k$	$k+1$	$k+2$	$k+2$
$\underset{k\geq 2}{\mathbf{BDM}_k}$	$k+1$	$k+2$	$k+2$

Recall that, for 3D problems, the $\mathbf{RT}_k$ method was introduced by Nédélec in [14] and the method $\mathbf{BDM}_k$ was introduced by Brezzi, Douglas, Durán and Fortin in [4]. The methods also include the $\mathbf{HDG}_k$ method with the two possible local stabilization operators proposed in [9].

The definition and approximation properties of the projections Π_h associated with the mixed methods can be found in [6]. Those of the projection associated with the $\mathbf{HDG}_k$ method with $\boldsymbol{\alpha} := \tau\, \boldsymbol{Id}$ in [10]. The projection Π_h associated with the $\mathbf{HDG}_k$ method with $\boldsymbol{\alpha} = \tau \boldsymbol{r}$ has not been previously considered, but its properties are not essentially different from those of the case $\boldsymbol{\alpha} := \tau\, \boldsymbol{Id}$, as we are going to show in Section 5 for HDG methods on squares and cubes.

3.4. Methods using squares and cubes. Here, we consider methods for which the element K is a square ($n = 2$) or a cube ($n = 3$). To describe them we use the following notation: $Q^k(D)$ denotes the space of polynomials of degree k in each variable defined on D, $\boldsymbol{Q}^k(F)$ denotes the space $[Q^k(D)]^n$, $\widetilde{P}^k(D)$ denotes the space of homogeneous polynomials of degree k defined on D, and $P^{\ell_1,\ell_2}(D)$ for $n = 2$ and P^{ℓ_1,ℓ_2,ℓ_3} for $n = 3$ denote the space of polynomials of degree ℓ_i on the $i - th$ variable, $i = 1, \ldots, n$.

In Table 3, we display the methods using squares and in Table 4, those using cubes. Their orders of convergence are given by Table 2 if we simply replace *simplex*

TABLE 3

Methods satisfying assumptions A, B, and C for which $M(F) = P^k(F), k \geq 1,$ and K is a square.

method	$\boldsymbol{V}(K)$	$W(K)$	$\widetilde{\boldsymbol{V}}(K)$	$\widetilde{W}(K)$
$\mathbf{BDFM}_{[k+1]}$	$P^{k+1}(K)\backslash\{y^{k+1}\}$ $\times(P^{k+1}(K)\backslash\{x^{k+1}\})$	$P^k(K)$	$\boldsymbol{P}^{k-1}(K)$	$P^k(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k}(K)$ $\times P^{k,k+1}(K)$	$Q^k(K)$	$P^{k-1,k}(K)$ $\times P^{k,k-1}(K)$	$Q^k(K)$
$\mathbf{HDG}_{[k]}$	$\boldsymbol{P}^k(K)$ $\oplus\nabla\times(xy\widetilde{P}^k(K))$	$P^k(K)$	$\boldsymbol{P}^{k-1}(K)$	$P^{k-1}(K)$
$\mathbf{BDM}_{[k]}$ $k \geq 2$	$\boldsymbol{P}^k(K)$ $\oplus\nabla\times(xy\,x^k)$ $\oplus\nabla\times(xy\,y^k)$	$P^{k-1}(K)$	$\boldsymbol{P}^{k-2}(K)$	$P^{k-1}(K)$

TABLE 4

Methods satisfying assumptions A, B, and C for which $M(F) = P^k(F), k \geq 1,$ and K is a cube.

method	$\boldsymbol{V}(K)$	$W(K)$	$\widetilde{\boldsymbol{V}}(K)$	$\widetilde{W}(K)$
$\mathbf{BDFM}_{[k+1]}$	$P^{k+1}(K)\backslash\widetilde{P}^{k+1}(y,z)$ $\times P^{k+1}(K)\backslash\widetilde{P}^{k+1}(x,z)$ $\times P^{k+1}(K)\backslash\widetilde{P}^{k+1}(x,y)$	$P^k(K)$	$\boldsymbol{P}^{k-1}$	$P^k(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k,k}(K)$ $\times P^{k,k+1,k}(K)$ $\times P^{k,k,k+1}(K)$	$Q^k(K)$	$P^{k-1,k,k}(K)$ $\times P^{k,k-1,k}(K)$ $\times P^{k,k,k-1}(K)$	$Q^k(K)$
$\mathbf{HDG}_{[k]}$	$\boldsymbol{P}^k(K)$ $\oplus\nabla\times(yz\widetilde{P}^k(K),0,0)$ $\oplus\nabla\times(0,zx\widetilde{P}^k(K),0)$ $\oplus\nabla\times(0,0,xy\widetilde{P}^k(K))$	$P^k(K)$	$\boldsymbol{P}^{k-1}(K)$	$P^{k-1}(K)$
$\mathbf{BDM}_{[k]}$ $k \geq 2$	$\boldsymbol{P}^k(K)$ $\oplus\nabla\times(0,0,xy\widetilde{P}^k(y,z))$ $\oplus\nabla\times(0,yz\widetilde{P}^k(x,z),0)$	$P^{k-1}(K)$	$\boldsymbol{P}^{k-2}(K)$	$P^{k-1}(K)$

by *square* or *cube*, and $\mathbf{NAME}_k$ by $\mathbf{NAME}_{[k]}$.

Note that although the mixed methods $\mathbf{BDFM}_{[k+1]}$, $\mathbf{RT}_{[k]}$, and $\mathbf{BDM}_{[k]}$ are well known, the $\mathbf{HDG}_{[k]}$ methods are *new*. It is easy to verify that these mixed methods satisfy *Assumptions A, B, C*; see, for example, [6]. This also holds true for the $\mathbf{HDG}_{[k]}$ method, as we show in Section 5.

Let us end by pointing out that in [13], the convergence properties of the HDG method defined by $\boldsymbol{V}(K) := \boldsymbol{Q}^k(K)$, $W(K) := Q^k(K)$ and $M(F) := Q^k(F)$ (in two dimensions) were numerically studied. It was shown that the approximate flux $\boldsymbol{q}_h$ converges with order k and that the postprocessing u_h^* does not superconverges with order $k+2$ but converges with order $k+1$, just as u_h does– it does provide a better approximation though. This is to be contrasted with the convergence properties of our $\mathbf{HDG}_{[k]}$ method which achieves a full order of convergence more in both $\boldsymbol{q}_h$ and u_h^* with a significantly smaller space.

4. Proofs of the error estimates. In this section we provide detailed proofs for our a priori error estimates. The main idea is to work with the following projection of the errors:

$$\begin{aligned}\boldsymbol{e}_q &:= \boldsymbol{\Pi}_V \boldsymbol{q} - \boldsymbol{q}_h,\\ e_u &:= \Pi_W u - u_h,\\ \boldsymbol{e}_{\widehat{q}} \cdot \boldsymbol{n} &:= P_M(\boldsymbol{q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}),\\ e_{\widehat{u}} &:= P_M u - \widehat{u}_h.\end{aligned}$$

Here, P_M is the L^2-projection from $L^2(\mathcal{E}_h)$ into M_h. We abuse the notation for the sake of simplicity and denote with the *same* symbol the L^2-projection from $L^2(\partial\Omega_h)$ into the space

$$\{w \in L^2(\partial\Omega_h) : (w|_{\partial K})|_F \in M(F) \text{ for all faces } F \text{ of } K \text{ and all } K \in \Omega_h\}.$$

We begin by obtaining the equations satisfied by these projections. We then use an energy argument to obtain an estimate of $\boldsymbol{e}_q$; this would prove Theorem 2.1. To obtain an estimate of e_u and prove Theorem 2.2, we employ an elliptic duality. Finally, we obtain the estimate of $u - u_h^*$ of Theorem 2.3 by using a simple element-by-element argument.

Step 1: The equations for the projection of the errors. We begin our error analysis with the following auxiliary result.

LEMMA 4.1. *Suppose that the orthogonality properties of the projection Π_h and the properties of the traces of the local spaces of Assumption A are satisfied. Then, we have*

$$(\boldsymbol{e}_q\,,\,\boldsymbol{v})_{\Omega_h} - (e_u\,,\,\nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle e_{\widehat{u}}\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = (\boldsymbol{\Pi}_V\boldsymbol{q} - \boldsymbol{q}\,,\,\boldsymbol{v})_{\Omega_h}, \tag{4.1a}$$

$$-(\boldsymbol{e}_q\,,\,\nabla w)_{\Omega_h} + \langle \boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,w\rangle_{\partial\Omega_h} = 0, \tag{4.1b}$$

$$\langle \boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h\setminus\partial\Omega} = 0, \tag{4.1c}$$

$$\langle e_{\widehat{u}}\,,\,\mu\rangle_{\partial\Omega} = 0, \tag{4.1d}$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$. Moreover,

$$\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n} = \boldsymbol{e}_q\cdot\boldsymbol{n} + P_M(\boldsymbol{\alpha}((e_u - e_{\widehat{u}})\boldsymbol{n})\cdot\boldsymbol{n}) \quad \text{on} \quad \partial\Omega_h. \tag{4.2}$$

Proof. Let us begin by noting that the exact solution $(\boldsymbol{q}, u)$ satisfies the equations

$$\begin{aligned}(\boldsymbol{q}\,,\,\boldsymbol{v})_{\Omega_h} - (u\,,\,\nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle u\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0,\\ -(\boldsymbol{q}\,,\,\nabla w)_{\Omega_h} + \langle \boldsymbol{q}\cdot\boldsymbol{n}\,,\,w\rangle_{\partial\Omega_h} &= (f\,,\,w)_{\Omega_h},\\ \langle \boldsymbol{q}\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\ \langle u\,,\,\mu\rangle_{\partial\Omega} &= \langle g\,,\,\mu\rangle_{\partial\Omega},\end{aligned}$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$. By the orthogonality properties *(A.1)* and *(A.2)* of the projection $\Pi_h = (\boldsymbol{\Pi}_V, \Pi_W)$, we obtain that

$$\begin{aligned}(\boldsymbol{q}\,,\,\boldsymbol{v})_{\Omega_h} - (\Pi_W u\,,\,\nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle u\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0,\\ -(\boldsymbol{\Pi}_V\boldsymbol{q}\,,\,\nabla w)_{\Omega_h} + \langle \boldsymbol{q}\cdot\boldsymbol{n}\,,\,w\rangle_{\partial\Omega_h} &= (f\,,\,w)_{\Omega_h},\\ \langle \boldsymbol{q}\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\ \langle u\,,\,\mu\rangle_{\partial\Omega} &= \langle g\,,\,\mu\rangle_{\partial\Omega},\end{aligned}$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$. Moreover, since P_M is the L^2-projection into M_h, we get, by the properties $(A.4)$ and $(A.5)$ of the traces of the local spaces, that

$$\begin{aligned}
(\boldsymbol{q}\,,\,\boldsymbol{v})_{\Omega_h} - (\Pi_W u\,,\,\nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle P_M u\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\boldsymbol{\Pi}_V\boldsymbol{q}\,,\,\nabla w)_{\Omega_h} + \langle P_M(\boldsymbol{q}\cdot\boldsymbol{n})\,,\,w\rangle_{\partial\Omega_h} &= (f\,,\,w)_{\Omega_h},\\
\langle P_M(\boldsymbol{q}\cdot\boldsymbol{n}),\mu\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\
\langle P_M u\,,\,\mu\rangle_{\partial\Omega} &= \langle g\,,\,\mu\rangle_{\partial\Omega},
\end{aligned}$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$. Subtracting the first four equations defining the weak formulation of the HDG method (1.2) from the above equations, respectively, we obtain the equations for the projection of the errors.

It remains to prove the identity for $\boldsymbol{e}_{\widehat{q}}$. We have

$$\begin{aligned}
\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n} &= P_M(\boldsymbol{q}\cdot\boldsymbol{n}) - P_M(\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n})\\
&= P_M(\boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n} + \boldsymbol{\alpha}((\Pi_W u - P_M u)\boldsymbol{n})\cdot\boldsymbol{n}) - P_M(\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}),
\end{aligned}$$

by the orthogonality property $(A.3)$ of the projection Π_h. Inserting the definition of the numerical trace $\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}$, (1.3), we get

$$\begin{aligned}
\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n} &= P_M(\boldsymbol{e}_q\cdot\boldsymbol{n} + \boldsymbol{\alpha}((e_u - e_{\widehat{u}})\boldsymbol{n})\cdot\boldsymbol{n})\\
&= \boldsymbol{e}_q\cdot\boldsymbol{n} + P_M(\boldsymbol{\alpha}((e_u - e_{\widehat{u}})\boldsymbol{n})\cdot\boldsymbol{n}),
\end{aligned}$$

by the property $(A.4)$ of the trace of the local spaces. This completes the proof. □

Step 2: The energy argument for $\boldsymbol{e}_q$. We are now ready to obtain the upper bound of the L^2-norm of $\boldsymbol{e}_q$. We proceed as follows. Taking $\boldsymbol{v} := \boldsymbol{e}_q$ in the error equation (4.1a), $w := e_u$ in the error equation (4.1b), $\mu := -e_{\widehat{u}}$ in the error equation (4.1c), and $\mu := -P_M(\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n})$ in the error equation (4.1d), and adding the resulting equations up, we obtain

$$(\boldsymbol{e}_q\,,\,\boldsymbol{e}_q)_{\Omega_h} + \Theta_h = (\boldsymbol{\Pi}_V\boldsymbol{q} - \boldsymbol{q}\,,\,\boldsymbol{e}_q)_{\Omega_h},$$

where

$$\begin{aligned}
\Theta_h := &\langle e_{\widehat{u}}\,,\,\boldsymbol{e}_q\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,e_u\rangle_{\partial\Omega_h}\\
&- \langle\boldsymbol{e}_q\cdot\boldsymbol{n}\,,\,e_u\rangle_{\partial\Omega_h} - \langle\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}, e_{\widehat{u}}\rangle_{\partial\Omega_h\setminus\partial\Omega} - \langle P_M(\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}), e_{\widehat{u}}\rangle_{\partial\Omega}.
\end{aligned}$$

By the definition of the projection P_M, we get that

$$\begin{aligned}
\Theta_h &= \langle e_{\widehat{u}}\,,\,\boldsymbol{e}_q\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,e_u\rangle_{\partial\Omega_h}\\
&\quad - \langle\boldsymbol{e}_q\cdot\boldsymbol{n}\,,\,e_u\rangle_{\partial\Omega_h} - \langle\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}, e_{\widehat{u}}\rangle_{\partial\Omega_h\setminus\partial\Omega} - \langle\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}, e_{\widehat{u}}\rangle_{\partial\Omega}\\
&= \langle(\boldsymbol{e}_{\widehat{q}} - \boldsymbol{e}_q)\cdot\boldsymbol{n}\,,\,e_u - e_{\widehat{u}}\rangle_{\partial\Omega_h}\\
&= \langle P_M(\boldsymbol{\alpha}((e_u - e_{\widehat{u}})\boldsymbol{n}))\,,\,(e_u - e_{\widehat{u}})\boldsymbol{n}\rangle_{\partial\Omega_h},
\end{aligned}$$

by the identity (4.2) of Lemma 4.1. Finally, by the definition of the projection P_M and the property $(A.5)$ of the traces of the local spaces, we obtain that

$$\Theta_h = \langle\boldsymbol{\alpha}((e_u - e_{\widehat{u}})\boldsymbol{n})\,,\,(e_u - e_{\widehat{u}})\boldsymbol{n}\rangle_{\partial\Omega_h}.$$

Since $\Theta_h \geq 0$, by the semipositivity property $(A.6)$ of the local stabilization operator $\boldsymbol{\alpha}$, we have that

$$\begin{aligned}
\|\boldsymbol{e}_q\|^2_{\Omega_h} &\leq (\boldsymbol{\Pi}_V\boldsymbol{q} - \boldsymbol{q}\,,\,\boldsymbol{e}_q)_{\Omega_h}\\
&\leq \|\boldsymbol{\Pi}_V\boldsymbol{q} - \boldsymbol{q}\|_{\Omega_h}\|\boldsymbol{e}_q\|_{\Omega_h},
\end{aligned}$$

and the result follows. This completes the proof of Theorem 2.1.

Step 3: The elliptic duality argument for e_u. The estimate of e_u will follow from the following identity.

LEMMA 4.2. *Suppose that the assumptions of Lemma 4.1 are satisfied. Then, we have*

$$(e_u\,,\,\eta)_{\Omega_h}=(\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\,,\,\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{e}_q\,,\,\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h},$$

where $(\phi,\boldsymbol{\theta})$ is the solution of dual problem (2.2).

Proof. We begin by using the second equation (2.2b) of the dual problem to write that

$$\begin{aligned}(e_u\,,\,\eta)_{\Omega_h}&=(e_u\,,\,\nabla\cdot\boldsymbol{\theta})_{\Omega_h}\\&=(e_u\,,\,\nabla\cdot\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{e}_q\,,\,\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{e}_q\,,\,\nabla\phi)_{\Omega_h},\end{aligned}$$

by the first equation (2.2a) of the dual problem. This implies that

$$\begin{aligned}(e_u\,,\,\eta)_{\Omega_h}&=(e_u\,,\,\nabla\cdot\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{e}_q\,,\,\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{e}_q\,,\,\nabla\Pi_W^*\phi)_{\Omega_h}\\&\quad+(e_u\,,\,\nabla\cdot(\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta}))_{\Omega_h}-(\boldsymbol{e}_q\,,\,\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{e}_q\,,\,\nabla(\phi-\Pi_W^*\phi))_{\Omega_h}.\end{aligned}$$

Taking $\boldsymbol{v}:=\boldsymbol{\Pi}_V^*\boldsymbol{\theta}$ in the first error equation, (4.1a), and $w:=\Pi_W^*\phi$ in the second, (4.1b), we obtain that

$$\begin{aligned}(e_u\,,\,\eta)_{\Omega_h}&=(\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\,,\,\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}+\langle e_{\widehat{u}}\,,\,\boldsymbol{\Pi}_V^*\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,\Pi_W^*\phi\rangle_{\partial\Omega_h}\\&\quad+(e_u\,,\,\nabla\cdot(\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta}))_{\Omega_h}-(\boldsymbol{e}_q\,,\,\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{e}_q\,,\,\nabla(\phi-\Pi_W^*\phi))_{\Omega_h},\end{aligned}$$

and, after simple algebraic manipulations, that

$$(e_u\,,\,\eta)_{\Omega_h}=(\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\,,\,\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{e}_q\,,\,\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}+\mathbb{T},$$

where

$$\begin{aligned}\mathbb{T}:=&\langle e_{\widehat{u}}\,,\,\boldsymbol{\Pi}_V^*\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,\Pi_W^*\phi\rangle_{\partial\Omega_h}\\&+(e_u\,,\,\nabla\cdot(\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta}))_{\Omega_h}-(\boldsymbol{e}_q\,,\,\nabla(\phi-\Pi_W^*\phi))_{\Omega_h}.\end{aligned}$$

It remains to prove that $\mathbb{T}=0$.

To do that, we integrate by parts and use the orthogonality properties (*A.1*) and (*A.2*) of the projection Π_h^* to get

$$\begin{aligned}\mathbb{T}&=\langle e_{\widehat{u}}\,,\,\boldsymbol{\Pi}_V^*\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,\Pi_W^*\phi\rangle_{\partial\Omega_h}\\&\quad+\langle e_u\,,\,(\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle\boldsymbol{e}_q\cdot\boldsymbol{n}\,,\,(\phi-\Pi_W^*\phi)\rangle_{\partial\Omega_h}\\&=\langle e_{\widehat{u}}-e_u\,,\,(\boldsymbol{\Pi}_V^*\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle(\boldsymbol{e}_{\widehat{q}}-\boldsymbol{e}_q)\cdot\boldsymbol{n}\,,\,\Pi_W^*\phi-\phi\rangle_{\partial\Omega_h}\\&\quad+\langle e_{\widehat{u}}\,,\,\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle\boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,\phi\rangle_{\partial\Omega_h}\\&=\langle e_{\widehat{u}}-e_u\,,\,(\boldsymbol{\Pi}_V^*\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle(\boldsymbol{e}_{\widehat{q}}-\boldsymbol{e}_q)\cdot\boldsymbol{n}\,,\,\Pi_W^*\phi-\phi\rangle_{\partial\Omega_h}.\end{aligned}$$

Indeed, the fact that $\langle e_{\widehat{u}}\,,\,\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}=0$ is proven as follows. Since $e_{\widehat{u}}$ is single valued on $\mathcal{E}_h$ and $\boldsymbol{\theta}$ lies in $\boldsymbol{H}(div)$, we have that

$$\begin{aligned}\langle e_{\widehat{u}}\,,\,\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}&=\langle e_{\widehat{u}},\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\Omega}\\&=\langle e_{\widehat{u}},P_M(\boldsymbol{\theta}\cdot\boldsymbol{n})\rangle_{\partial\Omega},\end{aligned}$$

by the definition of the projection P_M. The above quantity is equal to zero by the fourth error equation (4.1d) with $\mu:=P_M(\boldsymbol{\theta}\cdot\boldsymbol{n})$.

We can prove that $\langle \boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,\phi\rangle_{\partial\Omega_h}=0$ as follows. By the definition of the projection P_M,

$$\begin{aligned}\langle \boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,\phi\rangle_{\partial\Omega_h}&=\langle \boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n}\,,\,P_M(\phi)\rangle_{\partial\Omega_h}\\&=\langle \boldsymbol{e}_{\widehat{q}}\cdot\boldsymbol{n},\phi\rangle_{\partial\Omega},\end{aligned}$$

by the third error equation (4.1c) with $\mu:=P_M(\phi)$. Finally, by the third equation (2.2c) of the dual problem, we have that $\phi=0$ on $\partial\Omega$ and the result follows.

Now, inserting the expression for $\boldsymbol{e}_{\widehat{q}}$, (4.2), given by Lemma 4.1 in the last expression for $\mathbb{T}$, we get

$$\begin{aligned}\mathbb{T}&=\langle e_{\widehat{u}}-e_u\,,\,(\boldsymbol{\Pi}_V^*\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle P_M(\boldsymbol{\alpha}((e_u-e_{\widehat{u}})\boldsymbol{n})\cdot\boldsymbol{n})\,,\,\Pi_W^*\phi-\phi\rangle_{\partial\Omega_h}\\&=\langle e_{\widehat{u}}-e_u\,,\,(\boldsymbol{\Pi}_V^*\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle \boldsymbol{\alpha}((e_u-e_{\widehat{u}})\boldsymbol{n})\cdot\boldsymbol{n}\,,\,\Pi_W^*\phi-P_M\phi\rangle_{\partial\Omega_h},\end{aligned}$$

by the definition of the projection P_M and the second property (*A.5*) of the traces of the local spaces. Then

$$\begin{aligned}\mathbb{T}&=\langle e_{\widehat{u}}-e_u\,,\,(\boldsymbol{\Pi}_V^*\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle e_u-e_{\widehat{u}}\,,\,\boldsymbol{\alpha}^*((\Pi_W^*\phi-P_M\phi)\boldsymbol{n})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\&=\langle e_{\widehat{u}}-e_u\,,\,(\boldsymbol{\Pi}_V^*\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}+\boldsymbol{\alpha}^*((\Pi_W^*\phi-P_M\phi)\boldsymbol{n})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\&=0,\end{aligned}$$

by the property (*A.5*) of the traces of the local spaces and the orthogonality property (*A.3*) of the projection Π_h^*. This completes the proof. $\square$

Step 4: The estimate for e_u. We are now ready to obtain the estimate of the L^2-norm of e_u and prove Theorem 2.2.

We start by taking $\eta=e_u$ in the identity of Lemma 4.2 to obtain

$$\begin{aligned}\|e_u\|^2_{\Omega_h}&=(\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\,,\,\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{e}_q\,,\,\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}\\&=(\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\,,\,\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{q}-\boldsymbol{q}_h\,,\,\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h}.\end{aligned}$$

If we now use the orthogonality property (*A.1*) of the projection Π_h and the property (*B.2*) that the space $W(K)$ is not too small, we get

$$\|e_u\|^2_{\Omega_h}=(\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\,,\,\boldsymbol{\theta}-\boldsymbol{P}_0\boldsymbol{\theta})_{\Omega_h}-(\boldsymbol{q}-\boldsymbol{q}_h\,,\,\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta})_{\Omega_h},$$

where $\boldsymbol{P}_0$ is the L^2-projection into $\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega):\boldsymbol{v}|_K\in\boldsymbol{P}^0(K)\,\forall\,K\in\Omega_h\}$. By the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\|e_u\|^2_{\Omega_h}&\le\|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_{\Omega_h}\|\boldsymbol{\theta}-\boldsymbol{P}_0\boldsymbol{\theta}\|_{\Omega_h}+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{\Omega_h}\|\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta}\|_{\Omega_h}\\&\le(\|\boldsymbol{\theta}-\boldsymbol{P}_0\boldsymbol{\theta}\|_{\Omega_h}+2\,\|\boldsymbol{\theta}-\boldsymbol{\Pi}_V^*\boldsymbol{\theta}\|_{\Omega_h})\,\|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_{\Omega_h}\\&\le C\,h\,(|\boldsymbol{\theta}|_{1,\Omega_h}+|\phi|_{2,\Omega_h})\,\|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_{\Omega_h}\end{aligned}$$

by the standard approximation properties of the projection $\boldsymbol{P}_0$ and by the approximation property (*B.1*) of the dual projection Π_h^*. Finally, by the elliptic regularity property (2.1) with $\eta:=e_u$, we conclude that

$$\|e_u\|^2_{\Omega_h}\le C\,C^*_{app}\,h\,\|e_u\|_{\Omega_h}\,\|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_{\Omega_h},$$

and the estimate follows. This completes the proof of Theorem 2.2.

Step 5: The estimate for $u - u^*$. By the Poincaré-Friedrichs inequality, we have that

$$\|u - u_h^*\|_K \leq \|\overline{u - u_h^*}\|_K + C\, h_K\, \|\nabla(u - u_h^*)\|_K,$$

where $\overline{w}$ is the average of w over K. But $\overline{u_h^*} = \overline{u_h}$, by the second equation defining u_h^*, (1.4c), and $\overline{u} = \overline{\Pi_W u}$ by Assumptions $(A.2)$ and $(C.1)$. This implies that

$$\|u - u_h^*\|_K \leq \|\Pi_W u - u_h\|_K + C\, h_K\, \|\nabla(u - u_h^*)\|_K.$$

Now, for any $\omega \in \mathcal{W}(K)$, we have that

$$\begin{aligned}\|\nabla(u - u_h^*)\|_K^2 &= (\nabla(u - u_h^*), \nabla(u - \omega))_K + (\nabla(u - u_h^*), \nabla(\omega - u_h^*))_K \\ &= (\nabla(u - u_h^*), \nabla(u - \omega))_K - (\boldsymbol{q} - \boldsymbol{q}_h, \nabla(\omega - u_h^*))_K,\end{aligned}$$

by the first equation defining the postprocessing u_h^*, (1.4b). Applying the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}\|\nabla(u - u_h^*)\|_K^2 &\leq \|\nabla(u - u_h^*)\|_K\, \|\nabla(u - \omega)\|_K + \|\boldsymbol{q} - \boldsymbol{q}_h\|_K\, \|\nabla(\omega - u_h^*)\|_K, \\ &\leq \|\nabla(u - u_h^*)\|_K\, (\|\nabla(u - \omega)\|_K + \|\boldsymbol{q} - \boldsymbol{q}_h\|_K) + \|\boldsymbol{q} - \boldsymbol{q}_h\|_K\, \|\nabla(\omega - u)\|_K,\end{aligned}$$

and, after simple applications of Young's inequality and some algebraic manipulations, we get that

$$\|\nabla(u - u_h^*)\|_K^2 \leq 3(\|\boldsymbol{q} - \boldsymbol{q}_h\|_K^2 + \|\nabla(u - \omega)\|_K^2).$$

This implies that

$$\|u - u_h^*\|_K \leq \|\Pi_W u - u_h\|_K + C\, h_K\, (\|\boldsymbol{q} - \boldsymbol{q}_h\|_K + \|\nabla(u - \omega)\|_K),$$

and so,

$$\|u - u_h^*\|_{\Omega_h} \leq \|\Pi_W u - u_h\|_{\Omega_h} + C\, h\, (\|\boldsymbol{q} - \boldsymbol{q}_h\|_{\Omega_h} + \|\nabla(u - \omega)\|_{\Omega_h}).$$

This completes the proof of Theorem 2.3.

5. Study of the projection $\mathit{\Pi}_h$ for the $\mathbf{HDG}_{[k]}$ method.

In this section, we study the properties of the projection $\mathit{\Pi}_h$ for the $\mathbf{HDG}_{[k]}$ method for squares and cubes. We only give a detailed proof for cubic elements since the case of square elements is similar and easier to prove.

5.1. The system defining the projection is square.

We first show that the system defining the projection Π_h is square. We see that the number of equations is

$$\begin{aligned}dim\, \widetilde{\boldsymbol{V}}(K) + dim\, \widetilde{W}(K) + dim\, M(F) &= 4\, dim\, P^{k-1}(K) + 6\, dim\, P^k(F) \\ &= 4\, dim\, P^k(K) + 2\, dim\, P^k(F)\end{aligned}$$

On the other hand, the number of unknowns is

$$\begin{aligned}dim\, \boldsymbol{V}(K) + dim\, W(K) &= 4\, dim\, P^k(K) + dim\, \nabla \times \widetilde{\boldsymbol{B}_k} \\ &= 4\, dim\, P^k(K) + dim\, \widetilde{\boldsymbol{B}_k} - dim\, \mathbf{Ker}\{\nabla \times \widetilde{\boldsymbol{B}_k}\},\end{aligned}$$

where $\widetilde{\boldsymbol{B}_k} = \{(yzp_1, zxp_2, xyp_3) |\, p_1, p_2, p_3 \in \widetilde{P}^k(K)\}$. Since $dim \widetilde{\boldsymbol{B}_k} = 3\, dim\, \widetilde{P}^k(K)$, and

$$dim\, \mathbf{Ker}\{\nabla \times \widetilde{\boldsymbol{B}_k}\} = dim\, \widetilde{P}^k(K),$$

since $\mathbf{Ker}\{\nabla \times \widetilde{\boldsymbol{B}_k}\} = \{\nabla\,(xyzp_4) |\, p_4 \in \widetilde{P}^k(K)\}$, we finally have that

$$\begin{aligned} dim\, \widetilde{\boldsymbol{V}} + \dim\, \widetilde{W} &= 4\, dim\, P^k(K) + 3\, dim\, \widetilde{P}^k(K) - dim\, \widetilde{P}^k(K) \\ &= 4\, dim\, P^k(K) + 2\, dim\, \widetilde{P}^k(K) \\ &= 4\, dim\, P^k(K) + 2\, dim\, P^k(F), \end{aligned}$$

and we see that this is a square system.

5.2. A property of the local stabilization operator $\boldsymbol{\alpha} := \tau\, \boldsymbol{r}$. Next, we obtain a property of the local stabilization operator $\boldsymbol{\alpha} := \tau\, \boldsymbol{r}$ we are going to need to prove the well posedness of the projection $\mathit{\Pi}_h$.

LEMMA 5.1. *Let p be an element of $W(K)$ such that $(p, w)_K = 0$ for all $w \in \widetilde{W}(K)$. Then*

$$\|p\|_F \le C\, \|\boldsymbol{r}(p\, \boldsymbol{n}|_F)\|_F,$$

for any face $F \in \partial K$.

Proof. Let us pick an arbitrary face F of the cube K. Taking $\mu := p$ and $\boldsymbol{v} := p\, \boldsymbol{n}$ in the definition of the operator $\boldsymbol{r}$, (3.2), we get

$$\begin{aligned} \|p\|_F^2 &= \frac{|F|}{|K|}(p, \boldsymbol{r}(p\, \boldsymbol{n}|_{\boldsymbol{F}}))_K \\ &\le \frac{|F|}{|K|}\|p\|_K\, \|\boldsymbol{r}(p\, \boldsymbol{n}|_{\boldsymbol{F}})\|_K \\ &\le C\, h_K^{-1/2}\, \|p\|_F\, \|\boldsymbol{r}(p\, \boldsymbol{n}|_{\boldsymbol{F}})\|_K \end{aligned}$$

since p is an element of $W(K) := P^k(K)$ such that $(p, w)_K = 0$ for all $w \in \widetilde{W}(K) := P^{k-1}(K)$; see, for example, Lemma 3.1 in [8] for the case in which K is a simplex. This implies that

$$\|p\|_F \le C\, h_K^{-1/2}\ \|\boldsymbol{r}(p\, \boldsymbol{n}|_{\boldsymbol{F}})\|_K.$$

Now, taking $\mu = \eta := p$ in the positivity property of the operator $\boldsymbol{r}$, (3.3), we obtain

$$\begin{aligned} \|\boldsymbol{r}(p\, \boldsymbol{n}|_{\boldsymbol{F}})\|_K^2 &= \frac{|K|}{|F|}\langle \boldsymbol{r}(p\, \boldsymbol{n}|_{\boldsymbol{F}}), p\, \boldsymbol{n}\langle_F \\ &\le C\, h_K\, \|\boldsymbol{r}(p\, \boldsymbol{n}|_{\boldsymbol{F}})\|_F\, \|p\|_F. \end{aligned}$$

This implies that

$$\|p\|_F \le C\, \|\boldsymbol{r}(p\, \boldsymbol{n}|_{\boldsymbol{F}})\|_F^{1/2}\, \|p\|_F^{1/2},$$

and the result follows. This completes the proof. □

5.3. The system defining the projection is unisolvent.

Next we show that the system is unisolvent. It suffices to show that the only solution for the case of a zero right-hand side, that is, when we set $\boldsymbol{q} := \boldsymbol{0}$ and $u := 0$, is the trivial solution.

So, taking $\mu = \Pi_W u$ in the third equation defining the projection, (3.1c), we get that

$$\begin{aligned}\langle \boldsymbol{\alpha}(\Pi_W u\,\boldsymbol{n}), \Pi_W u\,\boldsymbol{n}\rangle_{\partial K} &= -\,\langle \boldsymbol{\Pi}_V \boldsymbol{q}\cdot\boldsymbol{n}, \Pi_W u\rangle_{\partial K}\\ &= -\,(\nabla\cdot(\boldsymbol{\Pi}_V\boldsymbol{q}), \Pi_W u)_K - (\boldsymbol{\Pi}_V\boldsymbol{q}, \nabla\,\Pi_W u)_K\\ &= 0,\end{aligned}$$

by the first and second equations defining the projection, (3.1a) and (3.1b), respectively, an by the inclusion properties (3.1d).

If $\boldsymbol{\alpha} := \tau\,\boldsymbol{Id}$, we have that $\tau\Pi_W u = 0$ on ∂K. This is also true for the choice $\boldsymbol{\alpha} := \tau\,\boldsymbol{r}$. Indeed, we have that

$$\langle \boldsymbol{\alpha}(\Pi_W u\,\boldsymbol{n}), \Pi_W u\,\boldsymbol{n}\rangle_{\partial K} = \sum_F \langle \boldsymbol{\alpha}(\Pi_W u\,\boldsymbol{n}), \Pi_W u\,\boldsymbol{n}\rangle_F = \sum_F \tau_F\,\frac{|F|}{|K|}\,\|\boldsymbol{r}(\Pi_W u\,\boldsymbol{n}|_F)\|_K^2,$$

by the property (3.3) of the function $\boldsymbol{r}$ with $\mu := \eta := \Pi_W u$. This implies that $\tau_F\,\boldsymbol{r}(\Pi_W u\,\boldsymbol{n}|_F) = 0$ on any face F of K, and, by Lemma 5.1, that $\tau\Pi_W u = 0$ on any face of K.

Without loss of generality, we can assume that two of the faces of the cube K lie on the planes $z = 0$ and $z = 1$. Since $\tau > 0$ at the face lying on the plane $z = 0$, we get that $\Pi_W u|_{z=0} = 0$ and hence that $\Pi_W u = z\,r$ for $r \in P^{k-1}(K)$. Taking $w = r$ in the second equation defining the projection, (3.1b), we immediately get that $\Pi_W u = 0$.

It remains to show that $\boldsymbol{\Pi}_V\boldsymbol{q} = \boldsymbol{0}$. To do that, we take $\mu := \boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}$ in the third equation defining the projection, (3.1c), to obtain that $\boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}|_{\partial K} = 0$. In particular, if we use the notation $(q_1, q_2, q_3) := \boldsymbol{\Pi}_V\boldsymbol{q}$, we get that $q_3|_{z=0,1} = 0$. Therefore we can write $q_3 = z(z-1)r'$ for $r' \in P^{k-1}(K)$. Then, taking $\boldsymbol{v} := (0, 0, r')$ in the first equation defining the projection, (3.1a), we conclude that $q_3 = 0$. By the same argument we can prove that $q_1 = q_2 = 0$. This implies that the projection is well defined.

5.4. Approximation properties of the projection.

For the sake of completeness, here we state a result that contains the approximation properties of the projection we are considering.

PROPOSITION 5.2. *Suppose $k \geq 0$, $\tau|_{\partial K}$ is nonnegative and $\tau_K^{\max} := \max\tau|_{\partial K} > 0$. Then there is a constant C independent of K and τ such that*

$$\begin{aligned}\|\boldsymbol{\Pi}_V\boldsymbol{q} - \boldsymbol{q}\,\|_K &\leq C\,h_K^{\ell_{\boldsymbol{q}}+1}\,|\boldsymbol{q}|_{\boldsymbol{H}^{\ell_{\boldsymbol{q}}+1}(K)} + C\,h_K^{\ell_u+1}\,\tau_K^*\,|u|_{H^{\ell_u+1}(K)},\\ \|\Pi_W u - u\|_K &\leq C\,h_K^{\ell_u+1}\,|u|_{H^{\ell_u+1}(K)} + C\,\frac{h_K^{\ell_{\boldsymbol{q}}+1}}{\tau_K^{\max}}\,|\nabla\cdot\boldsymbol{q}|_{H^{\ell_{\boldsymbol{q}}}(K)},\end{aligned}$$

for $\ell_u, \ell_{\boldsymbol{q}}$ in $[0, k]$. Here $\tau_K^* := \max\tau|_{\partial K\setminus F^*}$, where F^* is a face of K at which $\tau|_{\partial K}$ is maximum.

It can be proven by following the proof of the similar result for the projection for HDG methods on simplexes proposed in [10] almost work-by-word manner. It holds when $\boldsymbol{\alpha} := \tau\,\boldsymbol{Id}$ as well as when $\boldsymbol{\alpha} := \tau\,\boldsymbol{r}$.

Note that this implies that when τ is non-negative and is uniformly bounded, the orders of convergence of the projection are $k+1$ for both the scalar variable and the flux, for both squares and cubes.

6. Concluding remarks. We end this paper by discussing some variations on the theoretical results we have proposed in Section 2.

6.1. Other postprocessings. There are several ways to define a new approximation $u_h^* \in W^*(K)$ for which Theorem 2.3 does hold. The following example is particularly useful when working with the $p-version$ of the method; see also [10] and the references therein. On each element $K \in \Omega_h$, the postprocessing u_h^* is defined as the element of $W^*(K)$ such that

$$\begin{aligned} (\nabla u_h^*, \nabla \omega)_K &= -\,(\boldsymbol{q}_h, \nabla \omega)_K \quad && \forall\, \omega \in W^*(K):\ (\omega, \widetilde{w})_k = 0 \text{ for all } \widetilde{w} \in \widetilde{W}(K),\\ (u_h^*, w)_K &= (u_h, w)_K && \forall\, \widetilde{w} \in \widetilde{W}(K). \end{aligned}$$

6.2. Optimal convergence when the local space $W(K)$ is small. When the local space $W(K)$ is small, that is, when it does *not* satisfy *Assumption (B.2)*, the superconvergence of the projection of the error in the scalar variable, $\Pi_W u - u_h$, is not guaranteed, and in general it does not take place. Examples of these methods are the HDG_0 and the BDM_1 methods for simplexes. [HOW about squares?]

However, in this case we can still obtain the optimal order of convergence of $\Pi_W u - u_h$. Indeed, by the identity of Lemma 4.2 obtained by duality, we have

$$\begin{aligned} \|e_u\|^2_{\Omega_h} &= (\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\,,\, \boldsymbol{\Pi}_V^* \boldsymbol{\theta})_{\Omega_h} - (\boldsymbol{e}_q\,,\, \boldsymbol{\theta} - \boldsymbol{\Pi}_V^* \boldsymbol{\theta})_{\Omega_h}\\ &\le \|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\Omega_h} \|\boldsymbol{\Pi}_V^* \boldsymbol{\theta}\|_{\Omega_h} + \|\boldsymbol{e}_q\|_{\Omega_h} \|\boldsymbol{\theta} - \boldsymbol{\Pi}_V^* \boldsymbol{\theta}\|_{\Omega_h}\\ &\le \|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\Omega_h} (\|\boldsymbol{\Pi}_V^* \boldsymbol{\theta}\|_{\Omega_h} + \|\boldsymbol{\theta} - \boldsymbol{\Pi}_V^* \boldsymbol{\theta}\|_{\Omega_h}) && \text{by Theorem 2.1,}\\ &\le \|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\Omega_h} (\|\theta\|_{\Omega_h} + 2\, C_{app}\, h\, (|\phi|_{H^2(\Omega_h)} + |\boldsymbol{\theta}\|_{H^1(\Omega_h)})) \end{aligned}$$

by *Assumption (B.1)*. Finally, after a simple application of the elliptic regularity inequality (2.1) with $\eta := e_u$, we get that

$$\|e_u\|_{\Omega_h} \le C\, \|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\Omega_h}.$$

Thus, even though the *Assumption (B.2)* does not hold. the convergence in the scalar variable can be optimal.

6.3. Superconvergence when the local space $W(K)$ is small. Next, we show that it is still possible to obtain superconvergence of the projection of the error in the scalar variable when the local space $W(K)$ is small, that is, when it does *not* satisfy *Assumption (B.2)*. We do this for the RT_0 method, which is the only method for which this is known to happen. We begin by noting that, by the identity of Lemma 4.2, we have

$$\begin{aligned} \|e_u\|^2_{\Omega_h} &= (\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\,,\, \boldsymbol{\Pi}_V^* \boldsymbol{\theta})_{\Omega_h} - (\boldsymbol{e}_q\,,\, \boldsymbol{\theta} - \boldsymbol{\Pi}_V^* \boldsymbol{\theta})_{\Omega_h}\\ &= (\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\,,\, \boldsymbol{\theta})_{\Omega_h} - (\boldsymbol{q} - \boldsymbol{q}_h\,,\, \boldsymbol{\theta} - \boldsymbol{\Pi}_V^* \boldsymbol{\theta})_{\Omega_h}\\ &= -\,(\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\,,\, \nabla \phi)_{\Omega_h} - (\boldsymbol{q} - \boldsymbol{q}_h\,,\, \boldsymbol{\theta} - \boldsymbol{\Pi}_V^* \boldsymbol{\theta})_{\Omega_h} && \text{by (2.2a),}\\ &= -\,(\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\,,\, \nabla(\phi - \overline{\phi}))_{\Omega_h} - (\boldsymbol{q} - \boldsymbol{q}_h\,,\, \boldsymbol{\theta} - \boldsymbol{\Pi}_V^* \boldsymbol{\theta})_{\Omega_h}\\ &= (\nabla \cdot (\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q})\,,\, \phi - \overline{\phi})_{\Omega_h} - \langle (\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}) \cdot \boldsymbol{n}, \phi - \overline{\phi} \rangle_{\partial\Omega_h}\\ &\quad - (\boldsymbol{q} - \boldsymbol{q}_h\,,\, \boldsymbol{\theta} - \boldsymbol{\Pi}_V^* \boldsymbol{\theta})_{\Omega_h}\\ &= (\nabla \cdot (\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q})\,,\, \phi - \overline{\phi})_{\Omega_h} - \langle (\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}) \cdot \boldsymbol{n}, \phi - \overline{\phi} \rangle_{\partial\Omega_h \setminus \partial\Omega}\\ &\quad - (\boldsymbol{q} - \boldsymbol{q}_h\,,\, \boldsymbol{\theta} - \boldsymbol{\Pi}_V^* \boldsymbol{\theta})_{\Omega_h} \end{aligned}$$

by the boundary condition of the dual problem (2.2c). For the RT_0 method, we can write

$$\|e_u\|^2_{\Omega_h} = (\nabla\cdot\boldsymbol{q} - \overline{\nabla\cdot\boldsymbol{q}}\,,\,\phi - \overline{\phi})_{\Omega_h} - (\boldsymbol{q} - \boldsymbol{q}_h\,,\,\boldsymbol{\theta} - \boldsymbol{\Pi}^*_V\boldsymbol{\theta})_{\Omega_h},$$

and, proceeding in the previous subsection, we can obtain that

$$\|e_u\|_{\Omega_h} \leq C\,h^2\,|\nabla\cdot\boldsymbol{q}|_{H^1(\Omega)} + C\,h\,\|\boldsymbol{q} - \boldsymbol{\Pi}_V\boldsymbol{q}\|_{\Omega_h}.$$

Superconvergence of order two is thus achieved for the projection of the error e_u.

6.4. Other formulas for the numerical trace of the flux. The hybridizable DG method based on the use of the so-called interior penalty (IP) method on each element, see [9], does not use the formula for the numerical trace of the flux (1.3). Instead it uses the formula

$$\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = -\nabla u_h\cdot\boldsymbol{n} + \boldsymbol{\alpha}((u_h - \widehat{u}_h)\boldsymbol{n})\cdot\boldsymbol{n} \quad \text{on} \quad \partial\Omega_h.$$

The application of our approach to this method remains open. However, let us point out that since when $\boldsymbol{\alpha} = \tau\boldsymbol{Id}$, this method is well defined provided τ is of order $1/h$; see [9]. As a consequence, it seems very unlikely that optimal convergence will be attained for the approximate flux.

6.5. Conclusion. The projection-approach we have presented here provides a *simple, unified a priori error analysis* of a large class of finite elements methods including mixed and HDG methods. It provides *sufficient conditions* on the different local spaces and by the local stabilization operator $\boldsymbol{\alpha}$ that guarantee the *superconvergence* of the postprocessing u^*_h. In other words, it gives us guidelines for the devising of new superconvergent methods for elliptic problems.

Indeed, by following them, we showed that the $\mathbf{HDG}_k$ methods for simplexes with the local stabilization operator used in [3, 7, 9] are superconvergent, and uncovered two superconvergent $\mathbf{HDG}_{[k]}$ methods for squares and cubes, and the only finite element method known to be superconvergent for general quadrilaterals and hexahedra.

The extension of this approach to more general elliptic problems, to time-dependent convection-diffusion problems and to the Stokes system of incompressible fluid flow constitutes the subject of ongoing work.

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