Lecture 17. Black-Sholes Equation

Black-Sholes Equation

Itô Process

A generalized Wiener process in which the parameters a and b are functions of the value of the underlying variable x and t. An Itô process can be written as

$$dx = a(x,t)dt + b(x,t)dz$$

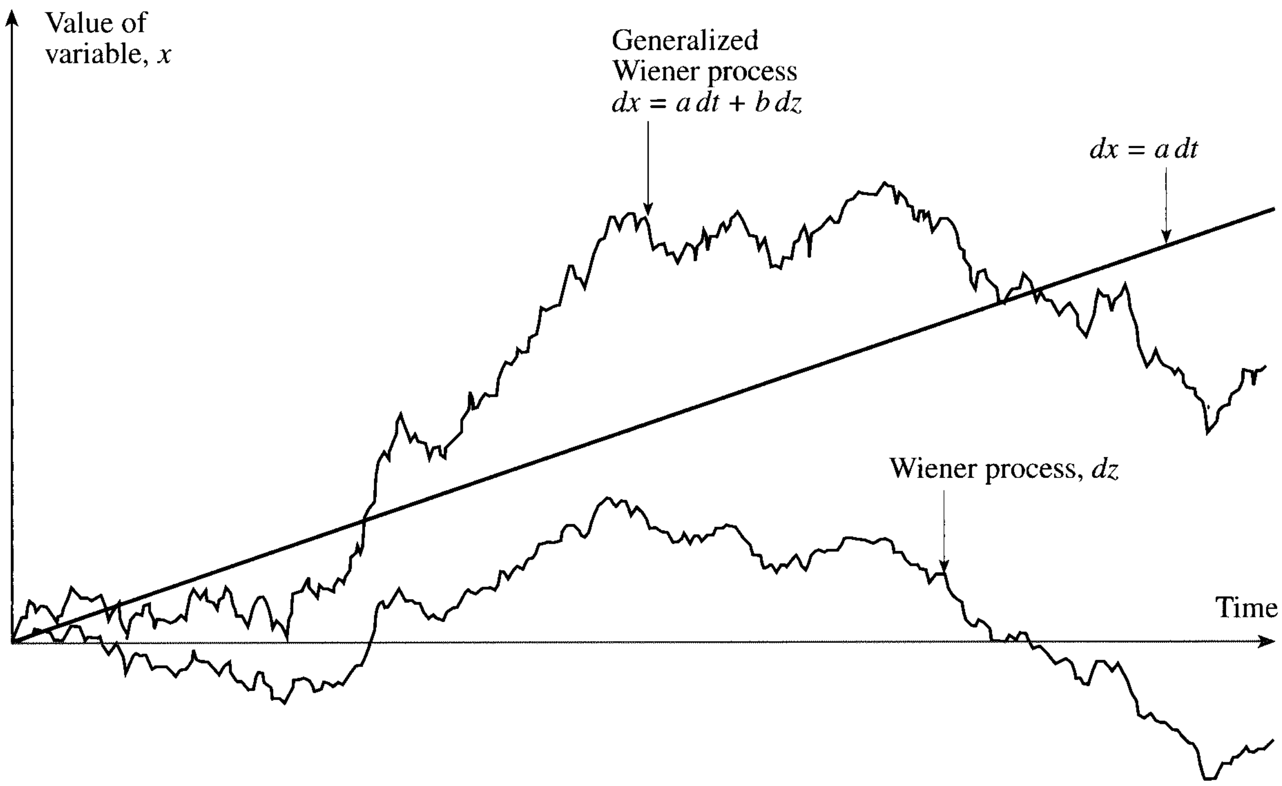
Both the expected drift rate and the variance rate of an Itô process are liable to change over time. In a small time interval between t and $t + \Delta t$, the variable changes from x to $x + \Delta x$, where

$$\Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$$

Thus b^2 is the variance and a is the mean during the interval between t and $t + \Delta t$.

Generalized Wiener Process

Figure 12.2 Generalized Wiener process with $a = 0.3$ and $b = 1.5$.



Process for a stock price

Including volatility then expect: variability of the percentage return in a short period of time Δt is the same regardless of the stock price.

Standard deviation of the change in a short period of time Δt should be proportional to the stock price and leads to

$$dS = \mu S dt + \sigma S dz$$

or

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{1}$$

We use (1) to price stocks. Here σ is the volatility and μ is the expected return rate.

Discrete-Time Model

Discrete-time version of the model is

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t} \tag{2}$$

so the change in the stock value over a short period of time is

$$\Delta S = \mu S \Delta t + \sigma \epsilon S \sqrt{\Delta t}$$

Discrete-Time Model

Left-hand-side of (2) is the return provided by the stock in a short period of time.

- Term $\mu\Delta t$ is the expected value of the return
- Term $\sigma\epsilon\sqrt{\Delta t}$ is the stochastic component of the return. Variance is $\sigma^2\Delta t$ (consistent with the definition of volatility defined earlier).

Then $\Delta S/S$ is normally distributed with mean $\mu\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$, so

$$\frac{\Delta S}{S} \sim \phi(\mu\Delta t, \sigma\sqrt{\Delta t})$$

Itô's Lemma

An Itô process is one in which the drift and the volatility depend on both x and t. Suppose x is an Itô's process then

$$dx = a(x,t)dt + b(x,t)dz$$

where dz is a Wiener process and a, b are functions of x and t. Then x has a variance b^2.

Itô's Lemma states that a function G of x and t follows the following process:

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$

In particular G is an Itô process with drift rate

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

and variance

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2$$

Itô's Lemma: Modeling stock movements

We argued that a reasonable model of stock movements should be

$$dS = \mu S dt + \sigma S dz$$

with μ and σ constants.

From Itô's Lemma we can consider a process G that depends on t and S. Then

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

so both S and G are affected by dz - the noise in the system.

Derivation of Black-Scholes-Merton Differential Equation

1. The stock price follows the process defined earlier for μ and σ:

$$\frac{dS}{S} = \mu dt + \sigma dz$$

2. Short selling of securities with full use of proceeds is permitted
3. There are no transactions costs or taxes. All securities are perfectly divisible
4. There are no dividends during the life of the derivative
5. There are no riskless arbitrage opportunities
6. Security trading is continuous
7. The risk-free rate of interest, r, is constant and the same for all maturities

Derivation of Black-Scholes-Merton Differential Equation

Recall our process for a continuous stock movement modeled on an Itô process with expected gain μ and volatility σ.

$$dS = \mu S dt + \sigma S dz$$

Let f be the price of a call option that depends on S. The variable f depends, then S and t. Then

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz$$

Derivation of Black-Scholes-Merton Differential Equation

We now build a portfolio that will eliminate the stochasticity of the process. The appropriate portfolio (as we will see) is

- -1 option
- $\frac{\partial f}{\partial S}$ shares (recall $\Delta = \frac{fu - f_d}{S_0 u - S_0 d}$ in the binomial tree)

which changes continuously over time. Let Π be the value of the portfolio then

$$\Pi = -f + \frac{\partial f}{\partial S}$$

and $\Delta\Pi$ be the value of the portfolio in the time interval Δt then

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

Derivation of Black-Scholes-Merton Differential Equation

Then

$$\begin{aligned}
\Delta\Pi &= -\Delta f + \frac{\partial f}{\partial S}\Delta S \\
&= -\left[\left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S\Delta z\right] \\
&\quad + \frac{\partial f}{\partial S}\left[\mu S\Delta t + \sigma S\Delta z\right] \\
&= -\left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right]\Delta t
\end{aligned}$$

Note that $\Delta\Pi$ does not depend on dz, therefore there is no risk during time Δt! Thus the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities.

Thus:

$$\Delta\Pi = r\Pi\Delta t$$

Derivation of Black-Scholes-Merton Differential Equation

where r is the risk-free rate. Then

$$-\left[\frac{\partial f}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right]\Delta t=\left[-f+\frac{\partial f}{\partial S}S\right]\Delta t$$

so

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}=rf \tag{3}$$

Equation (3) is the Black-Scholes partial differential equation. Any solution corresponds to the price of a derivative overlying a particular stock.

In order to specify further what the derivative is, we use a boundary condition to constrain it. Boundary conditions for European call options:

$$f=\max\{S-K,0\}$$

when $t=T$. Boundary conditions for European put options:

$$f=\max\{K-S,0\}$$

when $t=T$. The portfolio created is riskless only for infinitesimally short periods.

Black-Scholes Pricing Formulas

The Black-Scholes formulas for the price at time 0 of a European call option on a non-dividend-paying stock and for a European put option on a non-dividend paying stock are

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

and

$$p = K N(-d_2) - S_0 e^{-rT} N(-d_1)$$

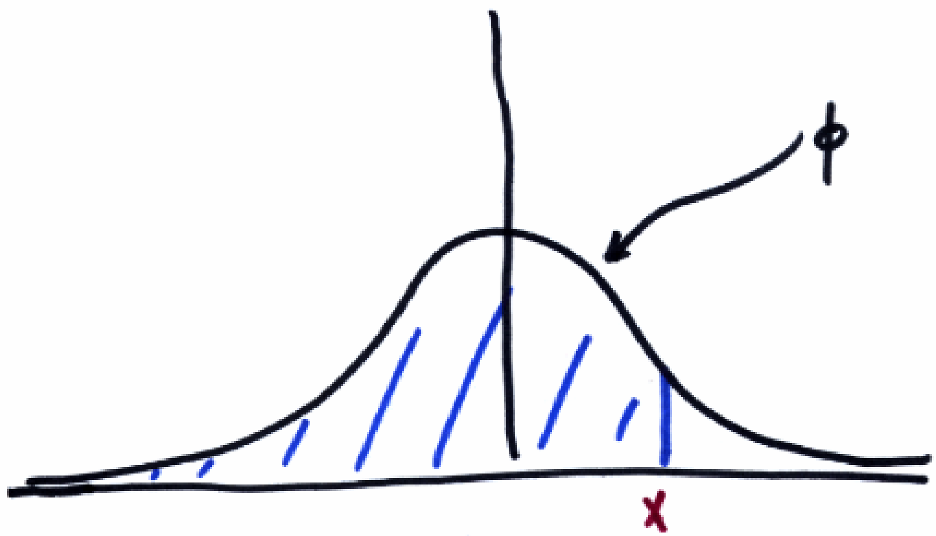
where

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$

and $N(x)$ is the cumulative probability distribution function.

Black-Scholes Pricing Formulas



- $N(x) = \int_{-\infty}^{x} \phi(t)\,dt$
- $1 - N(x) = N(-x)$

The variables c and p are the European call and put prices, S_0 is the current stock price at time 0, K is the strike price, r is the continuously compounded risk-free rate, σ is the stock price volatility, and T is the time to maturity of the option. Why?

Black-Scholes Pricing Formulas

Consider a European call option. The expected value of the option at maturity in a risk-neutral world is

$$\hat{E}[\max\{S_T - K\}, 0]$$

where $\hat{E}$ is the expected value in a risk-neutral world.

From the risk-neutral argument, the European call option price c is the expected valued discounted at the risk-free rate of interest, i.e.

$$c = e^{-rT}\hat{E}[\max\{S_T - K\}, 0] \tag{4}$$

Can check that (4) does indeed solve (3). We now compute the Black-Scholes formulas.

Solving BSM

Recall that we wish to solve the Black-Scholes-Merton PDE

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{5}$$

subject to the boundary conditions

$$f = \max\{S - K, 0\}$$

when $t = T$. If the solution of the European call option is shown to be

$$f = e^{-rT}\hat{E}[\max\{S_T - K\}, 0] \tag{6}$$

then the Black-Scholes Formula holds.

We now prove (6) is indeed the solution.

Solving BSM

We do a change of variables on (5). Set

$$x = \ln \frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)(T - t)$$

$$\tau = T - t$$

$$u = f e^{r(T-t)}$$

Blackboard Calculation

Volatility

In order to measure σ we use historical

Estimating Volatility from Historical Data

Define

- $n + 1$: Number of observations
- S: Stock price at end of the ith interval, with $i = 0, 1, \ldots, n$.
- τ: Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

for $i = 0, 1, \ldots, n$.

Then standard deviation of u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1} n\left(u_i - \overline{u}\right)^2}$$