# Lecture 6. Determining Forward and Futures Prices

Relating forward and future prices to the spot price

# Solutions

Problem 26: A company enters into a short futures contract to sell 5,000 buhels of wheat for 250 cents per bushel. The initial margin is $3,000 and the maintenance margin is $2000. What price change would lead to a margin call? Under what circumstance coud $1,500 be withdrawn from the margin account?

Solution: There is a Margin Call if $1000 is lost on the contract. This happens if the price of wheat futures rises by $0.20 from $2.50 to $2.70 per bushel. $ 1500 can be withdrawn if the futures price falls by $0.30 to $2.20 per bushel.

# Solutions

Problem 27: Suppose that there are no storage costs for corn and the interest rate for borrowing or lending is 5% per annum. How could you make money on Feb. 4, 2004, by trading March 2004 and May 2004 contracts.

Solution: The March 2004 settlement price for corn is 270.25 cents per bushel. The May 2004 settlement price for corn is 275.25 cents per bushel. You could go long one March 2004 corn contract and short one May 2004 contract. In March 2004 you take delivery of the corn borrowing 270.25 cents per bushel at 5% to meet cash outflows. The interest accumulated in two months is about $270.25 \times 0.05 \times \frac{2}{12} = 2.25$ cents per bushel. In May 2004 the corn is sold for 275.25 and $275.25 + 2.25 = 272.50$ cents per bushel has to be repaid on the loan. The strategy therefore leads to a profit of $275.25 - 272.50 = 2.75$ cents per bushel. The profit is independent of the spot price of corn in March 2004 or May 2004.

# Solutions

Problem 28: What position is equivalent to a long forward contract to buy an asset at $K$ on a certain date and a put option to sell it for $K$ on that date.

Solution: A long forward contract is equivalent to a long European call option and a short European put option, both with a strike price of $K$.

If $S_T$ is the final spot price and K is the option price then

If $S_T > K$ : the call option is exercised by the investor and the put expires. Payoff is $S_T - K$.

If $S_T < K$ : the call option is not exercised and the put option is exercised against the investor. Loss is $S_T - K < 0$

In all cases the value of the long call and short put is $S_T - K$, equal to the long forward contract.

# Investment vs. Consumption

- An investment asset is held for investment purposes by the significant portion of investors. Example:
  - Treasury bills
  - Stocks
  -
- A consumption asset is held primarily for consumption or manufacturing. Example
  - Corn
  - Cattle

Use arbitrage arguments to determine the forward & future prices of investment asset from its spot price and observable market variables.

Cannot do this for consumption assets.

# Short Selling Strategies

Short selling involves selling an asset that is not owned by the investor. This is not possible with every asset. Consider short selling a stock:

- Shares are borrowed from a different
- If the broker runs out of shares to borrow, the investor must close out the position immediately (short-squeezed)
- All income from the stock (dividends, etc) is returned to the stock's owner.
- Margin accounts are required.
- Stocks may be shorted only on an uptick in the market

# Assumptions

Assume the following holds for our major market participants:

- Market participants are subject to no transaction cost when they trade
- Market participants are subject to the same tax rate on all net trading profits
- Market participants can borrow money at the same risk-free rate of interest as they can lend money
- Market participants take advantage of arbitrage opportunities as they occur

# Notation

$$T \equiv \text{Time until delivery date in a forward or future contract in years}$$

$$S_0 \equiv \text{Price of the asset underlying the forward or future contracts today}$$

$$F_0 \equiv \text{Forward or future price today}$$

$r \equiv$ Zero-coupon risk-free rate of interest per annum with continuously compounding, for an investment maturing at the delivery date in $T$ years

# Pricing forward contracts on investments Asset with no income

Easiest forward contracts to price are non-income producing contracts. For example non-dividend-paying stocks and zero-coupon bonds.

**Example**: Consider a long forward contract to purchase a non-dividend-paying stock in 3-months. Assume the current stock price is $40 and the 3-month risk-free interest rate is 5% per annum. What should the forward price be?

Suppose the forward price is $43.

- Arbitrageur borrows $40 at the risk-free rate of 5% per year. Buy one share and short a forward contract to sell one share in 3 months.
- After three months, the arbitrageur delivers the share and earns $43 dollars.
- Her costs are $\$40e^{0.5\times 0.25} = \$40.50$.
- The arbitrageur profits $43-40.50 = $2.50 per share.

Arbitrage opportunity (with forward price $43) works so long as the price is above $40.50.

# Nonincome producing example

Suppose the forward price is $39.50.

- Arbitrageur shorts one share at $39.50.
- She invests the $39.50 in the risk-free investment at the 5% rate.
- Arbitrageur then takes a long 3-month forward contract.
- She earns in 3-months $39.50e^{0.5\times 0.25} = \$40$.
- At the end of the three months she takes delivery of the share in the long contract and closes out the short position for a gain of $40 - 39.50 = $0.50.

Arbitrage opportunity (with forward price $39.50) works so long as the price is below $40.50.

| Forward price = $ 43 | Forward price = $39.50 |
|---|---|
| *Action now* | *Action now* |
| Borrow $40 at 5% for 3 months | Short 1 unit of asset to earn $40 |
| Buy 1 unit of the asset | Invest $40 at 5% for 3 months |
| Enter into forward contract | Enter into a forward contract |
| to sell asset in 3 months for $43 | to buy the asset in 3 months for $ 39.50 |
| *Action in 3 months* | *Action in 3 months* |
| Sell asset for $43 | Buy asset for $39.50 |
| Use $ 40.50 to repay the loan with interest | Close short position |
| | Receive $40.50 from investment |
| Profit realized = $2.50 | Profit realized = $0.50 |

# General Rule for Nonincome producing Futures

- Consider a forward contract on investment asset with price $S_0$ with no income
- Let $F_0$ be the future position then

$$F_0 = S_0 e^{rT}$$

Why?

- If $F_0 > S_0 e^{rT}$ then arbitrageur follows the strategy:
  - Borrow $S_0$ dollars at an interest rate $r$ for $T$ years
  - Enter into a short forward contract on the asset
- If $F_0 < S_0 e^{rT}$ then arbitrageur follows the strategy:
  - Short the asset for $S_0$ dollars
  - Enter into a long forward contract on the asset

The formula follows from buying one unit of the asset and enter a short forward contract to sell it for $F_0$ at time $T$. This costs $S_0$ and leads to a cash influx of $F_0$ at time $T$. Therefore, $S_0$ must equal the present value of $F_0$, or

$$S_0 = F_0 e^{-rT} \equiv \text{ if and only if } F_0 = S_0 e^{rT}$$

# Use of pricing formula

Consider a 10 month forward contract to buy a zero-coupon bond that will mature 2 years from today (bond still has 14 months at the end of the contract). The current price of the bond is \$930. Assume the 10-month risk-free rate of interest, continuously compounded is 6% per annum.

Recall $F_0 = S_0 e^{rT}$ with $S_0 = 930$, $r = 0.06$, $T = \frac{10}{12}$. The forward price is

$$F_0 = 930 e^{0.06 \times \frac{10}{12}} = \$977.68$$

This is the delivery price in a contract negotiated today.

# What if short sales are not possible?

Many commodities do not allow short sales. Not important for pricing reasons.

If forward price is too low, then enough asset owners will sell their position and take a long position

- If $F_0 > S_0 e^{rT}$ then investor adopts the strategy:
  - Borrow $S_0$ dollars at interest rate $r$ for $T$ years
  - Buy one-contract's worth of the asset
  - Short a forward contract
  - Investor makes $F_0 - S_0 e^{rT}$ on the transaction
- If $F_0 < S_0 e^{rT}$ then any owner of the asset adopts the strategy:
  - Sell gold for $S_0$ dollars
  - Buy one-contract's worth of the asset
  - Invest the cash at interest rate $r$ for $T$ years
  - Investor makes $S_0 e^{rT} - F_0$ on the transaction

Forward price will adjust so that neither arbitrage opportunity exists.

# Pricing forward contracts on investments with known income

We now consider pricing forward contracts with fixed income as part of the contract. For example dividend-yielding stocks and coupon-bearing bonds.

**Example**: Consider a long forward contract to purchase a coupon-bearing bond whose current price is \$900. Suppose the forward contract matures in 9 months. Assume the coupon payment of \$40 is expected after 4 months. Assume that the 4-month and 9-month risk-free continuously compounded interest rate are 3% and 4% per annum, respectively.

Suppose the forward price is \$920.

- Arbitrageur borrows \$900 to purchase the bond and short a forward contract.
- To compute the present value of the first coupon, we discount $40e^{-0.03\times\frac{4}{12}} = 39.60$
- The remaining \$860.40 is borrowed at 4% annually for 9 months, so $860.40e^{0.04\times\frac{9}{12}} = \$886.60$
- The arbitrageur makes

$$910 - 886.60 = \$33.40$$

Arbitrage opportunity works so long as the price is above \$886.60.

# Forward price on investment with known income

Suppose the forward price is \$860.

- Arbitrageur shorts the bond and enters a long forward contract.
- Of the \$900 earned by shorting the bond, \$ 39.60 is invested for 4-months at 3% per annum. This matures at the correct time and pays for the coupon.
- The remaining \$860.40 is invested for 9 months at 4% per year and grows to \$886.60. \$900 to purchase the bond and short a forward contract.
- To compute the present value of the first coupon, we discount $40e^{-0.03\times\frac{4}{12}} = 39.60$
- The remaining \$860.40 is borrowed at 4% annually for 9 months.

$$860.40e^{0.04\times\frac{9}{12}} = \$886.60$$

- The arbitrageur makes

$$886.60 - 860 = \$26.60$$

Arbitrage opportunity works so long as the price is below \$886.60.

# General Rule for Income Producing Futures

- Consider a forward contract on investment asset with price $S_0$ with income with a present value of $I_0$ during the life of a forward contract.
- Let $F_0$ be the future position then

$$F_0 = (S_0 - I)\, e^{rT}$$

Why?

- If $F_0 > (S_0 - I)\, e^{rT}$ then an arbitrageur can buy the asset and short a forward contract.
- If $F_0 < (S_0 - I)\, e^{rT}$ then an arbitrageur can short the asset and take a long position in a forward contract.
- If short sales are not possible, the investors who own the asset will find it profitable to sell the asset and enter into long forward contracts.

# General Rule for Income Producing Futures, cont.

Alternative explanation via the following strategy

- Buy one unit of the asset
- Enter into a short forward contract to sell it for $F_0$ at time $T$.
- This costs $S_0$ and leads to a cash inflow of $F_0$ at time $T$ and income $I$.
- The initial outflow is $S_0$.
- The present value of the inflows is $I + F_0 e^{-rT}$. Hence

$$S_0 = I + F_0 e^{-rT} \text{ if and only if } F_0 = (I - S_0) e^{rT}$$

# Arbitrage Strategy

| Forward price = $ 910 | Forward price = $860 |
|---|---|
| *Action now* | *Action now* |
| Borrow $900: $39.50 for 4months | Short 1 unit of asset to realize $900 |
| and $860.40 for 9 months | Invest $39.60 for 4 months |
| Buy 1 unit of the asset | and $860.40 for 9 month |
| Enter into forward contract to sell | Enter into a forward contract to buy |
| asset in 9 months for $910 | asset in 9 months for $ 870 |
| *Action in 4 months* | *Action in 4 months* |
| Receive $40 of income on asset | Receive $40 of income on asset |
| Use $40 to repay first loan | Pay income of $ 40 on asset |
| with interest | |
| *Action in 9 months* | *Action in 9 months* |
| Sell asset for $910 | Receive $886.60 from 9 month investment |
| Use $886.60 to repay second load | Buy asset for $870 |
| with interest | Close out short position |
| Profit realized = $23.40 | Profit realized = $26.60 |

# Example

Consider a 10 month forward contract on a stock with a price of \$50. We assume that the risk-free rate of interest, continuously compounded, is 8% per annum for all maturities. We also assume that dividends of \$0.75 per share are expected after 3 months, 6 months, and 9 months. The present value of the dividends, $I$, is

$$I = 0.75e^{-0.08\times\frac{3}{12}} + 0.75e^{-0.08\times\frac{6}{12}} + 0.75e^{-0.08\times\frac{9}{12}} = 2.162$$

The variable $T$ is 10 months, so that the forward price, $F_0$ is

$$F_0 = (S_0 - I)\, e^{rT} = (50 - 2.162)\, e^{0.08\times\frac{10}{12}} = \$51.14.$$

If the forward price is less than \$51.14 then an arbitrageur wold short the stock spot and buy forward contracts. If the forward price is greater than \$51.14, an arbitrageur would short forward contracts and buy the stock spot.

# Pricing forward contracts on investments with known yield

We now consider pricing forward contracts with fixed known yield instead of a known income. In particular the income is expressed as a percentage of the asset's value at the time the income is paid.

Let $q$ be the average yield per annum on an asset during the life of a forward contract with continuous compounding, then

$$F_0 = S_0 e^{(r-q)T}.$$

This follows since

**Example**: Consider a 6 month forward contract on an asset that is expected to provide an income equal to 2% of the asset price once during a 6-month period. The risk-free rate of interest with continuous compounding is 10% per annum. The asset price is \$25. In this case, $S_0 = 25$, $r = 0.10$, and $T = 0.5$. The yield is 4% per annum with semiannual compounding. Then

$$r_C = 2 \ln \left( 1 + \frac{0.04}{2} \right) = 0.0396$$

Therefore, $q = 0.0396$,

$$F_0 = 25 e^{(0.10-0.0396)\times 0.5} = \$25.77$$

# Valuing Forward Contracts

Forward contracts initially have a value of zero, but the value changes daily with the *marking to market*.

$$\begin{aligned} K &\equiv \text{ delivery price for a contract that was negotiated in the past} \\ T &\equiv \text{ time to delivery date, in years} \\ r &\equiv \text{ the } T\text{-year risk-free interest rate} \\ F_0 &\equiv \text{ forward price that would be applicable if negotiated the contract today} \\ f &\equiv \text{ value of the forward contract today} \end{aligned}$$

$$f_0 = (F_0 - K)\, e^{-rT}$$

# Valuing Forward Contracts

- Value of forward contracts with no income

$$f = S_0 - Ke^{-rT}$$

- Value of forward contracts with known income

$$f = S_0 - I - Ke^{-rT}$$

- Value of forward contracts with known yield

$$f = S_0e^{-rT} - Ke^{-rT}$$

# Example of valuing forward contracts

A long forward contract on a non-dividend-paying stock was entered in the past. The currently has 6 months to maturity. The risk-free rate of interest with continuous compounding is 10% per annum, the stock price is \$25, and the delivery price is \$24. Here $S_0 = 25$, $r = 0.10$, $T = 0.5$, and $K = 24$. We have

$$F_0 = S_0 e^{rT} = 25 e^{0.1 \times 0.5} = \$26.28$$

The value of the forward contract is

$$f = (F_0 - K)\, e^{-rT} = (26.28 - 24)\, e^{-0.1 \times 0.5} = \$2.17$$

Therefore, this contract contains value, depending on the spot price

# Futures prices of Stock Indices

We consider how to determine the price of stock indices.

- Stock index can usually be regarded as the price of an investment asset that pays dividends.
- The investment asset is the portfolio of stocks underlying the index.
- Dividends paid by the investment asset are the dividends that would be received by the holder of the portfolio.
- Usually assumed that dividends provide a known yield rather than a known cash income.
- If $q$ is the dividend yield rate gives the future price $F_0$ as

$$F_0 = S_0 e^{(r-q)T}$$

**Example**: Consider a 3 month futures contracts on the S&P 500. Suppose that the stocks underlying the index provide a dividend yield of 1% per annum, that the current value of the index is 800, and that the continuously compounded risk-free interest rate is 6% per annum. Here $r = 0.06$, $S_0 = 800$, $T = 0.25$, and $q = 0.01$. The futures price $F_0$ is therefore,

$$F_0 = S_0 e^{(r-q)T} = 800 e^{(0.06-0.01)\times 0.25} = \$810.06.$$

# Futures prices of Stock Indices

The risk free rate is 7% per annum with continuous compounding, and the dividend yield on the stock index is 3.2% per annum. The current value of the index is 150. What is the 6 month futures price?

We set $r = 0.07$ , $q = 0.032$, and $T = 0.5$. Then

$$F_0 = S_0 e^{(r-q)T} = 150 e^{(0.07-0.032)\times 0.5} = 152.88$$

or $152.88.

# Futures prices of Stock Indices

**Remark**:

- In practice the dividend yield on the portfolio underlying an index varies week by week throughout the year.

**Index Arbitrage**:

- If $F_0 > S_0 e^{(r-q)T}$, profits can be made by buying the stocks underlying the index at the spot price and shorting future contracts. Often done by a corporation holding short-term money market investments.
- If $F_0 < S_0 e^{(r-q)T}$ profits can be made by shorting the stocks underlying the index and taking a long position in future contracts. Often done by a pension fund that owns an indexed portfolio of stocks.
- Such strategies are called index arbitrage
- Index arbitrage can be approximated by trading a sample of stocks whose movements closely track large index funds.
- Often accomplished via program trading using computers to institute trades

# Pricing Forward and Futures Contracts on Currencies

- Consider the price of contracts on currency exchanges
- Let $S_0$ define the current spot price, in dollars, of one unit of foreign currency.
- Let $F_0$ define the forward or futures price in dollars of one unit of foreign currency.
- This is **not necessarily** the way in which such contracts are quoted.
- Major exchange rates ( outside of the pound, euro, Australian dollar, and New Zealand dollar) quote number of units of currency equivalent to one dollar.
- Foreign currency earns interest at the risk-free rate in the foreign country.

Let $r_j$ be the value of foreign risk-free interest rate when money is invested for time $T$. Let $r$ be the US dollar risk-free rate when money is invested for time $T$.

$$F_0 = S_0 e^{(r - r_j)T}$$

# Pricing Forward and Futures Contracts on Currencies, cont.

Why?

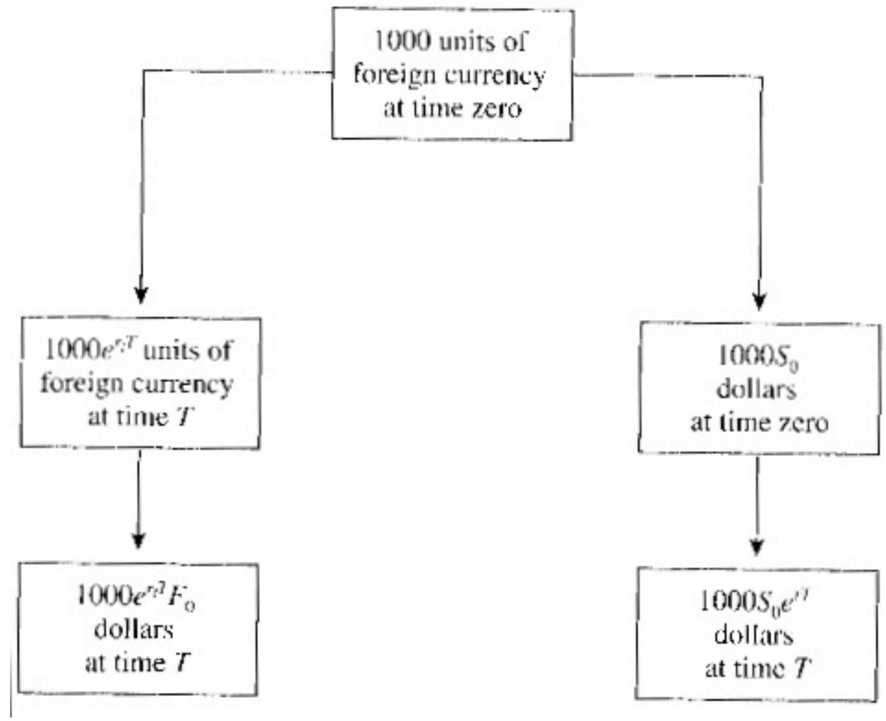
- Start with 1000 units of foreign currency. Two approaches to converting the money to USD.
- First, invest for $T$ years at $r_j$ and enter into a forward contract to sell the proceeds for dollars at time $T$. Generates $1000e^{r_jT}F_0$.
- Second, exchange on the spot market and invest the proceeds for $T$ years at rate $r$. This generates $1000S_0e^{rT}$.
- Absence of arbitrage yields

$$1000F_0e^{r_fT} = 1000S_0e^{rT}$$

or

$$F_0 = S_0e^{(r-r_j)T}$$

# Pricing Forward and Futures Contracts on Currencies, cont.



# Foreign currency as an asset providing known yield

**Example**: Suppose that the 2-year interest rates in Australia and the United States are 5% and 7%, respectively, (continuously compounded) and the spot exchange between the Australian dollar (AUD) and the US dollar (USD) is 0.6200 USD per AUD. The 2-year forward exchange rate should be

$$0.62e^{(0.07-0.05)\times 2} = 0.6453$$

Suppose that the 2-year forward exchange rate is less than this, say 0.6300. Arbitrageur should

- Borrow 1000 AUD at 5% per annum for 2 years, convert to 620 USD and invest the USD at 7%.
- Enter into a forward contract to buy 1,105.17 AUD for $1,105.17 \times 0.63 = 696.26$ USD.

The 620 USD that are invested at 7% grow to $620e^{0.07\times 2} = 713.17$ USD in 2 years. Of this 696.26 USD are used to purchase 1105.17 AUD under the terms of the forward contract. This is enough to repay principal and interest on the 1000 AUD that are borrowed, since

$$1000e^{0.05\times 2} = 1105.17$$

This yields a risk-less profit of

$$713.17 - 696.26 = 16.91$$

USD.

# Example, cont.

Suppose that the 2-year forward exchange rate is more than this, say 0.6600. Arbitrageur should

- Borrow 1000 USD at 7% per annum for 2 years, convert to 1000/0.6200 AUD, and invest the AUD at 5%.
- Enter into a forward contract to sell 1,782.53 AUD for $1,782.53 \times 0.66 = 1,16.47$ USD.

The 1,612.90 AUD that are invested at 5% grow to $1,612.90e^{0.05\times 2} = 1,782.53$ AUD in 2 years. The forward contract converts this to 1,176.47 USD. The amount needed to payoff the USD borrowings is $1000e^{0.07\times 2} = 1,150.27$ USD. The strategy gives a rise to a risk-less profit of

$$1,176.47 - 1,150.27 = 26.20$$

USD.

# Remark on Foreign Currency Futures

Note that the formula for forward contracts on assets with a known yield is

$$F_0 = S_0 e^{(r-q)T}$$

and the formula for currency forward contracts with different risk-less rates is

$$F_0 = S_0 e^{\left(r-r_j\right)T}$$

This means we can view foreign currencies as an investment asset paying a known yield. The yield is the risk-free rate of interest of the foreign currency.

# Futures on Commodities

We consider pricing futures contracts on commodities that are primarily investment assets, such as precious metals.

- Gold owners can earn income from leasing the gold.
- The interest charged from leasing gold is the gold lease rate
- Also holds for silver.

In the absence of storage and income, the forward price of the commodity that is an investment asset is

$$F_0 = S_0 e^{rT}$$

We treat storage costs as negative income, then set $U$ to be the present value of all storage costs, net of income, during the life of a forward contract, then

$$F_0 = (S_0 + U)\, e^{rT}$$

# Futures on Commodities, cont.

**Example**: Consider a 1 year futures contract on an investment asset that provides no income. It costs \$2 per unit to store the asset, with the payment being made at the end of the year. Assume that the spot price \$450 per unit and the risk-free rate is 7% per annum for all maturities. This corresponds to $r = 0.07$, $S_0 = 450$, $T = 1$, and

$$U = 2e^{-0.07\times 1} = 1.865$$

The futures price, $F_0$ is given by

$$F_0 = (450 + 1.865)\, e^{0.07\times 1} = \$484.63$$

If the actual futures price is greater than 484.63, an arbitrageur can buy the asset and short 1-year futures contracts to lock in a profit. If the actual futures price is less than 484.63, an investor who owns the asset can improve the return by selling the asset and buying futures contracts.

If storage costs are proportional to the price of the commodity then it can be treated as negative yield:

$$F_0 = S_0 e^{(r+u)T}$$

# Consumption Commodities

Commodities that are consumption assets (as opposed to investment assets) usually provide no income, but can be subject to significant storage costs. We consider if our pricing model holds:

$$F_0 > (S_0 + U)\, e^{rT}$$

Arbitrageur should follow the following strategy:

- Borrow an amount $S_0 + U$ at the risk-free rate and use it to purchase one unit of the commodity and to pay for storage.
- Short a forward contract on one unit of the commodity.

This would ideally lead to a profit of $F_0 - (S_0 + U)e^{rT}$ at time $T$. However, there is a tendency for $S_0$ to increase and $F_0$ to decrease until $F_0 > (S_0 + U)\, e^{rT}$ no longer holds. Therefore, $F_0 > (S_0 + U)\, e^{rT}$ cannot hold for long periods of time.

# Consumption Commodities

On the other hand, consider

$$F_0 < (S_0 + U)\, e^{rT}$$

Then an investor should do the following:

- Sell the commodity, save the storage costs, and invest the proceeds at the risk-free interest rate.
- Take a long position in a forward contract.

The result is a riskless rate of $(S_0 + U)e^{rT} - F_0$ relative to the position they would've been had they held the commodity. Therefore, we seem to get $F_0 = (S_0 + U)\, e^{rT}$ for the pricing of the commodity.

However, companies who keep consumption commodities in storage, do so because of its consumption value, not because of investment value. Such owners are reluctant to sell the commodity and buy forward contracts, because forward contracts cannot be consumed. There is nothing to stop $F_0 < (S_0 + U)\, e^{rT}$ from holding.

Users of the consumption asset place different value to the commodity than investors.