Lecture 16

Exotic Options

Exotic Options

American and European options are known as **plain vanilla options**. Recently there have developed nonstandard **exotic options** that have become parts of portfolios.

May be useful

- for hedging purposes
- for tax, accounting, legal, or regulatory reasons
- since they are designed to reflect a view on potential future movements in particular market variables

Throughout we assume the asset provides a known yield of q.

Package

A **package** is a portfolio consisting of standard European calls, standard European puts, forward contracts, cash, and the underlying asset itself.

Examples of such packages include the bull spreads, bear spreads, butterfly spreads, calendar spreads, straddles, etc.

Often the package is structured so as to have zero cost initially.

Nonstandard American Options

In practice American options traded in OTC markets have nonstandard features.

1. Early exercise may be restricted to certain dates. The instrument is then known as a **Bermudan option**.
2. Early exercise may be allowed during only part of the life of the option. For example, there may be an initial "lock-out" period with no early exercise
3. The strike price may change during the life of the option.

Nonstandard American options can usually be valued using a binomial tree. At each node the test for early exercise is adjusted to reflect the terms of the option.

Forward Start Options

A **forward start option** are options that will start at some time in the future.

Executive stock options can be viewed as a type of forward start option. This is because a company commits to granting at-the-money options to employees in the future.

- Consider a forward start at-the-money European call option that will start at time T_1 and mature at time T_2.
- Suppose that the asset price is S_0 at time zero and the S_1 at time T_1.
- To value the option, we note from the European option pricing formulas that the value of an at-the-money call option is proportional to the asset price, since

$$c = S_0 N(d_1) - K e^{-rT} N(d_2) = S_0 \left(N(d_1) - e^{-rT} N(d_2) \right)$$

where $d_1 = \ln \frac{S_0}{K} + (r + \frac{\sigma^2}{2})T = (r + \frac{\sigma^2}{2})T$ and $d_2 = (r - \frac{\sigma^2}{2})T$.
The value of the forward start option at time T_1 is therefore cS_1/S_0, where c is the value at time zero of an at-the-money option that lasts for $T_2 - T_1$.

- Using risk-neutral valuation the value of the forward start option at time zero is

$$e^{-rT_1}\hat{E}\left[c\frac{S_1}{S_0}\right]$$

where $\hat{E}$ is the expected value in the risk-neutral world.

- Since c and S_0 are known and $\hat{E}\left[S_1\right] = S_0e^{(r-q)T_1}$, then the value of the forward start option is ce^{-qT_1}. For a non-dividend-paying stock, $q = 0$ and the value of the forward start option is exactly the same as the value of a regular at-the-money option with the same life as the forward start option.

Compound Options

Compound options are options on options. There are four main kinds

- call on a call
  - At time T_1 the holder of the compound option is entitled to pay the first strike price K_1 and receive a call option.
  - The call option then gives the holder the right to buy the underlying asset for the second strike price K_2 on the second date T_2.
  - The compound option will be exercised on the first exercise date only if the value of the option on that date is greater than the first strike price.

  Let $M(a, b; \rho)$ be the cumulative bivariate normal distribution function that the first variable will be less than a and the second variable will be less than b when the coefficient of correlation between the two is ρ.

  Let S^* be the asset price at the time T_1 for which the option price at time T_1 equals K_1. If the actual asset price is above S^* at time T_1, the first option will be exercised; if it is not above S^*, the option expires worthless.

The price is

$$S_0 e^{-qT_2} M(a_1, b_1; \sqrt{T_1/T_2}) - K_2 e^{-rT_2} M(a_2, b_2; \sqrt{T_1/T_2}) - e^{-rT_1} K_1 N(a_2)$$

where

$$a_1 = \frac{\ln \frac{S_0}{S^*} + (r - q + \frac{\sigma^2}{2})T_1}{\sigma\sqrt{T_1}}, \qquad a_2 = a_1 - \sigma\sqrt{T_1}$$

$$b_1 = \frac{\ln \frac{S_0}{K_2} + (r - q + \frac{\sigma^2}{2})T_2}{\sigma\sqrt{T_2}}, \qquad b_2 = b_1 - \sigma\sqrt{T_2}$$

- put on a call
  The price is

$$K_2 M(a_1, b_1; \sqrt{T_1/T_2}) - S_0 e^{-qT_2} e^{-rT_2} M(a_2, b_2; \sqrt{T_1/T_2}) + e^{-rT_1} K_1 N(a_2)$$

- call on a put
- put on a put

Why?

Chooser Options

A **chooser** option has the feature that after a specified period of time, the holder of the option can chose whether the option is a call or a put.

Suppose that the time when the choice is made is T_1. THe value of the chooser option at this time is

$$\max\{c, p\}$$

where c is the value of the call underlying the option and p is the value of the put underlying the option.

If the options underlying the chooser option are both European and have the same strike price, put-call parity can be used to provide a valuation formula.

Suppose that S_1 is the asset price at time T_1, K is the strike price, T_2 is the maturity of the options, and r is the risk-free interest rate. Put-call parity implies

$$\begin{aligned}\max\{c, p\} &= \max\{c, c + Ke^{-r(T_2-T_1)} - S_1 e^{-q(T_2-T_1)}\} \\ &= c + e^{-q(T_2-T_1)} \max\{0, Ke^{-(r-q)(T_2-T_1)} - S_1\}\end{aligned}$$

Hence the chooser is a sum of

1. A call option with strike price K and maturity T_2
2. $e^{-q(T_2-T_1)}$ put options with strike price $Ke^{-(r-q)(T_2-T_1)}$ and maturity T_1.

We can use Black-Scholes to compute the exact values.

Barrier Options

Barrier options are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time

- A number of different types of barrier options regularly trade in the OTC market. They are attractive to some market participants because they are less expensive than the corresponding regular options.
- Barrier options can be classified as one of two types:
- A **knock-out option** ceases to exist when the underlying asset price reaches a certain barrier
- A **knock-in option** comes into existence only when the underlying asset price reaches a barrier

A **down-and-out call** is one type of knock-out option. It is a regular call option that ceases to exist if the asset price reaches a certain barrier level H. The barrier level is below the initial asset price.

The corresponding knock-in option is a **down-and-in call** which comes into existence once the asset reaches a certain barrier value.

Recall that

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

$$c = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

where

$$d_1 = \frac{\ln \frac{S_0}{K} + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

We can give a few pricing formulas. The price of a down-and-in call is

$$c_{di} = S_0 e^{-qT} \left(\frac{H}{S_0}\right)^{2\lambda} N(y) - K e^{-rT} \left(\frac{H}{S_0}\right)^{2\lambda-2} N(y - \sigma\sqrt{T})$$

where

$$\lambda = \frac{r - q + \frac{\sigma^2}{2}}{\sigma^2}$$

$$y = \frac{\ln \frac{H^2}{S_0 K}}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

Since the value of a regular call equals the value of a down-and-in call plus the value of a down-and-out call, the value of a down-and-out cll is given by

$$c_{do} = c - c_{di}$$

There are seven other similar barrier options.

Note that barrier options can have quite different properties from regular vanilla options. For example, the vega can sometimes be negative. Consider an up-and-out call option when the asset price is close to the barrier level.

As a volatility increases the probability that the barrier will be hit increases. As a result, a volatility increase can cause the price of the barrier option to decrease in these circumstances.

Binary Options

Binary options are options with discontinuous payoffs.

- A **cash-or-nothing call** pays nothing if the asset price ends up below the strike price at time T and pays a fixed amount Q if it ends up above the strike price.
- In a risk-neutral world, the probability of the asset price being above the strike price being above the strike price at the maturity of an option is

  $$N(d_2)$$

  The value of the cash-or-nothing call is therefore,

  $$Qe^{-rT}N(d_2)$$

- A **cash-or-nothing put** pays off Q if the asset price is below the strike price and nothing if it is above the strike price. The value of a cash-or-nothing put is

  $$Qe^{-rT}N(-d_2)$$

- An **asset-or-nothing call** pays nothing if the underlying asset ends up below the strike price and pays an amount equal to the asset price if it ends up above the strike price. Therefore, the asset-or-nothing call is worth
$$S_0e^{-rT}N(d_2)$$
- An **asset-or-nothing call** pays nothing if the underlying asset ends up above the strike price and pays an amount equal to the asset price if it ends up below the strike price. Therefore, the asset-or-nothing call is worth
$$S_0e^{-rT}N(-d_2)$$
- A regular European call option is equivalent to a long position in an asset-or-nothing call and a short position in a cash-or-nothing call where the cash payoff on the cash-or-nothing call equals the strike price.

Lookback Options

The payoffs from lookback options depend on the maximum or minimum asset price reached during the life of the option.

The payoff from a European lookback call is the amount that the final asset price exceeds the minimum asset price achieved during the life of the option.

The payoff from a European lookback put is the amount which the maximum asset price achieved during the life of the option exceeds the final asset price.

Valuation of a European lookback call at time zero is

$$S_0e^{-qT}N(a_1) - S_0e^{-qT}\frac{\sigma^2}{2(r-q)}N(-a_1) - S_{min}e^{-rT}\left[N(a_2) - \frac{\sigma^2}{2(r-q)}e^{Y_1}N(-a_3)\right]$$

where

$$a_1 = \frac{\ln\frac{S_0}{S_{min}} + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$a_2 = a_1 - \sigma\sqrt{T}$$

$$a_3 = \frac{\ln\frac{S_0}{S_{min}} + (-r + q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$Y_1 = -\frac{2(r - q - \frac{\sigma^2}{2})\ln\frac{S_0}{S_{min}}}{\sigma^2}$$

Shout Options

A **shout option** is a European option where the holder can "shout" to the writer at one time during its life. At the end of the option, the option holder receives ether the usual payoff from the European option or the intrinsic value at the time of the shout, whichever is greater.

Suppose the final strike price is \$50 and the holder of a call shouts when the price of the underlying asset is \$60. If the final asset price is less than \$60, the holder receives \$10. If it is greater than \$60, the holder receives the excess of the asset price over \$50.

A shout option has some of the same features as a lookback option, but is considerably less expensive.

To value the shout, note that if the holder shouts at time τ and the asset price is S_τ then the payoff from the option is

$$\max\{0, S_T - S_\tau\} + (S_\tau - K)$$

Thus the value at time τ if the holder shouts is the present value of $S_\tau - K$ plus the value of a European option with strike price S_τ. The latter can be computed using Black-Scholes.

To construct a numerical price, we roll back through a binomial or trinomial tree for the underlying asset in the usual way. We calculate at each node the value of the option if we shout and the value if we do not shout. The option's price is therefore similar to valuing a regular American option.

Asian Options

Asian options are options where the payoff depends on the average price of the underlying asset during at least some part of the life of the option.

The payoff from the an **average price call** and an **average price put** are

$$\max\{0, S_{ave} - K\} \text{ and } \max\{0, K - S_{ave}\},$$

respectively. Here S_{ave} is the average value of the underlying asset calculated over a predetermined averaging period.

- Average price options are less expensive than regular options
- Someways more appropriate for hedging portfolios.

The payoff from the an **average strike call** and an **average strike put** are

$$\max\{0, S_T - S_{ave}\} \text{ and } \max\{0, S_{ave} - S_T\},$$

respectively. Here S_{ave} is the average value of the underlying asset calculated over a predetermined averaging period.

The underlying asset price S is assumed to be lognormally distributed and S_{ave} is a geometric average of the S's, analytic formulas are available for valuing European average price options. This is because the geometric average of a set of lognormally distributed variables is also lognormal.

In a risk-neutral world, it can be shown that the probability distribution of the geometric average of an asset price over a certain period is the same as that of the asset price at the end of the period if the asset's expected growth rate is set equal to

$$\frac{r - q - \frac{\sigma^2}{6}}{2}$$

rather than $r - q$. Its volatility is set equal to

$$\frac{\sigma}{\sqrt{3}}$$

rather than σ.

The geometric average price option can, therefore, be treated like a regular option with the volatility set equal to $\frac{\sigma}{\sqrt{3}}$ and the dividend yield equal to

$$r - \frac{1}{2}\left(r - q - \frac{\sigma^2}{6}\right) = \frac{1}{2}\left(r + q + \frac{\sigma^2}{6}\right)$$

On the other hand when Asian options are defined in terms of arithmetic averages, there are no exact pricing formulas.

Options in Exchange one asset for another

The **exchange option** is an options to exchange one asset for another.

Consider a European option to give up an asset worth U_T at time T and receive in return an asset worth V_T. The payoff from the option is

$$\max\{V_T - U_T, 0\}$$

A formula for valuing this option can be done by Black-Scholes. Suppose that the asset prices U and V both follow geometric Brownian motion with volatilities σ_U and σ_V. Suppose further that the instantaneous correlation between U and V is ρ and the yields provided by U and V are q_U and q_V respectively. The value of the option at time zero is

$$V_0 e^{-q_V T} N(d_1) - U_0 e^{-q_U T} N(d_2)$$

where

$$d_1 = \frac{\ln \frac{V_0}{U_0} + (q_U - q_V + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

and

$$\sigma = \sqrt{\sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V}$$

with U_0 and V_0 are the initial values of U and V. To be discussed later....

Options involving several assets

Options involving two or more risky assets are sometimes referred to as **rainbow options**.

One example is the bond futures contract traded on CBOT. The party with the short position is allowed to choose between a large number of different bonds when making delivery.

Probably the most popular option involving several assets is the **basket option**. This is an option where the payoff is dependent on the value of a portfolio (basket) of assets.

The assets are usually either individual stocks or stock indices or currencies. A European basket option can be valued with Monte Carlo simulation, by assuming that the assets follow correlated geometric Brownian motion processes.

A much faster approach is to calculate the first two moments of the basket at the maturity of the option in a risk-neutral world, and then assume that value of the basket is lognormally distributed at that time.

The option can be then regarded as an option on a futures contract.

Static Options Replication

If we try using exotic options to hedge, it may be difficult. Instead we may try to search for a portfolio that approximately replicates the exotic hedge.

Shorting on this position provides the hedge. The basic principle underlying static options replication is as follows. If two portfolios are worth the same on a certain boundary, they are also worth the same at all interior points of the boundary.

Consider as an example a 9-month up-and-out call option on a non-dividend-paying stock where the stock price is 50, the strike price is 50, the barrier is 60, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum. Suppose that $f(S, t)$ is the value of the option at time t for a stock price of S.

We can use an boundary in (S, t) space for the purposes of producing the replicating portfolio.

A boundary can be drawn as....

It is defined by $S = 60$ and $t = 0.75$. The values of the up-and-out option on the boundary are given by

$$f(S, 0.75) = \max\{S - 50, 0\} \text{ when } S < 60$$

and

$$f(60, t) = 0 \text{ when } 0 \leq t \leq 0.75$$

There are many ways that we can approximately match these boundary values using regular options. The natural instrument to match the first boundary is a regular 9-month European call option with a strike price of 50.

The first instrument (option A) introduced into the replicating portfolio is therefore likely to be one unit of this option. One way of proceeding is as follows. We divide the life of the option into a number of time steps and choose options that satisfy the second boundary condition at the beginning of each time step.

Suppose that we choose time steps of 3 months. The next instrument (option B) we choose should lead to the second boundary being matched at $t = 0.25$. In other words it should lead to the value of the complete replicating portfolio zero when $t = 0.25$ and $S = 60$. The option should have the property that it has zero value on the first boundary since this has already been matched. One possibility is a regular 9-month European call option with a strike price of 60. Black-Scholes formulas show that this is worth 4.33 at the 6-month point when $S = 60$. They also show that the position in option A is worth 11.54 at this point. The position we require in option B is therefore $-\frac{11.54}{4.33} = -2.66$.

We next move on to matching the second boundary condition at $t = 0.25$. The option used should have the property that it has zero value on all boundarys that have been matched thus far. One possibility is a regular 6-month European call option (option C) with a strike price of 60. This is worth 4.33 at the 3-month point when $S = 60$. Our position in options A and B is worth -4.21 at this point. The position in option C should be therefore $\frac{4.21}{4.33} = 0.97$.

Finally, we match the second boundary condition at $t = 0$. For this we use a regular 3-month European option with strike price of 60 (option D). Similar calculation shows the required position is 0.28. It is initially worth 0.73.

Option	Strike Price	Maturity	Position	Initial Value
A	50	0.75	1.00	+ 6.99
B	60	0.75	-2.66	- 8.21
B	60	0.50	0.697	+ 1.78
B	60	0.25	0.28	+ 0.17

Computing using the formula from before gives a value of 0.31 for an up-and-out call.

To hedge the portfolio we short the portfolio that replicates its boundary conditions. This has the advantage over delta hedging that it does not require frequent rebalancing.