REFINED JACOBIAN ESTIMATES FOR GINZBURG-LANDAU FUNCTIONALS

ROBERT JERRARD AND DANIEL SPIRN

Abstract. We prove various estimates that relate the Ginzburg-Landau energy $E_\varepsilon(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{(|u|^2-1)^2}{4\varepsilon^2}\,dx$ of a function $u \in H^1(\Omega;\mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$, to the distance in the $W^{-1,1}$ norm the Jacobian $J(u) = \det \nabla u$ and a sum of point masses. These are interpreted as quantifying the precision with which "vortices" in a function u can be located via measure-theoretic tools such as the Jacobian; and the extent to which variations in the Ginzburg-Landau energy due to translation of vortices can be detected using the Jacobian. We give examples to show that some of our estimates are close to optimal.

1. INTRODUCTION

In this paper we establish some estimates that provide a basis for quantitative versions of Γ-limit theorems and associated compactness results relating the Jacobian $J(u)$ (see (2.9) for the definition) and the Ginzburg-Landau energy

$$E_\varepsilon(u) = \int_\Omega e_\varepsilon(u)\,dx, \qquad e_\varepsilon(u) := \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2. \tag{1.1}$$

Here, and throughout this paper, Ω is a bounded, open subset of $\mathbb{R}^2$ and $u \in H^1(\Omega;\mathbb{C})$.

1.1. quantitative compactness.

A typical compactness result of the sort we seek to quantify states that if $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is a sequence of functions such that

$$E_\varepsilon(u^\varepsilon) \le M|\ln\varepsilon|, \tag{1.2}$$

then $\{J(u^\varepsilon)\}$ is precompact in suitable weak topologies, including for example in the $W^{-1,1}(\Omega)$ norm; and moreover, every limit of a convergent subsequence is a measure of the form

$$\pi\sum_{i=1}^{n_0} d_i\delta_{a_i} \tag{1.3}$$

for some $a = (a_1,\dots,a_{n_0}) \in \Omega^{n_0}$ and $d_i \in \{\pm1\}^{n_0}$, with $n_0 \le \frac{M}{\pi}$. This is a simple special case of results proved in [7], [1].

R. Jerrard was partially supported by the National Science and Engineering Research Council of Canada under operating Grant 261955. D. Spirn was partially supported by NSF grant DMS–0306398. While working on this paper he visited the University of Toronto several times, and he gratefully acknowledges its hospitality and support.

A quantitative version of such a result would give an estimate, for an arbitrary $u \in H^1(\Omega;\mathbb{C})$ in which the distance (in $W^{-1,1}$ say) between $J(u)$ and the set of measures of the form $\pi\sum_{i=1}^n d_i\delta_{a_i}$ is controlled by the Ginzburg-Landau energy $E_\varepsilon(u)$.

We have two main results in this direction. The first is

Theorem 1. *There exists a constant C such that for any bounded, open $\Omega \subset \mathbb{R}^2$, any $u \in H^1(\Omega;\mathbb{C})$, any $\varepsilon \in (0,1]$, and any $n > \frac{1}{\pi|\ln\varepsilon|}\int_\Omega e_\varepsilon(u)\,dx - 1$, there exist an integer $n_0 \le n$, points $a_1,\dots,a_{n_0} \in \Omega$, not necessarily distinct, and $d_1,\dots,d_{n_0} \in \{\pm 1\}$ such that*

(1.4)
$$\|J(u)-\pi\sum_{i=1}^{n_0} d_i\delta_{a_i}\|_{\dot W^{-1,1}(\Omega)} \le \varepsilon C(n+1)\left(\int_\Omega e_\varepsilon(u)\,dx\right)\exp\left[\frac{1}{\pi(n+1)}\int_\Omega e_\varepsilon(u)\,dx\right].$$

Theorem 1 implies in particular that if u^ε is a function satisfying (1.2), then for $n > \frac{M}{\pi} - 1$, there exist a_i, d_i as above such that

$$\|J(u^\varepsilon) - \pi\sum_{i=1}^{n_0} d_i\delta_{a_i}\|_{\dot W^{-1,1}(\Omega)} \le C(n+1)(M|\ln\varepsilon|)\varepsilon^{1-\frac{M}{\pi(n+1)}}.$$

For M, n fixed, we demonstrate in Lemma 18 that the scaling in ε is almost optimal; that is, for fixed M, n, we construct sequences of functions that satisfy (1.2) (up to error terms that are negligible as $\varepsilon \to 0$) and such that

$$\inf_{a_i\in\Omega, |d_i|\ \le 1} \|J(u^\varepsilon) - \pi\sum_{i=1}^{n_0} d_i\delta_{a_i}\|_{\dot W^{-1,1}(\Omega)} \ge c\varepsilon^{1-\frac{M}{\pi(n+1)}} \quad \text{for all } \varepsilon \in (0,1].$$

In particular, if (1.2) holds with $M \approx \pi$, then for $n = 1$, the best we can hope for is

$$\inf_{a\in\Omega,|d|\le 1} \|J(u) - \pi d\delta_a\|_{\dot W^{-1,1}} \approx C\varepsilon^{1/2}. \tag{1.5}$$

Our second quantitative compactness result shows that a much stronger estimate is possible if we impose an additional hypothesis to the effect that $J(u)$ is reasonably close to an isolated point mass. By restricting our attention to a neighborhood of this point mass, it suffices to consider an open ball U_r in $\mathbb{R}^2$ of radius r. In the statement of the theorem, and throughout much of this paper, we use the notation

$$K_0 = K_0{}^{\varepsilon,r}(u) = \int_{U_r} e_\varepsilon(u)dx \ - \ \pi\ln\frac{r}{\varepsilon}. \tag{1.6}$$

(We will always write simply K_0, suppressing the dependence of K_0 on u, ε, r.)

Theorem 2. *There exists an absolute constant C such that if $r > \varepsilon$ and $u \in H^1(U_r;\mathbb{C})$ satisfies*

$$\|J(u) - \pi\delta_0\|_{\dot W^{-1,1}(U_r)} < \frac{r}{4}, \tag{1.7}$$

then there exists some $\xi \in U_{r/2}$ such that

$$\|J(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(U_r)} \le \varepsilon C(C+K_0)\left[(C+K_0)e^{K_0/\pi} + \sqrt{\ln(r/\varepsilon)}\right] \tag{1.8}$$

and

$$\|J'(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(U_r)} \le \varepsilon C(C+K_0)^2 e^{K_0/\pi}. \tag{1.9}$$

Here $J'(u)$ denotes the modified Jacobian, defined in (2.15).

When $K_0 = O(1)$, we interpret (1.9) as showing that one can use the modified Jacobian to determine the location of a "vortex" to length scales of order ε. For this reason, we refer to Theorem 2 (and similarly Theorem 1) as "localization" theorems.

We show in Lemma 17 that the scaling in (1.8), (1.9) is sharp in the sense that there exists a sequence $\{u^\varepsilon\}_{\varepsilon\in(0,1]} \subset H^1(U_r;\mathbb{C})$ such that K_0 as defined in (1.6) is bounded uniformly in ε, and

$$\inf_{\xi\in U_r} \|J(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(U_r)} \ge c\varepsilon\sqrt{\ln(r/\varepsilon)}, \qquad \inf_{\xi\in U_r} \|J'(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(U_r)} \ge c\varepsilon.$$

In particular, it is necessary to introduce the above-mentioned modified Jacobians to get the desired $O(\varepsilon)$ scaling.

1.2. quantitative lower bounds.

In view of the results described above, it is reasonable to ask: what can one say about $E_\varepsilon(u)$, if $u \in H^1(\Omega;\mathbb{C})$ satisfies an estimate of the form provided by Theorem 1, ie

$$\|J(u) - \pi\sum_{i=1}^{n_0} d_i\delta_{\xi_i}\|_{W^{-1,1}(\Omega)} \le s_\varepsilon \tag{1.10}$$

for some small number s_ε?

In a sequel [8] to this paper we prove that if $d_i = \pm 1$ for all i, and if s_ε is sufficiently small compared to $\rho_\xi := \min\{\min_{i\neq j}\{|\xi_i - \xi_j|, \min_i\{\operatorname{dist}(\xi_i,\partial\Omega\}\}$, then

$$E_\varepsilon(u) \ge n_0 I(\varepsilon,1) + W(\xi,d) - C\left[\frac{n^5}{\rho_\xi}(s_\varepsilon + \varepsilon E_\varepsilon(u))\right]^{1/2} \tag{1.11}$$

where $W(\xi,d)$ is the renormalized energy introduced by Bethuel, Brezis, and Hélein [2], and $I(\varepsilon,1) \approx \pi\ln\frac{1}{\varepsilon}$ is defined in (6.1). The constant C depends on the domain Ω. and the lower bound is sharp up to errors of the same order of magnitude. The leading-order terms in (1.11) are sharp in that there exist u satisfying (1.10) with $s_\varepsilon = Cn\varepsilon(1+\frac{n^3}{\rho_\xi^2}$ and such that

$$E_\varepsilon(u) \le n_0 I(\varepsilon,1) + W(\xi,d) + Cn_0\varepsilon(1+\varepsilon\frac{n_0^3}{\rho_\xi^2}).$$

The full version of (1.11) in [8] includes additional positive terms on the right-hand side and is valid for arbitrary $n_0 \ge 0$. The scaling of the error terms in (1.11) is not optimal.

This should be compared with estimates in [5, 10] which show that for a sequence of functions u^ε,

$$\text{if } \|J(u^\varepsilon) - \pi\sum_{i=1}^{n_0} d_i\delta_{\xi_i}\|_{W^{-1,1}(\Omega)} \to 0,$$

$$\text{then } \liminf_{\varepsilon\to 0}\int_\Omega e_\varepsilon(u)\,dx - n_0 I(\varepsilon,1) \ge W(a,d), \tag{1.12}$$

if $\xi_i \ne \xi_j$ whenever $i \ne j$. Estimate (1.11) as proved in [8] establishes a quantitative analog of (1.12), with error estimates, valid for a fixed function rather than a sequence.

The final main result of this paper is in effect the basic case of the above estimate (1.11), when the domain Ω is an open ball U_r of radius r, and ξ is the center of the ball. In this situation we show that the error terms are bounded by $\frac{C}{r}(\varepsilon\sqrt{\ln\frac{r}{\varepsilon}} + s_\varepsilon)$.

Theorem 3. *There exists a constant $C > 0$ such that if $u \in H^1(U_r;\mathbb{C})$ satisfies*

$$\|Ju - \pi\delta_0\|_{\dot W^{-1,1}(U_r)} \le \frac{r}{4},$$

then

$$I(r,\varepsilon) - \int_{U_r} e_\varepsilon(u) \ \le\ C\frac{\varepsilon}{r}\sqrt{\ln\frac{r}{\varepsilon}} + \frac{C}{r}\|Ju - \pi\delta_0\|_{\dot W^{-1,1}(U_r)}$$

and

$$I(r,\varepsilon) - \int_{U_r} e_\varepsilon(u) \ \le\ C\frac{\varepsilon}{r} + \frac{C}{r}\|J'(u) - \pi\delta_0\|_{\dot W^{-1,1}(U_r)}$$

where $J'(u)$ denotes the modified Jacobian as defined in (2.15).

This result is proved in Section 6. We remark that the proof of (1.11) in [8] relies very heavily on Theorem 3 from this paper.

1.3. **other remarks.** Throughout this paper we mostly use the $\dot W^{-1,1}$ norm, defined in (2.3), when formulating Jacobian estimates. Once estimates in $\dot W^{-1,1}$ are established, one can use interpolation arguments to obtain estimates in certain other negative Sobolev norms or in dual Hölder norms; an example of this sort of argument is given in the proof of Lemma 15.

We conclude this introduction by describing some aspects of our proofs and sketching the organization of this paper:

We introduce notation and recall some background concerning the Jacobian and modified Jacobian in Section 2.

The proofs of Theorems 1 and 2 occupy Sections 3 through 5. Both these theorems are proved by constructing families of balls (sometimes called "vortex balls") in which the Jacobian $J(u)$ of a given function u is concentrated, and then converting information about these vortex balls to estimates of the form (1.4), (1.8), etc. The techniques we use for obtaining Jacobian estimates from the vortex ball constructions rely on arguments developed by [1], [12] for example. The main new point, particularly in the proof of Theorem 2, is in the construction of vortex balls, which implements a number of improvements over earlier such constructions as introduced in [11], [6]. Various attributes of the resulting vortex balls (for example, the sum of the radii) scale in an optimal way as the parameter ε varies, and this makes it possible to deduce from the vortex balls estimates with sharp or almost-sharp scaling.

Techniques used in the construction of vortex balls are introduced in Section 3. These are used to give the proof of Theorem 2 in Section 4 and Theorem 1 in Section 5. In both cases we also prove some some additional estimates, in which we show that the norms in (1.4), (1.9) can be strengthened if one modifies the domain slightly; see Theorem $2'$ and Theorem $1'$ for the precise statements. These technical refinements are very useful in applications in [8].

In Section 6 we prove the energy lower bound of Theorem 3, as described above. The main point is to show that a given function $u \in H^1(U_r;\mathbb{C})$ can be modified to create a new function $\tilde{u}$ such that $\tilde{u} = e^{i\theta}$ on ∂U_r, and with $E_\varepsilon(\tilde{u}) - E_\varepsilon(u)$ controlled by $\|J(u) - \pi\delta_0\|_{\dot{W}^{-1,1}(U_r)}$. This control requires both upper bounds on $E_\varepsilon(\tilde{u})$, and lower bounds on $E_\varepsilon(u)$. The former are derived essentially by explicit calculations, using the exact form of the construction of $\tilde{u}$, whereas the latter rely on arguments developed in the earlier part of the paper.

The final section contains two examples that prove the optimality and near-optimality of the localization theorems.

2. NOTATION AND BACKGROUND

2.1. general notation.

For $v, w \in \mathbb{R}^2$ we write $v \times w := v_1 w_2 - v_2 w_1$.

For $w : \mathbb{R}^2 \to \mathbb{R}^2$ we define $\nabla \times w := w_{2,x_1} - w_{1,x_2}$.

For $\phi : \mathbb{R}^2 \to \mathbb{R}$ we use the notation $\nabla \times \phi := (\phi_{x_2}, \phi_{x_1})$.

We denote open and closed balls, respectively by the notation

$$U_r(x) := \{y \in \mathbb{R}^2 : |x - y| < r\}, \qquad U_r := U_r(0),$$

$$B_r(x) := \{y \in \mathbb{R}^2 : |x - y| \le r\}, \qquad B_r := B_r(0).$$

For a ball labeled with sub- and superscripts, such as B^a_b, our default notation designates r^a_b as the radius, and x^a_b as the center.

We speak of the functions that we consider as $\mathbb{C}$-valued rather than $\mathbb{R}^2$-valued, mainly because in explicit calculations it is often convenient to use the multiplicative structure of $\mathbb{C}$. However, in practice we are not completely consistent, as for example when we write $J(u) = \det \nabla u$; here we think of ∇u as a real 2×2 matrix. We believe that this abuse of notation does not lead to any ambiguity anywhere.

For an open set $U \subset \mathbb{R}^n$ and a closed set Γ (typically a subset of ∂U) we use the notation

$$W^{1,p}_\Gamma(U) := \{\phi \in W^{1,p}(U) \ : \ \phi = 0 \text{ on } \Gamma\}, \tag{2.1}$$

or more precisely the closure in $W^{1,p}(U)$ of the set of smooth functions that vanish on Γ. For $\Gamma = \emptyset$ we use the convention that $W^{1,p}_\emptyset = W^{1,p}(\Omega)$. We also define the dual norms

$$\|\mu\|_{\dot W^{-1,q}_\Gamma(U)} := \sup\{\int \phi \, d\mu \ : \ \|\nabla \phi\|_{L^p} \le 1, \ \phi \in W^{1,p}_\Gamma(U)\}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{2.2}$$

In this paper we will only consider $\|\mu\|_{\dot W^{-1,q}_\Gamma(U)}$ for $\frac{1}{q} > 1 - \frac{1}{n}$ and μ a (finite signed) measure; in this situation $\|\mu\|_{\dot W^{-1,q}_\Gamma(U)}$ is always finite, by the Sobolev Embedding Theorem and the Riesz Representation Theorem. Note that these norms scale nicely if μ, Γ, and U are all dilated. We use special notation for certain norms that are employed frequently throughout the paper:

$$\|\mu\|_{\dot W^{-1,q}(U)} := \|\mu\|_{\dot W^{-1,q}_{\partial U}(U)}, \qquad \|\mu\|_{Lip^*(U)} := \|\mu\|_{\dot W^{-1,1}_\emptyset(U)}. \tag{2.3}$$

Note that $\|\mu\|_{Lip^*(U)} = +\infty$ unless $\int_U \mu = 0$. Clearly $\|\mu\|_{\dot W^{-1,1}(U)} \le \|\mu\|_{Lip^*(U)}$ for every measure μ on every open set U.

Throughout this paper we implicitly sum over repeated indices.

We write $\mathcal{H}^1$ to denote 1-dimensional Hausdorff measure.

For $a = (a_1, \dots, a_n) \in \Omega^n$, define

$$\rho_a = 4 \min\left(\{|a_i - a_j| : 1 \le i < j \le n\} \cup \{\operatorname{dist}(a_i, \partial\Omega) : 1 \le i \le n\}\right) \tag{2.4}$$

Define $\Omega^{n*} := \{(a_1, \dots, a_n) \in \Omega^n : \rho_a > 0\}$. And given $a \in \Omega^{n*}$, we will use the notation

$$\Omega_r(a) = \Omega \setminus \left(\cup_{i=1}^n B_r(a_i)\right). \tag{2.5}$$

We will sometimes write Ω_r when no confusion can result.

2.2. **Jacobian.**

For $\Omega \subset \mathbb{R}^2$ open and $u \in H^1(\Omega; \mathbb{C})$ we define

$$j(u) := \Im(\bar u \nabla u) \tag{2.6}$$

where $\Im$ denotes the imaginary part. If we write u locally in the form $u = \rho e^{i\phi}$ for real-valued ρ, ϕ, then one easily checks that

$$j(u) = \rho^2 \nabla \phi \tag{2.7}$$

It is also useful to observe that

$$e_\varepsilon(v) = \frac{1}{2}\frac{|j(v)|^2}{|v|^2} \ + \ e_\varepsilon(|v|). \tag{2.8}$$

For u as above we next define the Jacobian

$$J(u) := \frac{1}{2}\nabla \times j(u) = u_{x_1} \times u_{x_2} = \det \nabla u. \tag{2.9}$$

Note that the 2-form $J(u)\, dx^1 \wedge dx^2$ is the pullback by u of the standard volume form on $\mathbb{C} \cong \mathbb{R}^2$. It is often useful to define $J(u)$ and $j(u)$ in terms of differential forms, particularly in higher dimensions.

If $|u|^2$ is constant in an open subset V of Ω, then $u^i_{x_j} u^i = 0$ in V, so that 0 is an eigenvalue of ∇u. Thus

$$\text{if } |u| \text{ is constant in } V, \text{ then } J(u) = 0 \text{ a. e. in } V. \tag{2.10}$$

2.3. **degree.** Given $\Omega \subset \mathbb{R}^2$ and $u \in H^1(\Omega; \mathbb{C})$, suppose that V is an open subset of Ω with Lipschitz boundary and that there exists $\alpha > 0$ such that $|u(x)| \geq \alpha$ for $\mathcal{H}^1$ a. e. $x \in \partial V$. Then the degree is defined by

$$\deg(u; \partial V) = \frac{1}{2\pi} \int_{\partial V} \frac{j(u)}{|u|^2} \cdot \tau \ d\mathcal{H}^1. \tag{2.11}$$

Here τ denotes the unit tangent to ∂V, with the standard orientation. For a proof that this definition makes sense for $u \in H^1(\Omega; \mathbb{C})$, see Brezis and Nirenberg [4]. If $|u|$ vanishes on ∂V then $\deg(u; \partial V)$ is not defined. Note that we can use Stokes' theorem and the identity $J(u) = \frac{1}{2}\nabla \times j(u)$ to find that

$$\deg(u; \partial V) = \frac{1}{2\pi} \int_{\partial V} j(u) \cdot \tau \ d\mathcal{H}^1 \ = \ \frac{1}{\pi} \int_V J(u) \ dx \quad \textbf{if} \ \ |u| \equiv 1 \text{ on } \partial V. \tag{2.12}$$

From (2.11) one can check that if $V_1, \ldots, V_k$ are pairwise disjoint open sets, then

$$\deg(u; \partial(\overline{\cup V_j})) = \sum \deg(u; V_j) \tag{2.13}$$

whenever both sides are well-defined. An important property of degree (see again [4]) is that

$$\text{if } u \text{ is continuous in } V \text{ and } \deg(u; \partial V) \neq 0, \text{ then } u \text{ has a zero in } V. \tag{2.14}$$

2.4. **modified Jacobian.** Following Alberti, Baldo and Orlandi [1], we introduce the *modified Jacobian* $J'(u)$, defined by

$$J'(u) = \zeta(|u|) J(u), \tag{2.15}$$

where $\zeta : [0, \infty) \to [0, \infty)$ is a smooth function with support in $[0, 1/2)$, and such that $\int_{\mathbb{R}^2} \zeta(|y|)\, dy = \pi$. In other words, the 2-form $J'(u)\ dx^1 \wedge dx^2$ is the pullback by u of $\zeta(|y|) dy^1 \wedge dy^1$. The choice of ζ implies that

$$\text{supp } J'(u) \subset \{x : |u(x)| < \frac{1}{2}\} \tag{2.16}$$

so that $J'(u)$ is more concentrated than $J(u)$. In addition, the following lemma implies that $J'(u)$ is close to $J(u)$ if $\int e_\varepsilon(u)$ is not too large.

Lemma 1. *([1], lemma 3.6) There exists a constant C such that for any bounded, open $\Omega \subset \mathbb{R}^2$ and any $u \in H^1(\Omega;\mathbb{C})$, the estimate*

$$\|J'(u) - J(u)\|_{\dot W^{-1,1}(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}\ \|1-|u|^2\|_{L^2(\Omega)}$$

holds.

The constant C above depends on the exact choice of the auxiliary function ζ appearing in the definition of $J'(u)$ but is independent of the domain Ω. It follows from standard facts about degree that

$$\deg(u;\partial V) \ = \ \frac{1}{\pi}\int_V J'(u)\ dx \qquad \text{if } |u| \ge 1/2 \text{ on } \partial V. \tag{2.17}$$

3. MACHINERY FOR LOWER BOUNDS

We will obtain bounds on the Jacobian by constructing collections of balls that contain the set $\{|u| \le 1/2\}$, and such that the radii and degree of these balls are controlled by the Ginzburg-Landau energy. These sorts of argument originate in [11], [6] and have since been extensively developed.

In this section we recall some of this machinery and adapt it to our current needs. These adaptations will later permit us to produce collections of vortex balls that are optimal in certain ways.

3.1. some notation.

In these arguments it is convenient to work with functions $u \in H^1$ that are continuous. This can always be achieved by an approximation argument.

We will use the notation

$$S = \{x \in \Omega : |u(x)| \le 1/2\}, \tag{3.1}$$

for the set where u is small. (We will always suppress the dependence of S on u.) The essential part S_E of S is defined to be

$$S_E := \cup\{\text{components } S_i \text{ of } S\ : \deg(u;\partial S_i) \neq 0\}. \tag{3.2}$$

For u continuous, S is closed and consists of finitely many connected components, so the definition of S_E makes sense. If V is an open subset of Ω such that $\partial V \cap S_E = \emptyset$, we define the *essential degree* of V, denoted $\mathrm{dg}(u;\partial V)$, by

$$\mathrm{dg}(u;\partial V) := \sum\{\deg(u;\partial S_i) : \text{components } S_i \text{ of } S_E \text{ such that } S_i \subset V\}. \tag{3.3}$$

From the definitions one can easily check that

$$\deg(u;\partial V) = \mathrm{dg}(u;\partial V) \quad \text{whenever } |u| > 1/2 \text{ on } \partial V. \tag{3.4}$$

The essential degree shares certain key properties of the degree, to wit:

$$\operatorname{dg}(u;\partial(\overline{\cup V_j}))=\sum \operatorname{dg}(u;V_j)\qquad \text{for } V_1,\dots,V_k \text{ pairwise disjoint open sets} \tag{3.5}$$

whenever both sides are well-defined, and

$$\text{if } u \text{ is continuous in } V \text{ and } \operatorname{dg}(u;\partial V)\neq 0, \text{ then } u \text{ has a zero in } V. \tag{3.6}$$

3.2. **lower bounds on circles and annuli; auxiliary function Λ_ε.** Our constructions of vortex balls are based on the lemmas in this subsection.

The following is proved [1] in [6], Proposition 3.2.

Lemma 2. *Suppose that* Ω *is an open subset of* $\mathbb{R}^2$ *and* $u\in H^1(\Omega;\mathbb{C})$, *and suppose that* $B_s(x)\subset\Omega$ *and that* $s\geq\varepsilon$. *For* $m:=\min_{\partial B_s(x)}|u|$,

$$\int_{\partial B_s(x)} e_\varepsilon(|u|)\geq \frac{2(1-m^2)^2}{c_0\varepsilon}, \tag{3.7}$$

where c_0 *is an absolute constant, independent of* u *and* ε. *In addition, if* $dg(u;\partial B_s(x))$ *is well-defined and nonzero, then*

$$\frac{1}{2}\int_{\partial B_s(x)}\frac{|j(u)|^2}{|u|^2}\geq\frac{m^2\pi}{s}. \tag{3.8}$$

Finally, if $\varepsilon\leq s_0$ *and* $dg(B_s(x))$ *is well-defined and nonzero for all* $s_0<s<s_1$, *then*

$$\int_{B_{s_1}\setminus B_{s_0}(\xi)} e_\varepsilon(u)\ d\mathcal{H}^1\ \geq\Lambda_\varepsilon(s_1)-\Lambda_\varepsilon(s_0), \tag{3.9}$$

where

$$\Lambda_\varepsilon(r)=\pi\ln\Big(\frac{r}{c_0\varepsilon}+1\Big)=\int_0^r\lambda_\varepsilon(s)\ ds\qquad \textit{for } \lambda_\varepsilon(s):=\frac{\pi}{s+c_0\varepsilon}. \tag{3.10}$$

As a result of (3.7) and (3.8), there exists some $c_1>0$ such that if $r\geq c_1\varepsilon$ then

$$\begin{aligned}\int_{\partial B_r} e_\varepsilon(u)d\mathcal{H}^1\ &\geq\begin{cases}\frac{m^2\pi}{r}+\frac{2(1-m^2)^2}{c_0\varepsilon} & \text{if } m\geq 1/2 \text{ and } d(B_r(x))\neq 0\\ \frac{2(1-m^2)^2}{c_0\varepsilon} & \text{if } m\leq 1/2\end{cases}\\ &\geq\frac{m^2\pi}{r}+\frac{(1-m^2)^2}{c_0\varepsilon}\end{aligned} \tag{3.11}$$

---

[1] We recall the idea: Let $\rho:=|u|$. Writing $e_\varepsilon(\rho)=\frac{1}{2}|\nabla\rho|^2+\frac{1}{8\varepsilon^2}(\rho^2-1)^2+\frac{1}{8\varepsilon^2}(\rho^2-1)^2$ and using the inequality $a+b\geq 2\sqrt{ab}$, we find that $e_\varepsilon(\rho)\geq|\nabla H(\rho)|+\frac{1}{8\varepsilon^2}(\rho^2-1)^2$ for $H(\rho)=\frac{1}{\varepsilon}\left(\frac{\rho}{2}-\frac{\rho^3}{6}\right)$. If $M=\max_{\partial B_s}(\min(1,\rho(x)))$ it follows that $\int_{\partial B_s(x)}e_\varepsilon(\rho)\geq 2(H(M)-H(m))+\frac{2\pi s}{r\varepsilon^2}(1-M^2)^2$. To deduce (3.7), take the infimum over $M\geq m$ and rewrite suitably. Hölder's inequality and (2.11) imply (3.8). The definition of λ_ε in (3.10) is such that $\lambda_\varepsilon(s)\geq\inf_{m\in[0,1]}(\frac{2(1-m^2)^2}{c_0\varepsilon}+\frac{m^2\pi}{s})$; as a result, (3.9) follows from (3.7), (3.8).

if either $m \le 1/2$ or $d(B_r) \ne 0$.

We will need to use some elementary properties of Λ_ε:

Lemma 3. *If* $a_i > b_i \ge 0$ *for* $i = 1, \dots, m$ *then*

$$\sum_1^m [\Lambda_\varepsilon(a_i) - \Lambda_\varepsilon(b_i)] \ge \Lambda_\varepsilon(\sum_1^m a_i) - \Lambda_\varepsilon(\sum_1^m b_i). \tag{3.12}$$

And second, there exists $c_2 > 0$ *such that if* $r_1, \dots, r_m \ge \varepsilon$ *then*

$$\sum_{j=1}^m \Lambda_\varepsilon(r_j) - \Lambda_\varepsilon(\sum_{j=1}^m r_j) \ \ge \ \pi \ln \left[1 + \frac{1}{c_2 \varepsilon} \left(\sum_{j=1}^m r_j - \max_j r_j \right) \right] \tag{3.13}$$

We will later construct "vortex balls" with energy is bounded below by Λ_ε(radius). The inequality (3.13) quantifies the intuition that many small balls (with radii r_j) have more energy than one large ball (with radius $\sum_j r_j$). It is used the proof of Theorem $2'$.

Proof. Since λ_ε is a decreasing function, one sees by inspection that

$$\sum \int_{b_i}^{a_i} \lambda_\varepsilon(s)\, ds \ \ge \ \int_{\sum b_i}^{\sum a_i} \lambda_\varepsilon(s)\, ds.$$

This is (3.12).

Next, given $r_1, \dots, r_m \ge \varepsilon$, assume for concreteness that $r_1 = \max\{r_j\}$, and set $r_0 := 0$, $R_j := \sum_{k=1}^{j-1} r_j$ for $j \ge 1$. Then

$$\sum_{j=1}^m \Lambda_\varepsilon(r_j) - \Lambda_\varepsilon(\sum_{j=1}^m r_j) = \sum_{j=1}^m \left[\Lambda_\varepsilon(r_j) + \Lambda_\varepsilon(R_{j-1}) - \Lambda_\varepsilon(R_j) \right].$$

The $j = 1$ term does not contribute. In the other terms, using the definition (3.10),

$$\begin{aligned} \Lambda_\varepsilon(r_j) + \Lambda_\varepsilon(R_{j-1}) - \Lambda_\varepsilon(R_j) &= \pi \ln \left[\frac{(r_j + c_0\varepsilon)(R_{j-1} + c_0\varepsilon)}{c_0\varepsilon(R_j + c_0\varepsilon)} \right] \\ &= \pi \ln \left[\frac{r_j R_{j-1}}{c_0\varepsilon(R_j + c_0\varepsilon)} + 1 \right]. \end{aligned}$$

It is easy to see that $R_{j-1}/(R_j + c_0\varepsilon) \ge 1/(2 + c_0)$, using the facts that $r_j \le r_1 \le R_{j-1}$ for $j \ge 2$ and $r_j \ge \varepsilon$ for all j. Thus there exists some c_2 such that

$$\Lambda_\varepsilon(r_j) + \Lambda_\varepsilon(R_{j-1}) - \Lambda_\varepsilon(R_j) \ge \pi \ln \left[\frac{r_j}{c_2\varepsilon} + 1 \right]$$

for all $j \ge 2$. Now the argument used to prove (3.12) shows that

$$\sum_{j=1}^m \Lambda_\varepsilon(r_j) - \Lambda_\varepsilon(\sum_{j=1}^m r_j) \ \ge \ \sum_{j=2}^m \pi \ln \left[\frac{r_j}{c_2\varepsilon} + 1 \right] \ \ge \ \pi \ln \left[\frac{\sum_{j=2}^m r_j}{c_2\varepsilon} + 1 \right].$$

□

In the constructions that follow, we will need

Lemma 4. *If* $\{B_{r_i}(x_i)\}_{i=1}^M$ *is a collection of balls in the plane, then there exists a collection of pairwise disjoint balls* $\{B_{s_j}(y_j)\}_{j=1}^N$ *such that* $\cup_i B_{r_i}(x_i) \ \subset \cup_s B_{s_i}(y_j)$ *and* $\sum s_j = \sum r_i$.

The easy proof is left to the reader, or can be found in [6] for example.

3.3. **coverings of S_E.** In this subsection, we construct families of balls that cover the set S_E, as defined in (3.2). We restrict our attention to functions in $C \cap H^1(\Omega;\mathbb{C})$ (ie, continuous functions in H^1) to avoid any subtleties in the definitions of S_E and the essential degree dg.

We first quote a result that supplies an initial covering of S_E, and then we expand and merge these balls to construct families of larger balls with certain desirable properties.

The starting point of the construction is provided by the following lemma[2] , from [6], Proposition 3.3:

Lemma 5. *For* $u \in C\cap H^1(\Omega;\mathbb{C})$ *there exists a collection of balls* $\mathcal{B}^0 = \{B^0_j\}_{j=1}^{k^0}$, *with radii* r^0_j *and centers* x^0_j, *such that* $S_E \subset \cup B^0_j$, $r^0_j \ge \varepsilon$ *for all* j, *and*

$$\int_{B^0_j \cap \Omega} e_\varepsilon(u)\, dx \ \ge\ \frac{1}{C\varepsilon} r^0_j \ \ge\ \Lambda_\varepsilon(r^0_j) \qquad \text{for all } j \tag{3.14}$$

where C *is a universal constant.*

Referring to notation from Lemma 5, let

$$\sigma^0 = \min\{r^0_j : B^0_j \cap \partial\Omega = \emptyset, d^0_j \ne 0\}, \qquad \text{where } d^0_j := \mathrm{dg}(B^0_j). \tag{3.15}$$

The following lemma is crucial for the proof of Theorem 2′.

Lemma 6. *Let* $u \in C\cap H^1(\Omega;\mathbb{C})$ *for* $\Omega \subset \mathbb{R}^2$. *For every* $\sigma \ge \sigma^0$, *there exists a collection* $\mathcal{B}^\sigma$ *of closed, pairwise disjoint balls* $\{B^\sigma_j\}_{j=1}^{k(\sigma)}$ *such that*

$$S_E \subset \cup B^\sigma_j, \tag{3.16}$$

$$\sigma \le \min\{r^\sigma_j : d^\sigma_j \ne 0, B^\sigma_j \cap \partial\Omega = \emptyset\}, \qquad d^\sigma_j := dg(u;\partial B^\sigma_j). \tag{3.17}$$

---

[2] The idea of the proof is as follows: Around each component S_i of S_E, place a small ball of radius $\max\{\varepsilon, \operatorname{diam} S_i\}$. Note that

$$\int_{S_i} |\nabla u|^2 \ge C^{-1} \int_{S_i} |J(u)| \ge C^{-1} \left| \int_{S_i} J(u)\, dx \right| \ge C$$

since $\deg(u;\partial S_i) \ne 0$. This implies (3.14) for any ball with radius less than 2ε. For larger balls, (3.14) follows from (3.7). If two or more balls intersect, they can be combined into larger balls, with the Besicovitch covering theorem used to control the overlap and preserve (3.14).

Moreover, if $\sigma^0 \le \sigma \le \tau$*, then* $\cup_j B_j^\sigma \subset \cup_j B_j^\tau$*. Finally,*

$$\int_{B_k^\tau \cap U} e_\varepsilon(u)\,dx \;\ge\; \Lambda_\varepsilon(r_k^\tau) + \sum_{B_j^\sigma \subset B_k^\tau} \int_{B_j^\sigma} e_\varepsilon(u)\,dx - \Lambda_\varepsilon\left(\sum_{B_j^\sigma \subset B_k^\tau} r_j^\sigma\right) \tag{3.18}$$

$$\ge\; \Lambda_\varepsilon(r_k^\tau) + \sum_{B_j^\sigma \subset B_k^\tau} \Lambda_\varepsilon(r_j^\sigma) - \Lambda_\varepsilon\left(\sum_{B_j^\sigma \subset B_k^\tau} r_j^\sigma\right). \tag{3.19}$$

If we take $\sigma = \tau$ then the right-hand side of (3.19) reduces to $\Lambda_\varepsilon(r_k^\tau)$; estimates much like this appear in [11], [6], etc. The new terms are the ones involving the smaller balls B_j^σ. These capture additional energy that is present if many of these small balls are "swallowed" by a larger ball B_k^τ; note that these terms are always nonnegative, due to (3.13).

Proof. 1. Let

$$\mathcal{C} := \{\tau \ge \sigma^0 : \exists \text{ balls } B_j^\sigma \text{ for all } \sigma^0 \le \sigma \le \tau, \text{ with the stated properties}\}.$$

Lemma 5 guarantees that $\sigma^0 \in \mathcal{C}$, with $\mathcal{B}^{\sigma^0} = \mathcal{B}^0$ so that $\mathcal{C}$ is nonempty.

2. We now claim that if $\tau_0 \in \mathcal{C}$ then $[\tau_0, \tau_0 + \delta) \subset \mathcal{C}$ for some $\delta > 0$. For δ sufficiently small, to be chosen, and $\tau < \tau_0 + \delta$, we define $k(\tau) = k(\tau_0)$, and $\mathcal{B}^\tau = \{B_j^\tau\}_{j=1}^{k(\tau)}$, where B_j^τ has the same center as $B_j^{\tau_0}$ and radius

$$r_j^\tau = \begin{cases} \max\{\tau, r_j^{\tau_0}\} & \text{if } d_j^{\tau_0} \ne 0 \text{ and } B_j^{\tau_0} \cap \partial U = \emptyset \\ r_j^{\tau_0} & \text{otherwise.} \end{cases} \tag{3.20}$$

We take δ to be so small that the balls thus defined remain pairwise disjoint for $\tau < \tau_0 + \delta$, and such that none of the expanded balls intersect ∂U. We now verify that (3.18), (3.19) hold for all $\sigma^0 \le \sigma \le \tau$, for every $\tau \in (\tau_0, \tau_0 + \delta)$; it is clear by construction that all the other required properties of the balls are satisfied.

If $\tau_0 \le \sigma \le \tau < \tau_0 + \delta$, then (3.9) applies to $B_k^\tau \setminus B_k^\sigma$ to give

$$\int_{B_k^\tau} e_\varepsilon(u)\,dx \;\ge \Lambda_\varepsilon(r_k^\tau) - \Lambda_\varepsilon(r_k^\sigma) + \int_{B_k^\sigma} e_\varepsilon(u)\,dx. \tag{3.21}$$

This is exactly (3.18) in the case $\tau_0 \le \sigma \le \tau < \tau_0 + \delta$, since by construction B_k^σ is the unique ball from the collection $\mathcal{B}^\sigma$ contained in B_k^τ. If $\sigma < \tau_0$, then $B_j^\sigma \subset B_k^\tau$ if and only if $B_j^\sigma \subset B_k^{\tau_0}$, so (3.18) follows from combining (3.21) (with σ replaced by τ_0) and (3.18) (with τ replaced by τ_0).

We deduce (3.19) by noting that it follows from (3.18) together with the estimate $\int_{B_k^\sigma} e_\varepsilon(u)\,dx \ge \Lambda_\varepsilon(r_k^\sigma)$. For $\sigma \le \tau_0$, this holds by taking $\tau = \sigma$ in (3.18). For $\sigma \in (\tau_0, \tau_0 + \delta)$, it holds by taking (3.21) with τ replaced by σ and σ replaced by τ_0. This shows that $[\tau_0 + \delta) \subset \mathcal{C}$ as claimed.

3. We next claim that if $[\tau_0, \tau_1) \subset \mathcal{C}$, then $\tau_1 \in \mathcal{C}$. Together with our earlier claim, this will show that $\mathcal{C} = [\sigma^0, \infty)$, completing the proof of the lemma.

To do this, start by defining balls with radius $r_j^{\tau_1}$ exactly as in (3.20), which we designate $\{\tilde{B}_j^{\tau_1}\}$. These balls satisfy (3.18), (3.19) for all $\sigma \le \tau_1$, but they may fail to be pairwise disjoint. If this is the case, use Lemma 4 to form larger, pairwise disjoint balls $\{B_k^{\tau_1}\}$ with the property that

$$r_k^{\tau_1} = \sum_{\tilde{B}_j^{\tau_1} \subset B_k^{\tau_1}} \tilde{r}_j^{\tau_1}.$$

Using (3.12) and the fact that (3.18), (3.19) already hold for each $\tilde{B}_j^{\tau_1}$, one can check that each $B_k^{\tau_1}$ satisfies (3.18), (3.19) for all $\sigma \in [\sigma^0, \tau_1]$. ☐

3.4. **covering of** S**.** In this subsection, we present a procedure that produces a collection of balls complementing the one from Lemma 5. Unlike that lemma, the balls here cover all of S, rather than just S_E. However, due to poor control over the energy on very small balls, this lemma does not provide a suitable starting point for the expand-and-merge algorithm of Lemma 6. Thus both procedures are needed.

Lemma 7. *For* $u \in H^1 \cap C(\Omega; \mathbb{C})$, *there exists a collection* $\widetilde{\mathcal{B}}^0 = \{\widetilde{B}_i^0\}$ *of pairwise disjoint balls such that*

$$S \subset \cup_i \widetilde{B}_i^0, \tag{3.22}$$

$$\sum_i \widetilde{r}_i^0 \le C\,\varepsilon \int_\Omega e_\varepsilon(|u|)\,dx, \qquad \textit{where } \widetilde{r}_i^0 \textit{ denotes the radius of } \widetilde{B}_i^0. \tag{3.23}$$

The constant in (3.23) *is independent of* Ω, ε, *and* $\int_\Omega e_\varepsilon(|u|)$.

Proof. For the proof we write $\rho := |u|$. (In fact the lemma is really a statement about nonnegative functions.)

Note that $\frac{1}{2}|\nabla\rho|^2 + \frac{1}{4\varepsilon^2}(1-\rho^2)^2 \ge \frac{1}{\sqrt{2}\varepsilon}|1-\rho^2|\,|\nabla\rho|$, and so

$$\begin{aligned}\int_U \frac{1}{2}|\nabla\rho|^2 + \frac{1}{4\varepsilon^2}(1-\rho^2)^2 &\ge \frac{1}{\varepsilon\sqrt{2}}\int_U |1-\rho^2|\,|\nabla\rho| \\ &= \frac{1}{\varepsilon\sqrt{2}}\int_0^\infty |1-s^2|\,\mathcal{H}^1(\rho^{-1}(s))\,ds\end{aligned}$$

by the coarea formula. In particular,

$$\int_{1/2}^1 (1-s^2)\mathcal{H}^1(\rho^{-1}(s))\,ds \le C\,\varepsilon \int_U e_\varepsilon(\rho)\,dx.$$

For any set $A \subset U$, let $\mathcal{H}^1_\infty(A) := \inf\{2\sum s_i : A \subset \cup B_{s_i}(y_i)\}$. As noted by Sandier [11], for any open subset $A \subset U$,

$$\mathcal{H}^1_\infty(A) \le \mathcal{H}^1(\partial A \cup U).$$

This is easy to see if A is connected, and the general case is a straightforward consequence. Note that $s \mapsto \mathcal{H}^1_\infty(\{x : \rho(x) < s\})$ is an increasing function, so for any $\alpha \in (0,1)$,

$$\begin{aligned} C\varepsilon \int_U e_\varepsilon(\rho)dx &\geq \int_\alpha^1 (1-s^2)\mathcal{H}^1_\infty(\{x : \rho(x) < s\})ds \\ &\geq \mathcal{H}^1_\infty(\{x : \rho(x) < \alpha\}) \int_\alpha^1 (1-s^2)ds. \end{aligned}$$

Thus in particular $\mathcal{H}^1_\infty(\{x : \rho(x) \leq 1/2\}) \leq C\varepsilon \int_U e_\varepsilon(\rho)\, dx$, which immediately implies that there exists a collection of balls $\widetilde{\mathcal{B}}^0$ covering $\{x : \rho(x) \leq 1/2\}$ and satisfying (3.23). In view of Lemma 4, these balls can be taken to be pairwise disjoint. ☐

4. LOCALIZATION OF JACOBIAN NEAR A SINGLE VORTEX

In this section we prove Theorem 2. In fact we prove a result that gives a bit more information. The two additional conclusions recorded here assert, first, that the energy density is concentrated around a point where the Jacobian concentrates; and second, that by perturbing the ball U_r slightly, we can obtain a ball U_s with s very close to r, on which a stronger estimate than (1.9) holds. The stronger estimate we seek is

$$\|J'(u) - \pi\delta_\xi\|_{Lip^*(U_s)} \leq \varepsilon C(C+K_0)^2 e^{K_0/\pi}. \tag{4.1}$$

This is an improvement over (1.9) in that the *Lip** norm allows for test functions that do not vanish on ∂U_s. We will prove

Theorem 2′. *There exists an absolute constant C such that for any $u \in H^1(U_r;\mathbb{C})$ satisfing* (1.7)*, if we write $K_0 = \int_{U_r} e_\varepsilon(u)\,dx - \pi\ln\frac{r}{\varepsilon}$ as in* (1.6)*, then there exists a point $\xi \in U_{r/2}$ such that* (1.8) *and* (1.9) *hold. In addition, for any $\tau < r - |\xi|$ and $\varepsilon \leq \sigma < \tau$,*

$$\int_{B_\tau(\xi)\setminus B_\sigma(\xi)} e_\varepsilon(u)\,dx \geq \pi\ln\left(\frac{\tau}{\sigma}\right) - K_0{}^+ - C \tag{4.2}$$

where $K_0{}^+ = \max\{K_0, 0\}$. Finally, if $s_\varepsilon := \varepsilon\, C(C+K_0)e^{K_0/\pi}$, then for any $\sigma < r - 2s_\varepsilon$, the set

$$\{s \in [\sigma, r] : \text{ (4.1) } \textit{above holds for } U_s\} \tag{4.3}$$

has measure at least $r - \sigma - 2s_\varepsilon$, and in particular is nonempty.

Remark 1. It is easy to see from the proof that the conclusions still hold if the hypothesis (1.7) is replaced by the condition $\|J'(u) - \pi\delta_0\|_{\dot{W}^{-1,1}(U_r)} \leq \frac{r}{4}$.

We interpret the theorem as asserting that $J(u)$ is localized near the point ξ.

By interpolating between the easy estimate $\|J(u) - \pi\delta_0\|_{C^{0,*}} \leq C(1+\|\nabla u\|^2_{L^2})$ and (1.8), one can get estimates in the dual spaces $(C_0^{0,\gamma})^*$ for all $0 < \gamma \leq 1$, and by the Sobolev embedding theorem these imply estimates in $\dot{W}^{-1,q}$ for all $1 \leq q < 2$. We do not record such a result here, but we will prove and use something in this spirit in Section 6.2.

The conclusions of the theorem are trivial for $K_0 \geq \pi \ln \frac{r}{\varepsilon}$, since then for example the right-hand side of (1.8) is larger than the right-hand side of (1.7). Thus in the proof we may assume without loss of generality that $K_0 \leq \pi \ln \frac{r}{\varepsilon}$.

The statement of the theorem is invariant under the rescaling $u \in H^1(U_r; \mathbb{C}) \mapsto u_r \in H^1(U_1; \mathbb{C})$, $\varepsilon \mapsto \frac{\varepsilon}{r}$ where $u_r(x) = u(rx)$. Thus we will fix $r = 1$ in all subsequent argument in this section. In view of the above invariance, this does not entail any loss of generality.

Since we are working on the ball of radius 1, the assumed bound $K_0 \leq \pi \ln \frac{r}{\varepsilon}$ becomes

$$\int_{U_1} e_\varepsilon(u)\, dx \;\leq\; 2\pi \ln \frac{1}{\varepsilon}. \tag{4.4}$$

Also, since our arguments all focus on U_1 with $r = 1$, we feel free in the remainder of this section to let r denote a parameter unrelated to the r in the statement of the theorem.

The main part of the proof involves obtaining a detailed description of a collection of vortex balls that cover the set S. We will prove:

Proposition 1. *Assume that* $u \in C \cap H^1(U_1; \mathbb{C})$ *satisfies* (1.7), *and define* K_0 *as in* (1.6). *Then there exists a collection of balls* $\mathcal{B}^{**} = \{B_j^{**}\}_{j=1}^M$ *satisfying*

$$S := \{x \in \Omega \;:\; |u(x)| \leq 1/2\} \;\subset \cup B_k^{**} \tag{4.5}$$

(4.6)
$$\deg(u; \partial B_1^{**}) = 1; \qquad \deg(u; \partial B_j^{**}) = 0 \textit{ for all } k \geq 2 \textit{ such that } B_j^{**} \cap \partial U_1 = \emptyset,$$

$$\sum r_k^{**} \leq \varepsilon\; C(C + K_0) e^{K_0/\pi} \tag{4.7}$$

and

$$\sum_k \int_{B_k^{**} \cap \Omega} e_\varepsilon(u)\, dx \;\leq C(C + K_0). \tag{4.8}$$

for an absolute constant C. *Moreover, the center* ξ *of* B_1^{**} *satisfies* $|\xi| < 1/2$.

For the proof we need the following, which improves upon [5], Lemma 3.2.2.

Lemma 8. *There exists* $C > 0$ *such that if* $\varepsilon < 1$ *and* $u \in H^1(U_1; \mathbb{C})$ *satisfies* (1.7) *and* (4.4), *then*

$$\begin{aligned} \mathcal{L}^1(\{r \in (0,1) : |u| > \tfrac{1}{2} \textit{ on } \partial B_r, \deg(u; \partial B_r) = 1\}) \\ \geq 1 - \|\frac{1}{\pi} J(u) - \delta_0\|_{\dot{W}^{-1,1}(U_1)} - C\varepsilon |\ln \varepsilon|. \end{aligned} \tag{4.9}$$

Proof. Recall from (2.16) that the modified Jacobian $J'(u)$ is supported in S. Moreover, if we define $d(r) := \frac{1}{\pi}\int_{B_r} J'(u)$, then it follows from (2.17) that

$$d(r) = \deg(u;\partial B_r) \in \mathbb{Z} \qquad \text{if } \partial B_r \cap S = \emptyset. \tag{4.10}$$

Also, Lemma 7 and (4.4) imply that

$$|\{0 < r < 1 : B_r \cap S \neq \emptyset\}| \leq C\varepsilon|\ln\varepsilon|. \tag{4.11}$$

It follows from Lemma 1 and (4.4) that that

$$\begin{aligned} \|\pi\delta_0 - J'(u)\|_{\dot W^{-1,1}(U_1)} &\leq \|\pi\delta_0 - J(u)\|_{\dot W^{-1,1}(U_1)} + \|J(u) - J'(u)\|_{\dot W^{-1,1}(U_1)} \\ &\leq \|\pi\delta_0 - J(u)\|_{\dot W^{-1,1}(U_1)} + C\varepsilon|\ln\varepsilon|. \end{aligned} \tag{4.12}$$

Now consider a test function of the form $\phi(x) = f(|x|)$ for Lipschitz $f : \mathbb{R} \to \mathbb{R}$ with $f(1) = 0$, and note that for such ϕ,

$$\int_{U_1} \phi J'(u) = \int_0^1 f(r)\int_{\partial B_r} J'(u)\, d\mathcal{H}^1\, dr = \pi\int_0^1 f(r)d'(r)\, dr = -\pi\int_0^1 f'(r)d(r)\, dr.$$

It therefore follows from (4.12) that

$$\pi\int_0^1 f'(r)(1 - d(r))dr \leq \|\pi\delta_0 - J(u)\|_{\dot W^{-1,1}(U_1)} + C\varepsilon|\ln\varepsilon|$$

whenever $|f'| \leq 1$ a..e.Hence $\pi\int_0^1 |1 - d(r)|dr \leq \|\pi\delta_0 - J(u)\|_{\dot W^{-1,1}(U_1)} + C|\ln\varepsilon|$. The conclusion follows from combining this with (4.10) and (4.11). ☐

We now present the

Proof of Proposition 1. **Step 1**. In this step we construct a collection of balls $\mathcal{B}^*$ satisfying (4.5) with S replaced by S_E as defined in (3.2); (4.6) with the degree replaced by the essential degree $\mathrm{dg}\,(u;\partial B_j^*)$; and (4.7).

For $\sigma \geq \sigma^0$, let $\{B_k^\sigma\}$ be the family of balls generated by Lemma 6, where σ^0 is defined in (3.15). We will eventually define $\mathcal{B}^*$ to be $\{B_k^{\sigma_1}\}$ for a suitable value σ_1.

Step 1a First consider $\tau = \sigma^0$, and recall that $\mathcal{B}^{\sigma^0} = \mathcal{B}^0$ as constructed in Lemma 5. From (3.14) and (4.4) it follows that

$$\sum r_j^0 \leq \; C\varepsilon|\ln\varepsilon|.$$

Also, (4.9) and (1.7) imply that for ε sufficiently small, there exists $r < 1/2$ such that $|u| > 1/2$ on ∂B_r, $\deg(u;\partial B_r) = 1$, and $\partial B_r \cap B_j^0 = \emptyset$ for all j. Then

$$1 = \deg(u;\partial B_r) = \mathrm{dg}(u;\partial B_r) = \sum_{B_j^0 \subset B_r} \mathrm{dg}(u;\partial B_j^0) \tag{4.13}$$

using (3.4) and (3.5). In particular there is some j such that $B_j^0 \subset B_r$ and $\mathrm{dg}(u;\partial B_j^0) \neq 0$.

We now show that

$$\text{if } \tau > 1, \text{ then } \sum r_i^\tau > 3/8. \tag{4.14}$$

To achieve this, fix any $\tau > 1$, and assume toward a contradiction that $\sum r_j^\tau \leq 3/8$. Then, in view of (4.9), we can find some $r < 1$ such that $|u| > \frac{1}{2}$ on $\partial B_r, \deg(u; \partial B_r) = 1$, and $\partial B_r \cap B_j^\tau = \emptyset$ for all j. Exactly as in (4.13), this implies that there is some j such that $B_j^\tau \subset B_r$ and $\mathrm{dg}(u; \partial B_j^\tau) \neq 0$. However, if this is the case, then (3.17) implies that $r_j^\tau \geq \tau > 1$, which is impossible in view of the fact that that $B_j^\tau \subset B_r \subset U_1$.

Step 1b. We next use the refined vortex balls estimate of Lemma 6 to show that

$$\sum_j \Lambda_\varepsilon(r_j^\sigma) - \Lambda_\varepsilon(\sum r_j^\sigma) \ \leq \ \sum_j \int_{B_j^\sigma} e_\varepsilon(u)\,dx - \Lambda_\varepsilon(\sum r_j^\sigma) \ \leq \ K_0 + C \tag{4.15}$$

for all $\sigma \geq \sigma^0$. The first inequality follows immediately from (3.19) so we must only prove the second. To do this, note that for any $\sigma^0 \leq \sigma < \tau$,

$$\begin{aligned}
\pi \ln\frac{1}{\varepsilon} + K_0 &\geq \sum_k \int_{B_k^\tau \cap U_1} e_\varepsilon(u)\,dx && \text{by (1.6)} \\
&\geq \sum_k \left(\Lambda_\varepsilon(r_k^\tau) - \Lambda_\varepsilon(\sum_{B_j^\sigma \subset B_k^\tau} r_j^\sigma) \right) + \sum_k \int_{B_k^\sigma \cap U_1} e_\varepsilon(u)\,dx && \text{by (3.18)} \\
&\geq \Lambda_\varepsilon(\sum_k r_k^\tau) - \Lambda_\varepsilon(\sum_j r_j^\sigma) + \sum_k \int_{B_k^\sigma \cap U_1} e_\varepsilon(u)\,dx && \text{by (3.12).}
\end{aligned}$$

From (4.14) we infer that $\Lambda_\varepsilon(\sum r_i^\tau) \geq \pi \ln \frac{1}{\varepsilon} - C$ when $\tau > 1$, so we have established (4.15).

Step 1c. We now set $\sigma = \sigma^0$ in (4.15) to get improved bounds on the initial collection of balls. Since according to Lemma 5, $\int_{B_j^0} e_\varepsilon(u)dx \geq \frac{r_j^0}{C\varepsilon}$, for each j, after remembering the definition (3.10) of Λ_ε we obtain

$$\sum_{j=1}^{k(\sigma^0)} \frac{1}{C\varepsilon} r_j^0 - \ln\left(\frac{\sum_{j=1}^{k(\sigma^0)} r_j^0}{C\varepsilon} + 1\right) \leq K_0 + C.$$

This implies that $\sum_{k=1}^{k(\sigma^0)} r_j^0 \leq C(C + K_0)\varepsilon$, and hence (recalling that $r_j^0 \geq \varepsilon$ for every j) that $k(\sigma^0) \leq C(C + K_0)$. Finally, since we have shown in Step 1a that there is at least one ball, say B_1^0, with nonzero degree, the definition (3.15) of σ^0 implies that $\sigma^0 \leq r_1^0 \leq C(C + K_0)\varepsilon$.

Step 1d. As a further consequence of (4.15), note that together with (3.13) it yields

$$\sum_{j=1}^{k(\sigma)} r_j^\sigma - \max\{r_j^\sigma\} \le Ce^{K_0/\pi}\varepsilon \tag{4.16}$$

for all $\sigma > \sigma^0$. It follows that for $\sigma > Ce^{K_0/\pi}\varepsilon$, there is at most one ball B_k^σ such that $B_k^\sigma \cap \partial U = \emptyset$ and $d_k^\sigma \neq 0$, because if there are two or more such balls, each of them must have radius at least σ due to (3.17), and this is impossible by (4.16). Thus in view of Step 1a there is exactly one ball with center $\xi \in B_{r/2}$ and nonzero degree.

Step 1e. Fix $\sigma_1 = C'e^{K_0/\pi}\varepsilon$, for C' larger than the constant C in (4.16), and let $\mathcal{B}^* := \{B_j^{\sigma_1}\}$. We claim that this collection has the desired properties.

It is clear by construction that $S_E \subset \cup_j B_j^*$.

Since we have already shown that $\sum r_j^0 \le C(C+K_0)\varepsilon$ and $k(\sigma^0) \le C(C+K_0)$, the estimate

$$\sum r_j^* \le \ \varepsilon C(C+K_0)e^{K_0/\pi} \tag{4.17}$$

follows from the fact that

$$\sum_j r_j^\sigma \le \sum_j r_j^0 + k(\sigma^0)(\sigma - \sigma^0) \qquad \sigma > \sigma^0.$$

This asserts that as we increase σ from its initial value σ^0, the increase in the sum of the radii is bounded by the number of balls $k(\sigma^0)$ in the initial collection (which provides an upper bound for the number of balls in all subsequent collections) multiplied by the increase in the parameter σ. This is easily verified by inspecting the algorithm of Lemma 6 used to generate the balls.

Finally, we must show that after the balls are relabelled in a suitable way,

$$\mathrm{dg}(u;\partial B_1^*) = 1; \qquad \mathrm{dg}(u;\partial B_j^*) = 0 \text{ for all } k \ge 2 \text{ such that } B_j^* \cap \partial U_1 = \emptyset, \tag{4.18}$$

and the center ξ of B_1^* satisfies $|\xi| < 1/2$ To do this, let $r < 1/2$ by any number such that $\partial B_r \cap B_j^* = \emptyset$ for all j, and such that (4.13) holds. In view of (4.17) and the arguments of Step 1a, such an r exists. It follows from (3.5) that $\sum_{B_j^* \subset B_r} \mathrm{dg}(u;\partial B_j^*) = 1$ and hence that B_r contains at least one ball, say $B_1^* \subset B_r$, of nonzero degree. In view of Step 1d, nowever, there is at most one such ball, and so (4.18) follows.

Step 2. In this step we prove that most of the energy of u in U_1 is located on annuli centered at ξ that do *not* intersect the balls constructed in Step 1, where ξ denotes the center of the distinguished ball B_1^*. In particular this will demonstrate that $\mathcal{B}^*$ satisfies the analog of (4.8). We also show obtain separate bounds on $\int e_\varepsilon(|u|)$.

For $0 \le r \le \frac{1}{2}$, define $m(r) := \min_{\partial B_r(\xi)} |u|$. Also define

$$T_1 := \{r \in [c_1\varepsilon, 1/2] : \; m(r) \le 1/2 \text{ or } \deg(u; \partial B_r(\xi)) \ne 0\},$$
$$T_2 := [c_1\varepsilon, 1/2] \setminus T_1.$$

Here c_1 is the constant in (3.11), which implies in particular that

$$\int_{\partial B_r(\xi)} e_\varepsilon(u) \, d\mathcal{H}^1 \; \ge \frac{\pi m^2}{r} + \frac{(1-m^2)^2}{c_0\varepsilon} \tag{4.19}$$

for all $r \in T_1$. Note that if r is such that $\partial B_r(\xi)$ does not intersect any of the balls in the collection $\{B_j^*\}_{j=1}^M$ from Step 1, then $\mathrm{dg}(u; \partial B_r(\xi))$ is well-defined, and indeed $\mathrm{dg}(u; \partial B_r(\xi)) = \sum\{\mathrm{dg}(u; \partial B_j^*) : B_j^* \subset B_r(\xi)\} = \mathrm{dg}(u; \partial B_1^*) = 1$, since B_1^* is the unique ball with nonzero degree, and it must be contained in $B_r(\xi)$, given that the two balls are concentric and $\partial B_r(\xi) \cap B_1^* = \emptyset$. It follows that if $r \in T_2$, then $\partial B_r(\xi)$ must intersect B_j^* for some $j = 1, \ldots, M$. Then the estimate of $\sum r_i^*$ from Step 1 implies that

$$|T_2| \; \le \varepsilon C(C+K_0)e^{K_0/\pi}, \qquad \text{and as a result } |T_1| \ge 1/2 - \varepsilon C(C+K_0)e^{K_0/\pi}.$$

From the latter estimate it easily follows that

$$\int_{T_1} \frac{1}{r} \, dr \; \ge \ln\frac{1}{2} - \ln(\frac{1}{2} - |T_1|) \ge \ln\frac{1}{\varepsilon} - C - CK_0. \tag{4.20}$$

Now we set $A := \pi \int_{T_1} (1 - m^2(r))\frac{dr}{r}$ and $B := \int_{T_1} \frac{(1-m^2(r))^2}{c_0\varepsilon} dr$, where c_0 is the constant in (3.7). Then using (4.20), (4.19) and (1.6),

$$\begin{aligned} A - B \; &\ge \; \pi \ln\frac{1}{\varepsilon} \; - C(C+K_0) - \int_{T_1} \left(\frac{\pi m^2}{r} + \frac{(1-m^2)^2}{c_0\varepsilon} \right) \, dr \\ &\ge \pi \ln\frac{1}{\varepsilon} - C(C+K_0) - \int_{T_1} \int_{\partial B_s(\xi)} e_\varepsilon(u) d\mathcal{H}^1 \, ds \\ &\ge \pi \ln\frac{1}{\varepsilon} - \int_{U_1} e_\varepsilon(u) \, dx \; - C(C+K_0) = -C(C+K_0). \end{aligned} \tag{4.21}$$

In addition, by Hölder's inequality,

$$A \le \left(\int_{T_1} \frac{(1-m^2)^2}{c_0\varepsilon} \, dr \right)^{1/2} \left(\int_{T_1} \frac{c_0\varepsilon}{r^2} \, dr \right)^{1/2} \; \le \; C\sqrt{B}.$$

Combining these, we find that $A - CA^2 \ge -C(C+K_0)$, which implies that $A \le C(C+K_0)$, i.e.

$$\pi \int_{T_1} (1 - m^2(r)) \frac{dr}{r} \le C(C+K_0). \tag{4.22}$$

In addition, (4.22) and (4.21) imply that $B \le C(C+K_0)$, in other words that

$$\int_{T_1} \frac{(1-m^2(r))^2}{\varepsilon} dr \; \le C(C+K_0). \tag{4.23}$$

Note that (4.22), (4.20), and (3.8) imply that

$$\int_{T_1}\int_{\partial B_s(\xi)} \frac{1}{2}\frac{|j(u)|^2}{|u|^2} d\mathcal{H}^1\ ds \ \geq\ \pi \ln\frac{1}{\varepsilon} - C(C+K_0), \tag{4.24}$$

and this in combination with (1.6) yields

$$\int_{U_1} e_\varepsilon(|u|) + \int_{U_1\setminus B_{1/2}(\xi)} \frac{|j(u)|^2}{|u|^2} + \int_{T_2}\int_{\partial B_s(\xi)} \frac{|j(u)|^2}{|u|^2} d\mathcal{H}^1\ ds\ \leq C(C+K_0). \tag{4.25}$$

Step 3: So far we have constructed a collection of balls $\mathcal{B}^*$ that has all the properties that we want, except that it covers only S_E rather than S in (4.5), and with corresponding modifications in (4.6). In this final step we show that $\mathcal{B}^*$ can be modified to obtain a collection $\mathcal{B}^{**}$ satisfying (4.5), (4.6), as well as the other conclusions.

Let $\widetilde{\mathcal{B}}^0$ be the collection of balls generated by Lemma 7. It follows from (4.25) and (3.23) that

$$\sum \tilde{r}^0_j \ \leq \varepsilon C(C+K_0). \tag{4.26}$$

Let $\mathcal{B}^{**}$ be the collection formed by combining all the balls from the collections $\mathcal{B}^*$ and $\widetilde{\mathcal{B}}^0$, and using Lemma 4 to create a new pairwise disjoint collection that contains the union of all these balls and preserves the sum of the radii. One of these balls contains the point ξ (in fact it contains the ball B^*_1), and we label this ball B^{**}_1. Then (4.5) holds as a consequence of (3.22). From (4.5) and (3.4) it follows that

$$\deg(u;\partial B^{**}_j) = \sum_{B^*_k\subset B^{**}_j} \mathrm{dg}(u;\partial B^*_k).$$

This implies (4.6). The estimate (4.7) is a consequence of the analogous estimate for $\mathcal{B}^*$ and $\widetilde{\mathcal{B}}^0$, and Lemma 4. Finally, to verify (4.8), let

$$\widetilde{T}_1 := \{r\in[c_1\varepsilon, 1/2] : \partial B_r(\xi)\cap B^{**}_j = \emptyset \text{ for all } j\}, \qquad \widetilde{T}_2 = [\varepsilon, \frac{1}{2}]\setminus T_1.$$

It follows from (4.23) that $\{r:\ m(r)<\frac{1}{2}\}$ has measure bounded by $C(C+K_0)\varepsilon$. Thus $|\widetilde{T}_1| \geq |T_1| - C(C+K_0)\varepsilon \geq 1/2 - \varepsilon C(C+K_0)e^{K_0/\pi}$. Therefore we can repeat exactly the arguments of Step 2, to find that (4.24) and (4.25) hold with T_1 and T_2 replaced by $\widetilde{T}_1$ and $\widetilde{T}_2$ (and with larger constants C.) In view of the definition of $\widetilde{T}_2$, this establishes (4.8). □

Theorem 2′ will follow by converting the information about vortex balls to estimates of the Jacobian. To do this we will use the following straightforward

Lemma 9. *Suppose that* $\Omega\subset\mathbb{R}^2$ *is an open set and that* $\mu\in L^1(\Omega;\mathbb{R})$ *is supported in a union of balls* $\cup_{i=1}^M(\Omega\cap B_{r_i}(x_i))$. *Then, for* $a_i := \int_{B_{r_i}(x_i)}\mu dx$, *the* *following estimate holds:*

$$\left\| \mu - \sum_{i=1}^{M} a_i \delta_{x_i} \right\|_{Lip^*(\Omega)} \leq \sum r_i \int_{B_{r_i}(x_i)\cap\Omega} |\mu|\, dx. \tag{4.27}$$

Moreover, if Γ *is any set such that* $\partial\Omega \cap (\cup B_{r_i}(x_i)) \subset \Gamma$*, then*

$$\left\| \mu - \sum_{\{i: B_{r_i}(x_i)\subset\Omega\}} a_i \delta_{x_i} \right\|_{\dot W_\Gamma^{-1,1}(\Omega)} \leq \sum r_i \int_{B_{r_i}(x_i)\cap\Omega} |\mu|\, dx. \tag{4.28}$$

Clearly an analogous result still holds if μ is a signed measure.

Proof. Let $\mu_i(x) = \mu(x)$ if $x \in B_i := B_{r_i}(x_i)$ and 0 otherwise. Then for any Lipschitz function ϕ,

$$\int_\Omega \phi(x)\mu_i(x)\, dx \ = \ \phi(x_i)a_i + \int_{B_i\cap\Omega} (\phi(x) - \phi(x_i))\mu_i(x)\, dx.$$

If $\|\nabla\phi\|_\infty \leq 1$, the integral on the right is bounded by $r_i \int_{B_i\cap\Omega} |\mu|\, dx$. Thus

$$\|\mu_i - a_i\delta_{x_i}\|_{Lip^*(\Omega)} \leq r_i \int_{B_i\cap\Omega} |\mu|. \tag{4.29}$$

Since $\mu = \sum_i \mu_i$, (4.27) now follows by the triangle inequality. Next, note that if $B_{r_i}(x_i) \cap \partial\Omega \neq \emptyset$ and $\phi = 0$ on $B_{r_i}(x_i) \cap \partial\Omega$, then $|\phi(x)| \leq r_i$ on the support of μ_i, and so

$$\|\mu_i\|_{\dot W_\Gamma^{-1,1}} \leq r_i \int_{B_i\cap\Omega} |\mu|\, dx, \quad \text{when } B_{r_i}(x_i)\cap\partial\Omega \subset \Gamma. \tag{4.30}$$

Clearly $\|\nu\|_{\dot W_\Gamma^{-1,1}(\Omega)} \leq \|\nu\|_{Lip^*(\Omega)}$ for all ν, so we deduce (4.28) from (4.29), (4.30), and the triangle inequality. ☐

Finally, as promised, we combine the above lemma with the construction of vortex balls to complete the

proof of Theorem 2′. By an approximation argument, it suffices to prove the result for smooth u, so that we can use Proposition 1.

Step 1. Let $\mathcal{B}^{**}$ be the collection of balls from Proposition 1. We apply Lemma 9 to $\mu = J'(u)$, which in view of (2.16) is supported in $S \subset \cup B_j^{**}$. By (4.7), (4.8), and (4.28) with $\Gamma = \partial U_1$, we see that

$$\left\| J'(u) - \sum_{\{j\ :\ B_j^{**}\cap\partial U_1 = \emptyset\}} a_j \delta_{y_j} \right\|_{\dot W^{-1,1}(U_1)} \leq \varepsilon C (C + K_0)^2 e^{K_0/\pi} \tag{4.31}$$

where $a_j = \int_{U_1 \cap B_j^{**}} J'(u)\,dx$ and y_j denotes the center of B_j^{**}. If $\partial B_j^{**} \cap \partial U_1 = \emptyset$, then by (2.17) and (4.6),

$$a_j = \pi \deg(u; \partial B_j^{**}) = \begin{cases} \pi & \text{if } j = 1 \\ 0 & \text{if not.} \end{cases} \tag{4.32}$$

In addition $x_1 = \xi$. Hence (4.31) reduces to (1.9).

Step 2. Recall from Lemma 1 that

$$\|J'(u) - J(u)\|_{\dot{W}^{-1,1}(U_1)} \leq C \|\nabla u\|_2 \|1 - |u|^2\|_2.$$

Clearly

$$\|\nabla u\|_2 \leq C \left(\int_{U_1} e_\varepsilon(u) \right)^{1/2} = C \left(K_0 + \pi \ln \frac{1}{\varepsilon} \right)^{1/2},$$

and by (4.25),

$$\|1 - |u|^2\|_2 \leq C\varepsilon \left(\int_{U_1} e_\varepsilon(|u|) \right)^{1/2} = C\varepsilon \left(1 + K_0\right)^{1/2}.$$

It suffices to consider $\varepsilon < 1/2$, so the above inequalities imply that

$$\|J'(u) - J(u)\|_{\dot{W}^{-1,1}(U_1)} \leq C\varepsilon (K_0 + 1)\sqrt{\ln(1/\varepsilon)}. \tag{4.33}$$

We deduce (1.8) by combining the above inequality with (1.9).

Step 3. The proof of (4.2) is similar to arguments in Step 2 of the proof of Proposition 1, but easier. For $0 \leq \sigma < \tau$ such that $B_\tau(\xi) \subset U_1$, let

$$\mathcal{S} := \{ s \in (\varepsilon, r - |\xi|) \ : \ B_1^{**} \subset B_s, \ \partial B_s \cap B_i^{**} \ = \ \emptyset \ \forall i = 1, \ldots, M \},$$

and let $\mathcal{T} := (0, r - |\xi|) \setminus \mathcal{S}$. Then Lemma 2 implies that $\int_{\partial B_s(\xi)} e_\varepsilon(u) \ \geq \lambda_\varepsilon(s) = \frac{\pi}{s + c_0 \varepsilon}$ for $s \in \mathcal{S}$, and so

$$\int_{B_\tau(\xi) \setminus B_\sigma(\xi)} \ = \ \int_\sigma^\tau \int_{\partial B_s(\xi)} e_\varepsilon(u) d\mathcal{H}^1 \, ds \ \geq \int_{(\sigma,\tau)\setminus \mathcal{T}} \lambda_\varepsilon(s)\, ds.$$

Since $\lambda_\varepsilon(\cdot)$ is decreasing,

$$\int_{(\sigma,\tau)\setminus\mathcal{T}} \lambda_\varepsilon(s)\,ds \geq \int_\sigma^\tau \lambda_\varepsilon(s)\,ds - \int_0^{|\mathcal{T}|} \lambda_\varepsilon(s)\,ds.$$

From (4.7) we deduce that $|\mathcal{T}| \leq C\varepsilon(C + K_0)e^{K_0/\pi}$. Then (4.2) follows by evaluating the integrals and simplifying the resulting expressions.

Step 4. Finally we prove (4.3). To do this it suffices, in view of (4.7), to prove that (4.1) holds for U_s if $\partial U_s \cap B_j^{**} = \emptyset$ for all j. In this case we deduce (4.1) directly from (4.28) with $\Gamma = \emptyset$, together with (4.32). □

5. LOCALIZATION OF JACOBIAN FOR GENERAL CONFIGURATIONS

In this section, we present the proof of Theorem 1. As in the previous section, we actually prove a slightly stronger result than that stated in the introduction. To state this result, we introduce some notation: suppose that Ω is a bounded, open subset of $\mathbb{R}^2$ and that Γ is a fixed subset (possibly empty) of $\partial\Omega$. Let $\Sigma := \partial\Omega \setminus \Gamma$, and for $s > 0$ let

$$\Omega_{s,\Sigma} := \{x \in \Omega : \operatorname{dist}(x, \Sigma) > s\}. \tag{5.1}$$

(For $\Gamma = \partial\Omega$ we use the convention that $\Omega_{s,\emptyset} = \Omega$ for all s.) We define $\dot{W}^{-1,1}_{\Gamma}(\Omega_{s,\Sigma})$ as in (2.2). We will prove that for suitable $s \ll 1$,

$$\begin{aligned} &\|J'(u) - \pi \sum_{i=1}^{n_0} d_i \delta_{a_i}\|_{\dot{W}^{-1,1}_{\Gamma}(\Omega_{s,\Sigma})} \\ &\leq \varepsilon C(n+1) \left(\int_\Omega e_\varepsilon(u)\, dx \right) \exp\left[\frac{1}{\pi(n+1)} \int_\Omega e_\varepsilon(u)\, dx \right] \end{aligned} \tag{5.2}$$

and

$$|u| > \frac{1}{2} \quad \text{on } \partial\Omega_{s,\Sigma} \setminus \Gamma. \tag{5.3}$$

The point, as before, is that by modifying the domain slightly, we can obtain estimates in a stronger norm, and more generally better behavior at the boundary. The main result of this section is

Theorem 1′. *There exists an absolute constant C such that for any $\Omega \subset \mathbb{R}^2$ and $u \in H^1(\Omega;\mathbb{C})$, any $\varepsilon \in (0,1]$ and any $n > \frac{M}{\pi} - 1$, there exists an integer $n_0 \leq n$, points $a_1, \ldots, a_{n_0} \in \Omega$, and integers $d_1, \ldots, d_{n_0} \in \{\pm 1\}$, such that* (1.4) *holds.*

Moreover, if we write

$$M = \frac{1}{|\ln \varepsilon|} \int_\Omega e_\varepsilon(u)\, dx \tag{5.4}$$

and

$$r_{\varepsilon,n} := C(n+1)\varepsilon^{1-\frac{M}{\pi(n+1)}}, \tag{5.5}$$

then for any $\Gamma \subset \partial\Omega$ and any $\sigma > 2r_{\varepsilon,n}$,

$$|\{s \in [0,\sigma] \ : \ u \text{ satisfies } (5.2) \text{ and} (5.3) \text{ on } \Omega_{s,\Sigma}\}| \geq \sigma - 2r_{\varepsilon,n}, \tag{5.6}$$

where $|\cdots|$ denotes (Lebesgue) measure.

We have in mind applying this estimate for Ω of the form $\Omega = G \setminus \cup_i B_r(a_i)$, where G is connected and simply connected, and the balls $B_{2r}(a_i)$ are pairwise disjoint and contained in Ω. In this situation, we can take $\Gamma = \partial G$. We can then use (4.1) and (5.2) to glue together estimates on Ω obtained from Theorem $1'$ and estimates on the balls $B_{2r}(a_i)$ found via Theorem $2'$, thereby deriving good estimates on all of G. This argument is used in [8] to prove generalizations of Theorems 1 and 2.

As with Theorem $2'$, the main part of the proof consists in constructing a suitable collection of balls covering the set S around which $J(u)$ is concentrated. Unlike Theorem $2'$, the desired balls here can be found quite easily by invoking results already present in the literature. This is the content of the following

Proposition 2. *Assume that* $u \in C \cap H^1(\Omega;\mathbb{C})$ *satisfies* (5.4), *and let* $n > \frac{M}{\pi} - 1$ *be an integer. Then there exists a collection of balls* $\mathcal{B}^\# = \{B_j^\#\}_{j=1}^K$ *satisfying*

$$S \subset \cup B_k^\# \tag{5.7}$$

$$\sum_{\{j : B_j^\# \cap \partial\Omega = \emptyset\}} |\deg(u; \partial B_j^\#)| \le n \tag{5.8}$$

$$\sum r_k^\# \ \le\ r_{\varepsilon,n} \ =\ C(n+1)\varepsilon^{1-\frac{M}{\pi(n+1)}}, \tag{5.9}$$

where C is an absolute constant.

Proof. The proof relies on Proposition 6.4 from [7], in which it is shown that there exists some positive[3] number $\sigma^0 \le C\varepsilon \int_\Omega e_\varepsilon(u)\,dx$ and, for every $\sigma \ge \sigma^0$, a collection $\mathcal{B}^\sigma = \{B_j^\sigma\}_{j=1}^{k(\sigma)}$ of closed, pairwise disjoint balls with $r_j^\sigma \ge \varepsilon$ for all j, such that $S_E \subset \cup_j B_j^\sigma$ (where S_E is defined in (3.2)), and

$$\int_{\Omega \cap B_j^\sigma} e_\varepsilon(u)\,dx \ \ge\ \frac{r_j^\sigma}{\sigma}\Lambda_\varepsilon(\sigma) \tag{5.10}$$

and

$$r_j^\sigma \ge \sigma |d_j^\sigma| \quad \text{whenever } \partial\Omega \cap B_j^\sigma = \emptyset, \tag{5.11}$$

where $d_j^\sigma = \mathrm{dg}\,(u; B_j^\sigma)$; recall (3.3) for the definition.

We fix $\sigma^* := c_0\varepsilon^\beta$ for $\beta = 1 - M/\pi(n+1) > 0$. Here c_0 is the constant appearing in the definition of Λ_ε, see (3.10). Note in particular that $\Lambda_\varepsilon(\sigma^*) > \pi(1-\beta)|\ln\varepsilon|$.

We claim that

$$\sum_{\{j : B_j^{\sigma^*} \cap \partial\Omega = \emptyset\}} |\mathrm{dg}(u; \partial B_j^{\sigma^*})| \le n \tag{5.12}$$

and

$$\sum_j r_j^{\sigma^*} \ \le\ C(n+1)\varepsilon^{1-\frac{M}{\pi(n+1)}}. \tag{5.13}$$

---

[3] Strictly speaking, σ^0 is only finite if at least one ball in the collection $\mathcal{B}^0$ from Lemma 5 has nonzero essential degree. However, if $\mathrm{dg}\,(u; B_j^0) = 0$ for all j, then it is easy to see that the collection of balls $\mathcal{B}^0$ satisfies (5.10) and (5.11).

To prove these, we estimate

$$\begin{aligned}
M|\ln\varepsilon| \geq \; & \sum_j \int_{B_j^{\sigma^*}} e_\varepsilon(u)\; dx && \text{by (5.4)}\\
\geq \; & \sum_j \frac{r_j^{\sigma^*}}{\sigma^*}\Lambda_\varepsilon(\sigma^*) && \text{by (5.10)} \quad (5.14)\\
\geq \; & \sum_{\{j: B_j^{\sigma^*}\cap\partial\Omega=\emptyset\}} |\mathrm{dg}(u;\partial B_j^{\sigma^*})|\,\Lambda_\varepsilon(\sigma^*) && \text{by (5.11)}\\
> \; & \pi(1-\beta)|\ln\varepsilon| \sum_{\{j: B_j^{\sigma^*}\cap\partial\Omega=\emptyset\}} |\mathrm{dg}(u;\partial B_j^{\sigma^*})|. && (5.15)
\end{aligned}$$

Since $\frac{M}{\pi(1-\beta)} = n+1$, the first claim (5.12) follows from (5.15). Similarly, using the fact that $\Lambda_\varepsilon(\sigma^*) > \pi(1-\beta)|\ln\varepsilon|$ in (5.14) we obtain (5.13) .

Thus the balls $\mathcal{B}^{\sigma^*}$ satisfy the conclusions of the proposition, except that S in (5.7) is replaced by S_E as defined in (3.2), and the degree in (5.8) is replaced by the essential degree dg.

To finish the proof we argue as in Step 3 of the proof of Proposition 1: by appealing to Lemma 7 we obtain a collection of balls $\widetilde{\mathcal{B}}^0$ covering S and with the sum of the radii bounded by $C\varepsilon M|\ln\varepsilon|$. Using Lemma 4, we merge the collections of balls $\mathcal{B}^0$ and $\mathcal{B}^{\sigma^*}$ to obtain a larger collection $\mathcal{B}^\#$, and arguing exactly as in the proof of Proposition 1 we see that $\mathcal{B}^\#$ satisfies (5.7), (5.8), (5.9). □

The proof of the theorem now follows very closely the argument that deduces Theorem 2′ from the collection of balls provided by Proposition 1.

Proof of Theorem 1′. Let

$$n_0 := \sum_{\{j: B_j^\# \cap\partial\Omega=\emptyset\}} |\deg(u;\partial B_j^\#)|$$

and note that (5.8) states that $n_0 \leq n$. Choose integers $d_1,\ldots,d_{n_0}\in\{\pm1\}$ and points $x_1,\ldots,x_{n_0}\in\Omega$, in general not distinct, such that each x_i is the center of a ball $B_j^\#$ from Proposition 2 for which $\deg(u;\partial B_j^\#)\neq 0$ and $B_j^\#\cap\partial\Omega\neq\emptyset$; and such that for each such $B_j^\#$,

$$\sum_{\{i\,:\,x_i\in B_j^\#\}} d_i \;=\; \deg(u;\partial B_j^\#)$$

Now, exactly as in the proof of Theorem 2$'$, we can use Lemma 9 together with (5.9) and (2.17) to prove that

$$\left\| J'(u) - \pi \sum_{i=1}^{n_0} d_i \delta_{x_i} \right\|_{\dot W^{-1,1}(\Omega)} \le r_{\varepsilon,n} \int_\Omega e_\varepsilon(u)\, dx \ = \ r_{\varepsilon,n} M |\ln\varepsilon|, \tag{5.16}$$

where $J'(u)$ denotes the modified Jacobian.

Next, recall from Lemma 1 that

$$\|J'(u) - J(u)\|_{\dot W^{-1,1}(\Omega)} \le C\,\varepsilon \int_\Omega e_\varepsilon(u)\, dx \ = \ C\varepsilon M |\ln\varepsilon|.$$

In view of (5.9) and (5.4), these inequalities imply (1.4) that

$$\left\| J(u) - \pi \sum_{i=1}^{n_0} d_i \delta_{x_i} \right\|_{\dot W^{-1,1}(\Omega)} \le C(n+1) M |\ln\varepsilon|\, \varepsilon^{1-\frac{M}{\pi(n+1)}}.$$

This is exactly (1.4).

Finally, it is a direct consequence of (4.28) and the properties of $\mathcal{B}^\#$ that if

$$B_j^\# \cap (\partial\Omega_{s,\Sigma} \setminus \Gamma) = \emptyset \text{ for every ball } B_j^\#, \tag{5.17}$$

then $|u| > 1/2$ on $\partial\Omega_{s,\sigma} \setminus \Gamma$ (where $\Omega_{s,\Sigma}$ is defined in (5.1)) and

$$\|J(u) - \pi \sum_{i=1}^{n_0} d_i \delta_{a_i}\|_{\dot W_\Gamma^{-1,1}(\Omega_{s,\Sigma})} \le 2M |\ln\varepsilon| r_{\varepsilon,n}.$$

The above inequality is exactly (5.2), so to prove (5.6), it suffices to show that

$$|\{s > 0 \ : \ (5.17) \text{ is not satisfied } \}| \le 2 r_{\varepsilon,n}. \tag{5.18}$$

This however is clear from (5.9), since for each ball $B_i^\#$, the definition of $\Omega_{s,\Sigma}$ implies that

$$\left|\{s > 0 \ : \ B_i^\# \cap (\partial\Omega_{s,\Sigma} \setminus \Gamma) \ne \emptyset\}\right| \le 2 r_i^\#.$$

□

6. A LOCAL LOWER BOUND

We introduce the notation

$$I(r,\varepsilon) := \inf \left\{ \int_{B_r} e_\varepsilon(u) \ ; \ u \in H^1(B_r;\mathbb{C}), \ u = e^{i\theta} \text{ on } \partial B_r \right\} \tag{6.1}$$

and note that $I(r,\varepsilon) = I(r/\varepsilon, 1)$ for all r, ε. Next define

$$\gamma = \lim_{r\to\infty} (I(r,\varepsilon) - \pi \ln \frac{r}{\varepsilon}). \tag{6.2}$$

It is known that γ exists, is finite and is independent of ε. Moreover, at the end of this section, in Lemma 16 we prove that $\gamma - (I(r,\varepsilon) - \pi \ln \frac{r}{\varepsilon}) = O((\varepsilon/r)^2)$.

We recall the statement of Theorem 3, which is the main result of this section:

Theorem 3. *There exists a constant $C > 0$ such that if $u \in H^1(U_r; \mathbb{C})$ satisfies*

$$\|J(u) - \pi\delta_0\|_{\dot W^{-1,1}(U_r)} \le \frac{r}{4},$$

then

$$I(r,\varepsilon) - \int_{U_r} e_\varepsilon(u) \ \le\ C\frac{\varepsilon}{r}\sqrt{\ln\frac{r}{\varepsilon}} + \frac{C}{r}\|J(u) - \pi\delta_0\|_{\dot W^{-1,1}(U_r)} \tag{6.3}$$

and

$$I(r,\varepsilon) - \int_{U_r} e_\varepsilon(u) \ \le\ C\frac{\varepsilon}{r} + \frac{C}{r}\|J'(u) - \pi\delta_0\|_{\dot W^{-1,1}(U_r)} \tag{6.4}$$

where $J'(u)$ denotes the modified Jacobian as defined in (2.15).

The interesting case for the theorem is when $\frac{1}{r}\|J'(u) - \pi\delta_0\|_{\dot W^{-1,1}(U_r)}$ is very small; then (6.3), (6.4) provide good lower bounds for $\int_{B_r} e_\varepsilon(u)$.

The theorem improves on Lemma 4.1.1 in [5], which shows that if u is a sequence of fuctions such that $\|J(u)-\pi\delta_0\|_{\dot W^{-1,1}(U_r)} \to 0$, then $\limsup_{\varepsilon\to 0} I(r,\varepsilon) - \int_{U_r} e_\varepsilon(u) \le 0$. (It should also be noted that while the argument in [5] is basically okay, the proof is sloppily written and contains some errors that make it difficult to read.)

To prove the theorem it suffices to consider functions such that the left-hand side of (6.3), (6.4) is positive. The main point is to show that, given such a function u, we can construct a function $\tilde u$ such that $\tilde u = e^{i(\theta + const)}$ on ∂U_r, so that $\int_{U_r} e_\varepsilon(\tilde u)dx \ge I(r,\varepsilon)$; and with good estimates of $\int_{U_r}[e_\varepsilon(\tilde u) - e_\varepsilon(u)]dx$. This is carried out in Proposition 3. First we assume this result and use it to give the proof of the theorem. The statement of Proposition 3 immediately follows the proof of the theorem. The proof of this proposition is carried out in a series of lemmas that occupy the rest of Section 6

proof of Theorem 3. **Step 1**: It is useful to define

$$\mathcal{A}(r,s) := \inf\Big\{u \in H^1(U_r;\mathbb{C}), \|J'(u) - \pi\delta_0\|_{\dot W^{-1,1}(U_r)} \le s\Big\} \tag{6.5}$$

and

$$\delta(r,\varepsilon,s_\varepsilon) := \sup\left\{I(r,\varepsilon) - \int_{B_r} e_\varepsilon(u)\,dx \ : u \in \mathcal{A}(r,s_\varepsilon)\right\}. \tag{6.6}$$

Our goal is to prove an upper bound for $\delta(r,\varepsilon,s_\varepsilon)$. By an easy scaling argument,

$$\delta(r,\varepsilon,s_\varepsilon) = \delta(\lambda r, \lambda\varepsilon, \lambda s_\varepsilon) \qquad \text{for any } \lambda > 0. \tag{6.7}$$

Thus it suffices to estimate $\delta(1,\varepsilon,s_\varepsilon)$.

We claim that to prove the theorem it suffices to establish

$$\delta(1,\varepsilon,K_1\varepsilon) \le C\varepsilon \qquad \text{for all } \varepsilon \in (0,1] \tag{6.8}$$

for K_1 to be chosen momentarily. We first prove that (6.4) can be deduced from (6.8). To do this, assume that (6.8) holds, and fix u such that

$$\|J'(u) - \pi\delta_0\|_{\dot W^{-1,1}(U_1)} \le s_\varepsilon \le \frac{1}{4}. \tag{6.9}$$

We must prove that u verifies (6.3), (6.4) and so we can assume that

$$\int_{U_1} e_\varepsilon(u)\,dx \ \le\ I(1,\varepsilon). \tag{6.10}$$

It follows from Lemma 16, which is proved at the end of this section, that there exists a constant K_0 such that $I(\varepsilon, 1) \le \pi \ln \frac{1}{\varepsilon} + K_0$ for all $\varepsilon \in (0,1]$, and so (6.10) implies that

$$\int_{U_1} e_\varepsilon(u)\,dx - \pi \ln\frac{1}{\varepsilon} \le K_0 \tag{6.11}$$

We define K_1 to be the constant $K_1 := C(C+K_0)^2 e^{K_0/\pi}$ appearing on the right-hand side of (1.9), associated with the fixed constant K_0 from (6.11). Then in view of Remark 1 (which follows the statement of Theorem $2'$) and (6.9), there exists a point $\xi \in U_{1/2}$ such that

$$\|J'(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(U_1)} \le K_1\varepsilon \tag{6.12}$$

Note from (6.9) and (6.12) and the definition of the $\dot W^{1,1}$ norm that

$$|\xi| \ =\ \|\delta_0 - \delta_\xi\|_{\dot W^{-1,1}(U_1)} \le \frac{1}{\pi}(K_1\varepsilon + s_\varepsilon) \tag{6.13}$$

Another consequence of (6.12) is that, if we restrict u to $U_{1-|\xi|}(\xi)$ and then translate the resulting function to a ball centered at the origin, then we obtain a function that belongs to $\mathcal{A}(1-|\xi|, K_1\varepsilon)$. Thus from the definition (6.6) and from (6.7),

$$\begin{aligned}\int_{U_{1-|\xi|}} e_\varepsilon(u)\,dx \ &\ge I(1-|\xi|,\varepsilon) - \delta(1-|\xi|,\varepsilon,K_1\varepsilon)\\ &\ge I(1-|\xi|,\varepsilon) - \delta(1,\frac{\varepsilon}{1-|\xi|},K_1\frac{\varepsilon}{1-|\xi|}).\end{aligned}$$

Using (6.8) to estimate $\delta(1, \frac{\varepsilon}{1-|\xi|}, K_1\frac{\varepsilon}{1-|\xi|})$, we deduce from the above that

$$\int_{U_1} e_\varepsilon(u)\,dx - I(1,\varepsilon) \quad \ge I(1-|\xi|,\varepsilon) - I(1,\varepsilon) - C\frac{\varepsilon}{1-|\xi|}. \tag{6.14}$$

It follows from Lemma 16 that

$$I(1-|\xi|,\varepsilon) - I(1,\varepsilon) \ge -\pi\ln\frac{1}{1-|\xi|} - O(\varepsilon^2) \ge -C|\xi| - O(\varepsilon^2) \ge -C(s_\varepsilon + \varepsilon).$$

Clearly also $\frac{\varepsilon}{1-|\xi|} \le 2\varepsilon$, so (6.4) follows from (6.14).

We obtain (6.3) from (6.4) by observing that

$$\|\pi\delta_0 - J'(u)\|_{\dot W^{-1,1}(U_1)} \le \|\pi\delta_0 - J(u)\|_{\dot W^{-1,1}(U_1)} + \|J'(u) - J(u)\|_{\dot W^{-1,1}(U_1)}$$

and recalling from (4.33) that $\|J'(u) - J(u)\|_{\dot W^{-1,1}(U_1)} \le C\varepsilon\sqrt{|\ln\varepsilon|}$ when (6.10) holds.

Step 2. To prove (6.8), it is convenient to define

$$\delta_\star(\varepsilon) := \delta(1,\varepsilon,K_1\varepsilon). \tag{6.15}$$

We must show that $\delta_\star(\varepsilon) \le C\varepsilon$ for all $\varepsilon \in (0,1]$.

In this step we show that Proposition 3, which is stated and proved below, shows that there exists a constant C such that

$$\delta_\star(\varepsilon) \le C\frac{\varepsilon}{r} + Cr^2 \sup_{r_0\in[r,2r]} \delta_\star(\frac{\varepsilon}{r_0})^+ \tag{6.16}$$

for all $C\varepsilon \le r \le 1/4$, where $\delta_\star(s)^+ := \max\{\delta_\star(s), 0\}$. In other words, we must show that if $u \in \mathcal{A}(1, K_1\varepsilon)$ then $I(1,\varepsilon) - \int_{U_1} e(\varepsilon)$ is bounded by the right-hand side of (6.16). This is automatically satisfied if (6.10) is violated, so it suffices to consider functions satisfying (6.10).

We therefore fix $u \in \mathcal{A}(1, K_1\varepsilon)$ satisfying (6.10), and we observe that for any $r < 1$, the restriction of u to U_r belongs to $\mathcal{A}(r, K_1\varepsilon)$. Indeed, the definitions of the norms imply that

$$\|J'u - \pi\delta_0\|_{\dot W^{-1,1}(U_r)} \le \|J'u - \pi\delta_0\|_{\dot W^{-1,1}(U_1)} \le K_1\varepsilon.$$

Hence for any $r < 1$,

$$\begin{aligned}
\int_{U_1\setminus U_r} e_\varepsilon(u)\,dx &\le I(1,\varepsilon) - \int_{U_r} e_\varepsilon(u)\,dx && \text{by (6.10)}\\
&\le I(1,\varepsilon) - (I(r,\varepsilon) - \delta(r,\varepsilon,K_1\varepsilon)) && \text{by (6.6)}\\
&\le \pi\ln\frac{1}{r} + \delta_\star(\frac{\varepsilon}{r}) + C(\frac{\varepsilon}{r})^2 && \text{by (6.7) and Lemma 16.} \qquad (6.17)
\end{aligned}$$

In Proposition 3, we prove that for every $C\varepsilon \le r \le 1/4$, there exists $r_0 \in [r,2r]$ such that $I(1,\varepsilon) - \int_{U_1} u_\varepsilon(u)$ can be controlled by $\int_{U_1\setminus U_{r_0}} e_\varepsilon(u) - \pi\ln\frac{1}{r_0}$. The precise statement is given in (6.19) below, and combined with (6.17) it implies that for r_0 as described,

$$I(1,\varepsilon) - \int_{U_1} u_\varepsilon(u) \le C\frac{\varepsilon}{r_0} + C{r_0}^2\left(\delta_\star(\frac{\varepsilon}{r_0}) + C(\frac{\varepsilon}{r_0})^2\right) \le C\frac{\varepsilon}{r_0} + C{r_0}^2\delta_\star(\frac{\varepsilon}{r_0}).$$

It follows that

$$I(1,\varepsilon) - \int_{U_1} e_\varepsilon(u) \le \ C\frac{\varepsilon}{r} + Cr^2 \sup_{r_0\in[r,2r]} \delta_\star(\frac{\varepsilon}{r_0})$$

for $C\varepsilon \le r \le 1/4$. This proves (6.16).

Step 3. We next show that the relationship (6.16) implies that there exists some constant C such that $\delta_\star(\varepsilon) \le C\varepsilon$ for all $\varepsilon \in (0,1]$. This will complete the proof of the theorem (modulo Proposition 3.)

First, define $\delta^\star(\varepsilon) = \sup_{s\in[\varepsilon/2,\varepsilon]} \delta_\star(s)^+$. Then (6.16) implies that

$$\delta^\star(\varepsilon) \le C\frac{\varepsilon}{r} + Cr^2\delta^\star(\frac{\varepsilon}{r}).$$

Next, define $f(\varepsilon) := \frac{\delta^\star(\varepsilon)}{\varepsilon}$, so that the above inequality can be rewritten

$$f(\varepsilon) \le \frac{C}{r} + Crf(\frac{\varepsilon}{r}), \qquad C\varepsilon \le r \le 1/4.$$

We fix $r_\star < 1/4$ so small that $Cr_\star < \frac{1}{2}$, so that when $\varepsilon < r_\star/C$,

$$f(\varepsilon) \le \frac{C}{r_\star} + \frac{1}{2}f(\frac{\varepsilon}{r_\star})$$

or equivalently, writing $g(\varepsilon) = f(\varepsilon) - 2\frac{C}{r_\star}$,

$$g(\varepsilon) \le \frac{1}{2}g(\varepsilon/r\star), \qquad 0 < \varepsilon \le r_\star/C.$$

Clearly $\sup_{r_\star/C<\varepsilon\le 1} g(\varepsilon) < \infty$, so we deduce that $\limsup_{\varepsilon\searrow 0} g(\varepsilon) \le 0$. Thus $f(\varepsilon)$ is bounded for all $\varepsilon \in (0,1]$, completing the proof of (6.8). ☐

We now state the proposition used in the above proof:

Proposition 3. *Let* $u \in H^1(U_1;\mathbb{C})$ *satisfy* (6.10) *and*

$$\|J'(u) - \pi\delta_0\|_{\dot W^{-1,1}(U_1)} \le K_1\varepsilon. \tag{6.18}$$

Then there exists $C > 0$ *such that for every* $r \in [C\varepsilon, 1/4]$, *there exists* $r_0 \in [r, 2r]$ *such that*

$$I(1,\varepsilon) - \int_{U_1} e_\varepsilon(u) \ \le \ C\frac{\varepsilon}{r_0} + C{r_0}^2\left(\int_{U_1\setminus U_{r_0}} e_\varepsilon(u)\,dx - \pi\ln\frac{1}{r_0}\right). \tag{6.19}$$

The constant C in the above proposition depends on K_1, which however has been fixed in the proof of Theorem 3.

The following proof is really just an outline of the argument; most of the actual work is done in a series of lemmas, as described below.

proof of Proposition 3. Fix $u \in H^1(U_1;\mathbb{C})$ satisfying (6.10) and (6.18). We first prove in Lemma 10 that there exists K_2 such that for any $r \in [K_2\varepsilon, \frac{1}{2}]$, there exists $r_0 \in [r, 2r]$ such that

$$r_0\int_{\partial B_{r_0}} e_\varepsilon(u) \ \le K_2 \tag{6.20}$$

and

$$B_1^{**} \subset B_{r_0}, \quad \partial B_{r_0} \cap B_j^{**} = \emptyset \text{ for } j = 1, \dots, M \tag{6.21}$$

where $\{B_i^{**}\}_{i=1}^M$ denote the balls constructed in Proposition 1.

In the remainder of the proof we show that (6.19) holds for any r_0 satisfying (6.20) and (6.21). More precisely, we will show that for any such r_0, there exists $\tilde{u} \in H^1(U_1;\mathbb{C})$ such that $\tilde{u} = u$ in B_{r_0}, $\tilde{u} = e^{i(\theta+\gamma)}$ on ∂B_1 for some constant $\gamma \in \mathbb{R}$, and such that $\int_{U_1} e_\varepsilon(\tilde{u}) - e_\varepsilon(u)$ is bounded by the right-hand side of (6.19). From the definition (6.1) of $I(1,\varepsilon)$ it is clear that $\int_{U_1} e_\varepsilon(\tilde{u})\,dx \ \le I(1,\varepsilon)$, so this will complete the proof.

To prove this estimate we proceed as follows. We use the notation $A = U_1 \setminus B_{r_0}$. In the next section we will define harmonic maps $v_D, v_N : A \to S^1$, which we refer to, for reasons that will be obvious, as the Dirichlet and Neumann extensions of $e^{i(\theta+h)}$. We will define $\tilde{u}$ so that $\tilde{u} = u$ in B_{r_0}, $\tilde{u} = e^{i(\theta+const)}$ on ∂U_1 as required, and $\tilde{u}$ is a very small modification of v_D in A. Then

$$\begin{aligned}\int_{U_1} [e_\varepsilon(\tilde{u}) - e_\varepsilon(u)] &= \int_A \left[e_\varepsilon(\tilde{u}) - \frac{1}{2}|\nabla v_D|^2\right] \\ &+ \int_A \frac{1}{2}\left[|\nabla v_D|^2 - |\nabla v_N|^2\right] + \int_A \left[\frac{1}{2}|\nabla v_N|^2 - e_\varepsilon(u)\right].\end{aligned}$$

The three terms on the right-hand side are estimated in Lemmas 12, 11, and 14 respectively, and combining these estimates yields (6.19). Some of these estimates require either that $r_0 \ge C\varepsilon$ or that $r_0 \le 1/2$, and this leads to the condition $C\varepsilon \le r \le 1/4$ in the statement of the theorem. ☐

As described in the above outline, we first prove

Lemma 10. *There exists a constant $K_2 > 0$ such that if $u \in H^1(U_1;\mathbb{C})$ satisfies* (6.10) *and* (6.18) *then for every $r \in (K_2\varepsilon, 1/2]$, there exists $r_0 \in [r, 2r]$ satisfying* (6.21) *and* (6.20) *(with K_2 appearing in the latter.)*

Proof. Fix u satisfying (6.10) and (6.18), and let $\{B_i^{**}\}_{i=1}^M$ be the family of balls constructed in Proposition 1 (with K_0 equal to the fixed constant from (6.10), independent of ε) and as before let ξ denote the center of the distinguished ball B_1^{**}. Let

$$\mathcal{S} := \{s \in (0,1) \ : \ B_1^{**} \subset B_s, \ \partial B_s \cap B_i^{**} = \emptyset \ \forall i\}.$$

If $r_0 \in \mathcal{S}$, then (6.21) is satisfied. Recall from (6.12) that $\|J'(u) - \pi\delta_\xi\|_{\dot{W}^{-1,1}(U_1)} \le K_1\varepsilon$. As a result, (6.18) with the triangle inequality and the definition (2.3) of the $\dot{W}^{-1,1}$ norm imply that $|\xi| \le 2K_1\varepsilon$. It follows from this and from (4.7) that there exists K_2 such that if $r \ge K_2\varepsilon$, then

$$|\mathcal{S} \cap (r, 2r)| \ge \frac{r}{2}. \tag{6.22}$$

Next, in view of (4.2), we deduce from (6.18) and (6.10) that there exists a C such that

$$\int_{B_{r_2}\setminus B_{r_1}} e_\varepsilon(u) \ \geq \pi \ln \frac{r_2}{r_1} - C \tag{6.23}$$

for all $0 \leq r_1 < r_2 \leq 1$ and every $\varepsilon \in [0,1]$. It then follows from (6.10) that

$$\int_{B_{s_2}\setminus B_{s_1}} e_\varepsilon(u) \leq C + \pi \ln \frac{s_2}{s_1} \tag{6.24}$$

for $0 \leq s_1 < s_2 \leq 1$. In particular,

$$C \ \geq \ \int_{B_{2r}\setminus B_r} e_\varepsilon(u) \geq \int_{\mathcal{S}\cap(r,2r)} \left(\int_{\partial B_s} e_\varepsilon(u) d\mathcal{H}^1 \right) ds.$$

From this and (6.22) it easily follows that after taking K_2 larger if necessary,

$$\inf \left\{ s \int_{\partial B_s} e_\varepsilon(u) d\mathcal{H}^1 \ : s \in \mathcal{S} \cap (r, 2r) \right\} \ \leq K_2$$

for all $r \in [K_2\varepsilon, 1/2]$. □

6.1. **Dirichlet and Neumann S^1-valued extensions.** We now introduce some auxiliary functions that will be used in the construction of $\tilde{u}$ and associated estimates. Throughout this section we work with a fixed function u satisfying (6.10) and (6.18), and a fixed number r_0 satisfying (6.20), (6.21).

First, we define a real-valued function $h \in H^1(\partial B_{r_0}; \mathbb{R})$ by requiring that

$$u \ = \ |u| e^{i(\theta+h)} \qquad \text{on } \partial B_{r_0}. \tag{6.25}$$

This is possible as a result of (6.21), which also implies that $|u| > 1/2$ on ∂B_{r_0}. For constant $\gamma \in \mathbb{R}$, $e_\varepsilon(u) = e_\varepsilon(e^{i\gamma}u)$ and $J(u) = J(e^{i\gamma}u)$, so by replacing u by $e^{i\gamma}u$ for a suitable γ, we may assume that

$$\int_{\partial B_{r_0}} h \ = \ 0. \tag{6.26}$$

Note also that it is easy to check from (6.20) and (6.21) that

$$r_0 \int_{\partial B_{r_0}} |\nabla_\tau h|^2 \leq C(K_2). \tag{6.27}$$

For the duration of the proof we use the notation

$$A := U_1 \setminus B_{r_0}.$$

Let $H^1_{u/|u|}(A; S^1) := \{v \in H^1(A; \mathbb{C}) \ : \ |v| = 1 \ \ a.e, \ v = \frac{u}{|u|} \text{ on } \partial B_{r_0}\}$, and let v_N be the unique minimizer of the Dirichlet energy in $H^1_{u/|u|}(A; S^1)$. Similarly[4], let v_D denote the unique minimizer of the Dirichlet energy in the set of functions

---

[4] The subscripts $_N$ and $_D$ stand for Neumann and Dirichlet respectively, and refer to the boundary condition on ∂U_1.

$\{v \in H^1_{u/|u|}(A; S^1) : v = e^{i\theta}$ on $\partial B_1\}$. Note that v_N and v_D can be written in the form $e^{i(\theta+\phi_N)}$ and $e^{i(\theta+\phi_D)}$ respectively, where ϕ_N, ϕ_D are real-valued harmonic functions equalling h on ∂B_{r_0}, and with

$$\nu \cdot \nabla\phi_N = 0, \quad \text{on } \partial B_1, \qquad\qquad \phi_D = 0 \quad \text{on } \partial B_1.$$

The crucial factor of ${r_0}^2$ on the right-hand side of (6.19) appears in the next lemma, which is a straightforward explicit calculation:

Lemma 11. *If* $r_0 \le 1/2$*, then*

$$\int_A \frac{1}{2}\left[|\nabla v_D|^2 - |\nabla v_N|^2\right] \le 4{r_0}^2 \left(\int_A \frac{1}{2}|\nabla v_N|^2\,dx - \pi \ln\frac{1}{r_0}\right). \tag{6.28}$$

Proof. One easily checks that

$$\int_A \frac{1}{2}|\nabla v_D|^2 = \pi\ln\frac{1}{r_0} + \int_A \frac{1}{2}|\nabla\phi_D|^2, \qquad \int_A \frac{1}{2}|\nabla v_N|^2 = \pi\ln\frac{1}{r_0} + \int_A \frac{1}{2}|\nabla\phi_N|^2 \tag{6.29}$$

so we need to estimate $\frac{1}{2}\int_A(|\nabla\phi_D|^2 - |\nabla\phi_N|^2)$. We can write down explicit formulas for ϕ_N, ϕ_D in terms of the Dirichlet data h on ∂B_{r_0}. We will write h, ϕ_N, ϕ_D as maps $\mathbb{C} \to \mathbb{C} \supset \mathbb{R}$ in what follows. We represent h in the form

$$h(r_0 e^{i\theta}) = \sum \alpha_n e^{in\theta}. \tag{6.30}$$

Then one can check that

$$\phi_N(se^{i\theta}) = \sum_n \alpha_n \frac{(s^n + s^{-n})}{({r_0}^n + {r_0}^{-n})} e^{in\theta}, \tag{6.31}$$

$$\phi_D(se^{i\theta}) = \sum_n \alpha_n \frac{(s^n - s^{-n})}{({r_0}^n - {r_0}^{-n})} e^{in\theta}. \tag{6.32}$$

To verify these, one can simply check directly that ϕ_N, ϕ_D as defined are harmonic and satisfy the required boundary conditions when $s = r_0$ or $s = 1$. (Note also, condition (6.26) implies that $\alpha_0 = 0$, so that ϕ_D is well-defined.) Then using the fact that ϕ_N is harmonic and the boundary condition on ∂B_1, we compute

$$\begin{aligned} \int_A \frac{1}{2}|\nabla\phi_N|^2 = -\int_{\partial B_{r_0}} \frac{1}{2}\phi_N\ \partial_s\phi_N d\mathcal{H}^1 \ &= \ \pi\sum_n n|\alpha_n|^2 \frac{({r_0}^{-n} - {r_0}^n)}{({r_0}^{-n} + {r_0}^n)} \\ &= \ \pi\sum_n |n|\,|\alpha_n|^2 \left(1 - \frac{2{r_0}^{|n|}}{{r_0}^{-|n|} + {r_0}^{|n|}}\right). \end{aligned} \tag{6.33}$$

Similarly,

$$\int_A \frac{1}{2}|\nabla\phi_D|^2 \ = \ \pi\sum_n |n|\,|\alpha_n|^2 \left(1 + \frac{2{r_0}^{|n|}}{{r_0}^{-|n|} - {r_0}^{|n|}}\right) \tag{6.34}$$

and so

$$\int_A \frac{1}{2}(|\nabla\phi_D|^2 - |\nabla\phi_N|^2)dx \;=\; \pi\sum_n |n||\alpha_n|^2 \frac{4}{{r_0}^{-2|n|} - {r_0}^{2|n|}}$$

Note that $\frac{|n|}{{r_0}^{-2|n|}-{r_0}^{2|n|}} \le |n|\frac{{r_0}^2}{1-{r_0}^4} \le 2|n|{r_0}^2$ for every n, since $r_0 \le 1/2$. Thus

$$\int_A \frac{1}{2}(|\nabla\phi_D|^2 - |\nabla\phi_N|^2)dx \;\le\; 2{r_0}^2\sum_n |n|\,|\alpha_n|^2, \tag{6.35}$$

and so (6.28) is a consequence of (6.35), (6.33), and (6.29). □

For future reference we record the fact that, using the above notation,

$$\frac{1}{2}\int_{\partial B_{r_0}} |\tau\cdot\nabla h|^2 \;=\; \frac{\pi}{r_0}\sum_n |n|^2\,|\alpha_n|^2 \;\ge\; \frac{\pi}{r_0}\sum_n |n|\,|\alpha_n|^2. \tag{6.36}$$

This is easy to check, using the fact that $\tau\cdot\nabla h = \frac{1}{r_0}\partial_\theta h$.

The next lemma shows that we can find an extension $\tilde u$ whose energy on A is very close to that of v_D.

Lemma 12. *If r_0 satisfies* (6.20) *and $r_0 \le 1/2$, then there exists $\tilde u \in H^1(U_1;\mathbb{C})$ such that $\tilde u = u$ in B_{r_0} and*

$$\int_A \left[e_\varepsilon(\tilde u) - \frac{1}{2}|\nabla v_D|^2 \right] \;\le\; C(K_2)\frac{\varepsilon}{r_0}. \tag{6.37}$$

Proof. It suffices to show that we can construct $\tilde u$ on A such that (6.37) is satisfied and $\tilde u = |u|e^{i(\theta+h)}$ on ∂B_{r_0}.

We start by defining $\tilde u$ on the narrow annulus $U_{r_0+\varepsilon}\setminus B_{r_0}$. In this set we define

$$\tilde u(se^{i\theta}) \;=\; \tilde u(r_0e^{i\theta})\left(1 + \frac{s-r_0}{\varepsilon}\Big(\frac{1}{|u|(r_0e^{i\theta})} - 1\Big)\right)$$

so that $\tilde u = e^{i(\theta+h)}$ on $\partial B_{r_0+\varepsilon}$. Since $|u| \ge 1/2$ on ∂B_{r_0} and $|\partial_s\tilde u(se^{i\theta})| = \frac{|1-|u|(r_0e^{i\theta})|}{\varepsilon} \le \frac{|1-|u|^2(r_0e^{i\theta})|}{\varepsilon}$, it is easy to verify that $e_\varepsilon(\tilde u)(se^{i\theta}) \le Ce_\varepsilon(u)(r_0e^{i\theta})$ for $r_0 \le s \le r_0+\varepsilon$, and from this and (6.20) one easily checks that

$$\int_{r_0}^{r_0+\varepsilon}\int_{\partial B_s} e_\varepsilon(\tilde u)d\mathcal{H}^1(x)\; s\; ds \le C(K_2)\frac{\varepsilon}{r_0}. \tag{6.38}$$

In $B_1\setminus B_{r_0+\varepsilon}$ we set $\tilde u$ equal to the S^1-valued Dirichlet extension (on this slightly smaller annulus) of $e^{i(\theta+h)}$. It then follows from (6.38), (6.29) and (6.34) that

$$\int_A e_\varepsilon(\tilde u) - \frac{1}{2}|\nabla v_D|^2\; dx \le C\frac{\varepsilon}{r_0} + \pi\ln(\frac{r_0+\varepsilon}{r_0}) + g(r_0+\varepsilon) - g(r_0) \tag{6.39}$$

where $g(r) := \pi \sum_n |n|\,|\alpha_n|^2 \frac{(r^{-|n|}+r^{|n|})}{(r^{-|n|}-r^{|n|})} = \sum |n|\,|\alpha_n|^2 (1+\frac{2}{(r^{-2|n|}-1)})$. By calculus, since $r_0 \le 1/2$,

$$(1+\frac{2}{(r_0+\varepsilon)^{-2|n|}-1}) - (1+\frac{2}{{r_0}^{-2|n|}-1}) \le C|n|{r_0}^{2|n|}\frac{\varepsilon}{r_0},$$

and with (6.36) this implies that $g(r_0+\varepsilon) - g(r_0) \le C\frac{\varepsilon}{r_0}(r_0 \int_{\partial B_{r_0}} |\tau\cdot\nabla h|^2)$. The conclusion of the lemma now follows from (6.39) and (6.20). ☐

It remains to prove a lower bound for $\int_A e_\varepsilon(u)$ in terms of $\int_A |\nabla v_N|^2$. To do this we will need the following

Lemma 13. *For $p < 2 < 4$ there exists C (depending on p, K_2) such that the Neumann extension v_N satisfies*

$$\left(\int_A |\nabla v_N|^p\right)^{1/p} \le C{r_0}^{\frac{2}{p}-1}. \tag{6.40}$$

The proof shows that $C(p,K_2) \le C(\frac{1}{p-2} + \frac{1}{p-4})\sqrt{K_2}$. The bad behavior of $C(p,K_2)$ near $p=2$ is an artifact of the proof, whereas the behavior near $p=4$ probably is not.

Proof. **Step 1**: Writing $v_N = e^{i(\theta+\phi_N)}$ as above, it is clear that $|\nabla v_N|^p \le C_p(|\nabla\theta|^p + |\nabla\phi_N|^p)$. Since

$$\int_A |\nabla\theta|^p = \int_{r_0}^1 \int_{-\pi}^{\pi} r^{-p} d\theta\, r\, dr \le C{r_0}^{2-p}$$

we only need to estimate $\|\nabla\phi_N\|_p$. To do this we represent ϕ_N as in (6.31), and we split $|\nabla\phi_N|^p$ into radial and angular pieces:

$$|\nabla\phi_N|^p(se^{i\theta}) \le C_p(|\partial_s\,\phi_N|^p + |\frac{1}{s}\partial_\theta\phi_N|^p).$$

We estimate $\|\partial_s\phi_N\|_p^p$ first. From (6.31),

$$\partial_s\phi_N(se^{i\theta}) \;=\; \frac{1}{s}\sum_n n\alpha_n \frac{(s^n - s^{-n})}{({r_0}^n + {r_0}^{-n})}e^{in\theta} \;=\; \frac{1}{s}\sum_n |n|\alpha_n \frac{(s^{|n|} - s^{-|n|})}{({r_0}^{|n|} + {r_0}^{-|n|})}e^{in\theta}.$$

Thus $\partial_s\phi_N = \zeta^+ - \zeta^-$, where

$$\zeta^\pm = \frac{1}{s}\sum_n |n|\alpha_n \frac{s^{\pm|n|}}{{r_0}^{|n|} + {r_0}^{-|n|}}e^{in\theta}.$$

Step 2: In this step we prove that $\|\zeta^-\|_{L^p(A)} \le C_p{r_0}^{\frac{2}{p}-1}\left(r_0\int_{\partial B_{r_0}} |\tau\cdot\nabla h|^2\right)^{1/2}$. Note that

$$\zeta^-(se^{i\theta}) = \frac{1}{s}\sum_n \left(\frac{r_0}{s}\right)^n |n|\alpha_n \frac{1}{1+{r_0}^{2|n|}}e^{in\theta} = \frac{1}{s}(P_{r_0/s} * \gamma^-)(e^{i\theta})$$

where

$$\gamma^-(e^{i\theta}) = \sum_n |n|\alpha_n \frac{1}{1+{r_0}^{2|n|}} e^{in\theta}$$

and $P_\sigma(e^{i\theta})$ denotes the classical Poisson kernel,

$$P_\sigma(e^{i\theta}) = \sum_n \sigma^{|n|} e^{in\theta} \;=\; \frac{1-\sigma^2}{1-2\sigma\cos\theta+\sigma^2}, \qquad \text{for } 0 \le \sigma < 1.$$

The last equality follows by an easy calculation, or can be found in nearly any introductory harmonic analysis text. It is easy to see that $\|P_\sigma\|_{L^\infty(\mathbb{T})} = \frac{1+\sigma}{1-\sigma}$, so

$$\|P_\sigma\|_{L^q(\mathbb{T})} := \left(\int_{-\pi}^{\pi} |P_\sigma(e^{i\theta})|^q d\theta\right)^{1/q} \;\le\; \|P_\sigma\|_{L^1(\mathbb{T})}^{\frac{1}{q}} \, \|P_\sigma\|_{L^\infty(\mathbb{T})}^{1-\frac{1}{q}} \le C(1-\sigma)^{\frac{1}{q}-1} \tag{6.41}$$

for all $q \ge 1$ and $\sigma \in [0,1)$. Now we are in a position to estimate

$$\begin{aligned}
\|\zeta^-\|_{L^p(A)}^p &= \int_{r_0}^1 \int_{-\pi}^{\pi} s^{-p} |P_{r_0/s} * \gamma^-(e^{i\theta})|^p d\theta \; s\, ds \\
&= \int_{r_0}^1 s^{1-p} \|P_{r_0/s} * \gamma^-\|_{L^p(\mathbb{T})}^p \, ds \\
&\le \int_{r_0}^1 s^{1-p} \|P_{r_0/s}\|_{L^q(\mathbb{T})}^p \|\gamma^-\|_{L^2(\mathbb{T})}^p \, ds \qquad (6.42)
\end{aligned}$$

for $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{2}$, using Young's inequality for convolutions. From Parseval's identity, the definition of γ^-, and (6.36) it follows that

$$\|\gamma^-\|_{L^2(\mathbb{T})}^2 \le 2\pi \sum |n|^2 \, |\alpha_n|^2 \;=\; r_0 \int_{\partial B_{r_0}} |\tau \cdot \nabla h|^2.$$

Substituting this and (6.41) into (6.42), we get

$$\|\zeta^-\|_{L^p(A)} \le C \left(r_0 \int_{\partial B_{r_0}} |\tau \cdot \nabla h|^2 \right)^{1/2} \left(\int_{r_0}^1 s^{1-p} \left(1 - \frac{r_0}{s}\right)^{1-\frac{p}{2}} \, ds \right)^{1/p}.$$

Since

$$s^{1-p}\left(\frac{s-r_0}{s}\right)^{1-\frac{p}{2}} \le \begin{cases} Cs^{1-p} & \text{if } s \ge 2r_0, \\ C{r_0}^{-p/2}(s-r_0)^{1-\frac{p}{2}} & \text{if } r_0 \le s \le 2r_0. \end{cases}$$

one easily checks that

$$\left(\int_{r_0}^1 s^{1-p} \left(1 - \frac{r_0}{s}\right)^{1-\frac{p}{2}} \, ds \right)^{1/p} \le C_p {r_0}^{\frac{2}{p}-1},$$

and this completes Step 2. (We can write C_p more precisely as $C(\frac{1}{p-2} + \frac{1}{p-4})$, for C independent of p.)

Step 3. To estimate $\|\zeta^+\|_{L^p(A)}$ we write $\zeta^+ = \frac{1}{s}P_s * \gamma^+$, where

$$\gamma^+(e^{i\theta}) = \sum \frac{|n|\alpha_n}{{r_0}^{|n|} + {r_0}^{-|n|}} e^{in\theta}.$$

Then arguing as above we find that

$$\|\zeta^-\|_{L^p(A)} \le C \left(r_0 \int_{\partial B_{r_0}} |\tau \cdot \nabla h|^2 \right)^{1/2} \left(\int_{r_0}^1 s^{1-p} (1-s)^{1-\frac{p}{2}} \, ds \right)^{1/p},$$

and this leads to the estimate $\|\zeta^-\|_{L^p(A)} \le C_p {r_0}^{\frac{2}{p}-1} \left(r_0 \int_{\partial B_{r_0}} |\tau \cdot \nabla h|^2 \right)^{1/2}$.

Step 4. So far we have shown that

$$\|\partial_s \phi_N\|_{L^p(\mathbb{T})} \le C_p {r_0}^{\frac{2}{p}-1} \left(r_0 \int_{\partial B_{r_0}} |\tau \cdot \nabla h|^2 \right)^{1/2}.$$

It remains to prove the same estimate for $\|\frac{1}{s}\partial_\theta \phi_N\|_{L^p(\mathbb{T})}$. This can be accomplished by writing $\frac{1}{s}\partial_\theta \phi_N = \tilde{\zeta}^+ + \tilde{\zeta}^-$, with

$$\tilde{\zeta}^\pm(se^{i\theta}) = \frac{1}{s} \sum n\alpha_n \frac{s^{\pm|n|}}{{r_0}^{|n|} + {r_0}^{-|n|}} i e^{in\theta}.$$

Then $\|\tilde{\zeta}^-\|_{L^p(A)}$ and $\|\tilde{\zeta}^-\|_{L^p(A)}$ can be estimated by exactly the arguments of Step 2 and Step 3 respectively. □

6.2. lower bound on annulus: energy is close to Neumann extension.

In this section we continue to assume that u satisfies (6.10), (6.18), and that r_0 satisfies (6.20), (6.21), and we define v_N and so on as in the previous section. We will complete the proof of Proposition 3, and hence of Theorem 3, by proving

Lemma 14. *There exists a constant C if $r_0 \in (C\varepsilon, 1/2]$ satisfies the hypotheses of Proposition 3, then*

$$\int_A \left[\frac{1}{2}|\nabla v_N|^2 \; - e_\varepsilon(u) \right] dx \;\le\; C\frac{\varepsilon}{r_0}.$$

This is the most delicate of the estimates needed in the proof of Proposition 3, and employs the full machinery of the Jacobian estimates developed earlier in this paper, as well as the L^p estimates of ∇v_N from the previous section.

Proof. Throughout the proof of the lemma we use the notation

$$w = u/v_N.$$

Step 1. We first claim that

$$\frac{1}{2}|\nabla v_N|^2 - e_\varepsilon(u) = -\, e_\varepsilon(w) - jv_N \cdot j(w) + \frac{1}{2}|\nabla v_N|^2(1 - |w|^2). \tag{6.43}$$

This is most efficiently verified by using the fact that for any complex-valued H^1 function ζ, $|\nabla\zeta|^2 = |\nabla|\zeta||^2 + \frac{|j\zeta|^2}{|\zeta|^2}$. (This is easily checked by writing ζ locally as $\rho e^{i\phi}$ and $j\zeta$ as $\rho^2\nabla\phi$.) Then

$$|\nabla u|^2 = |\nabla(wv_N)|^2 = |\nabla|w||^2 + \frac{|j(wv_N)|^2}{|w|^2}$$

and $j(wv_N) = |v_N|^2 j(w) + |w|^2 jv_N = j(w) + |w|^2 jv_N$. The claim then follows by rewriting $e_\varepsilon(u)$ in terms of v_N and w.

Step 2: In this step we prove that there exists C such that

$$\left|\int_A jv_N \cdot j(w)\right| \;\le\; \frac{1}{2}\int_A e_\varepsilon(w) \qquad \text{whenever } r_0 \ge C\varepsilon, \tag{6.44}$$

with C depending only on K_2 and on the uniform bound for $\int_{U_1} e_\varepsilon(u) - \pi \ln\frac{1}{\varepsilon}$ implicit in the assumption (6.10). To prove (6.44), let $\psi_N : A \to \mathbb{R}$ satisfy

$$\nabla \times \psi_N = jv_N, \qquad \psi_N = 0 \text{ on } \partial U_1.$$

To verify that such a function exists, we must check that $\nabla\cdot jv_N = 0$, which is obvious in view of the fact that v_N is harmonic, and that

$$\int_{\partial B_s} jv_N \cdot \nu = 0$$

for some $s \in (r_0, 1)$ (and hence for every such s). The latter follows by using the Neumann condition for v_N on ∂U_1:

$$0 \;=\; \int_{U_1\setminus B_s} \nabla\cdot jv_N \;=\; \int_{\partial U_1} jv_N\cdot\nu - \int_{\partial B_s} jv_N\cdot\nu \;=\; -\int_{\partial B_s} jv_N\cdot\nu.$$

Hence ψ_N exists, and so we can compute

$$\int_A jv_N\cdot j(w) \;=\; \int_A \nabla\times\psi_N\cdot j(w) \;=\; 2\int_A \psi_N J(w).$$

Both boundary terms vanish in the above integration by parts[5] because $\psi_N = 0$ on ∂U_1, and on ∂B_{r_0}, the definitions of v_N, w imply that w is real-valued, and hence that $j(w)\cdot\tau = 0$. We control the integal of the Jacobian as follows: Let $\Gamma = \partial U_1 \subset \partial A$, and define $W^{1,p}_\Gamma$ and $\|\cdot\|_{\dot W^{-1,q}_\Gamma(A)}$ as in (2.1) and (2.2) respectively. Clearly $|\nabla\psi_N| \equiv |\nabla v_N|$, and so $\psi_N \in W^{1,p}_\Gamma(A)$ for $2 < p < 4$, according to Lemma 13. For $p\in(2,4)$ and $\frac{1}{q} = 1 - \frac{1}{p}$,

$$\left|\int_A \psi_N J(w)\,\right| \le \|\nabla\psi_N\|_{L^p(A)}\,\|J(w)\|_{\dot W^{-1,q}_\Gamma(A)}.$$

---

[5] If the same calculation is carried out with v_D instead of v_N, one is left with nonvanishing terms on ∂U_1, and these are difficult to control directly.

We prove in Lemma 15 below that $\|J(w)\|_{\dot{W}_\Gamma^{-1,q}(A)} \le C\varepsilon^{1-\frac{2}{p}}\int_A e_\varepsilon(w)$. Setting $p = 3$ and combining this with (6.20), and (6.40), we obtain

$$\left|\int_A \psi_N J(w)\ \right| \ \le\ C\left(\frac{\varepsilon}{r_0}\right)^{\frac{1}{3}}\int_A e_\varepsilon(w),$$

which easily implies (6.44).

Step 3: In this step we show that

$$\int_A |\nabla v|^2(1-|w|^2)\ \le\ C(p,K_2)\left(\frac{\varepsilon}{r_0}\right)^{p-2} + \frac{1}{2}\int_A e_\varepsilon(w). \tag{6.45}$$

In view of (6.44) and (6.43), this inequality with $p = 3$, say, will complete the proof of the lemma. To prove (6.45), first note that by Young's inequality

$$|\nabla v|^2(1-|w|^2) \le |\nabla v|^2(1-|w|^2)^+ \le C\varepsilon^{p-2}|\nabla v|^p + \frac{[(1-|w|^2)^+]^{p/(p-2)}}{8\varepsilon^2}.$$

Clearly $0 \le (1-|w|^2)^+ \le 1$, and $\frac{p-2}{2} > 2$ for $p \in (2,4)$, so

$$\frac{[(1-|w|^2)^+]^{p/(p-2)}}{8\varepsilon^2} \le \frac{1}{2}e_\varepsilon(w).$$

Thus (6.45) follows by integrating over A and again using Lemma 13. $\square$

It remains to prove the Jacobian estimate for w used above:

Lemma 15. *Assume that* $u \in H^1(U_1;\mathbb{C})$ *satisfies* (6.10) *and* (6.18)*, and define* $w = u/v_N$ *as above, for* r_0 *such that* (6.21), (6.20) *hold. Then for any* $1 \le q < 2$*,*

$$\|J(w)\|_{\dot{W}_\Gamma^{-1,q}(A)} \ \le C\varepsilon^{\frac{2}{q}-1}\int_A e_\varepsilon(w). \tag{6.46}$$

Here C depends on uniform constants implicit in assumption (6.10).

Proof. **Step 1**. First we claim that

$$\|J(w)\|_{\dot{W}_\Gamma^{-1,1}(A)} \ \le C\varepsilon\int_A e_\varepsilon(w)\,dx. \tag{6.47}$$

Recall that $\{B_i^{**}\}_{i=1}^M$ are balls satisfying the conclusions of Proposition 1 for some fixed, a priori bounded number K_0 determined by (6.10), By assumption (6.21), ∂B_{r_0} does not intersect any of these balls. The same assumption implies that the distinguished ball B_1^{**} of degree 1 is a subset of B_{r_0}. If we throw out the balls that are contained in B_{r_0} and relabel the remaining balls, we arrive at a collection, still denoted $\{B_i^{**}\}_{i=1}^M$, that in view of the definition of w satisfies $\{x \in A : |w(x)| \le 1/2\} \subset \cup B_i^{**}$,

$$\deg\,(w;\partial B_i^{**}) = 0 \text{ for all } i \text{ such that } B_i^{**} \cap \partial U_1\ = \emptyset, \tag{6.48}$$

and $\sum r_i^{**} \le C\varepsilon$. As in Step 1 of the proof of Theorem 2$'$, we can invoke (4.28) to conclude from the bound on $\sum r_i^{**}$ that

$$\Big\| J'w - \sum_{\{j : B_j^{**} \subset A\}} a_j \delta_{y_j} \Big\|_{\dot W_\Gamma^{-1,1}(A)} \le C\varepsilon \left(\int_A e_\varepsilon(w)\, dx \right)$$

where $J'w$ denotes the modified Jacobian, y_j is the center of B_j^{**}, and $a_j = \int_{B_j^{**}} J'w$. As in the earlier proof, it follows from (6.48) that $a_j = 0$ for all j appearing in the sum, so that the left-hand side of the above inequality reduces to $\|J'w\|_{\dot W_\Gamma^{-1,1}(A)}$. Then (6.47) follows from Lemma 1.

Step 2. We will need to use the interpolation inequality

$$\|J(w)\|^*_{C^{0,\alpha}_\Gamma(A)} \le C \|J(w)\|^\alpha_{C^{0,1}_\Gamma(A)^*} \, \|J(w)\|^{1-\alpha}_{C^0_\Gamma(A)^*} \tag{6.49}$$

where $\|\mu\|_{C^{0,\alpha}_\Gamma(A)^*} := \sup\{\int \phi\, d\mu \ : \ \phi = 0 \text{ on } \Gamma, [\phi]_{C^{0,\alpha}} \le 1\}$ for $\alpha \in (0,1]$, and $\|\mu\|_{C^0_\Gamma(A)^*} = \int_A |\mu|$. (Note that $\|\mu\|_{C^{0,1}_\Gamma(A)^*} = \|\mu\|_{\dot W_\Gamma^{-1,1}(A)}$ for all μ.) A very similar estimate is proved in Lemma 3.3 of [7], with Γ replaced by the whole boundary ∂A, and the proof given there, with very small modifications, establishes (6.49).

The Sobolev embedding theorem implies that for $1 < q < 2$ and $\alpha = \frac{2}{q} - 1$,

$$\|J(w)\|_{\dot W_\Gamma^{-1,q}(A)} \le C \|J(w)\|_{C^{0,\alpha}_\Gamma(A)^*} \tag{6.50}$$

It is obvious that $\|J(w)\|_{C^0_\Gamma(A)^*} \le C \|e_\varepsilon(w)\|_{L^1(A)}$. Combining this with (6.47), (6.50), and (6.49), we conclude (6.46). ☐

Finally, we prove the lemma used above, characterizing the rate of convergence of $I(r,\varepsilon) - \pi \ln(r/\varepsilon)$ as $\frac{r}{\varepsilon} \to \infty$.

Lemma 16. $\left|\gamma - (I(r,\varepsilon) - \pi \ln \frac{r}{\varepsilon})\right| \le C(\frac{\varepsilon}{r})^{-2}$.

(The notation $\gamma, I(r,\varepsilon)$ is introduced at the beginning of this section.)

Proof. Since $I(r,\varepsilon) = I(r/\varepsilon, 1)$, it suffices to prove the lemma for $\varepsilon = 1$. We do this, using the notation $I(r) = I(r,1)$, and writing $e(v)$ instead of $e_1(v) = \frac{1}{2}|\nabla v|^2 + \frac{1}{4}(|v|^2 - 1)^2$.

It is known that there is a unique nonzero function $S : [0,\infty) \to [0,1)$ such that $v(se^{i\theta}) = S(s)e^{i\theta}$ solves

$$-\Delta v + (|v|^2 - 1)v = 0$$

in $\mathbb{R}^2$. Moreover, Shafrir [13] proves that S satisfies

$$S(s) \ge 1 - c\frac{1}{s^2} \tag{6.51}$$

$$|\partial_r S(s)| \le c\frac{1}{s^3}. \tag{6.52}$$

Finally, by a comparison argument and arguments from [13], it can be shown that $v = Se^{-i\theta}$ satisfies

$$\int_{B_r} e(v)\, dx \le \int_{B_r} e(u)\, dx$$

for any $u \in H^1(U_r, \mathbb{C})$ such that $u = v$ on ∂B_r. We claim that

$$|I(r) - \int_{B_r} e(v)| \ \le\ Cr^{-2}. \tag{6.53}$$

To prove this, fix $r > 0$ and define $\tilde{S} : [0, r] \to [0, 1]$ by

$$\tilde{S}(s) = \begin{cases} S(s) & \text{for } 0 < s \le r-1, \\ (r-s)S(r-1) + (s-r+1) & \text{for } r-1 \le s \le r, \end{cases}$$

and define also $\tilde{v}(se^{i\theta}) = \tilde{S}(s)e^{i\theta}$. Then it is easy to check that

$$I(r) \le \int_{B_r} e(\tilde{v})\, dx \ \le\ \int_{B_r} e(v)\, dx + Cr^{-2}.$$

The opposite inequality is proved by similarly considering a minimizer for the variational problem implicit in the definition of $I(r)$, and modifying it to obtain a function that equals $S(r)e^{i\theta}$ on ∂B_r, then using the fact that v minimizes the energy for its own boundary data. Thus (6.53) is established.

We complete the proof by showing that

$$\left| \int_{B_r} e(v)\, dx - (\gamma + \pi \ln \frac{1}{r}) \right| \ \le\ Cr^{-2}. \tag{6.54}$$

In fact, it follows from the definition of γ and from (6.53) that

$$\begin{aligned} \int_{B_r} e(v)\, dx - \pi \ln \frac{1}{r} - \gamma &= \lim_{s \to \infty} [\int_{B_r} e(v)\, dx - \pi \ln \frac{1}{r} \ - \int_{B_s} e(v)\, dx - \pi \ln \frac{1}{s}] \\ &= \int_r^\infty \left| \frac{(S')^2}{2} + \frac{S^2 - 1}{2s^2} + \frac{(1-S^2)^2}{4} \right| 2\pi s\ ds. \end{aligned}$$

The claim (6.54) then follows from Shafrir's asymptotics (6.51), (6.52) for S. ☐

7. EXAMPLES

In this section we present a few examples that show that the bounds proven in Theorems 1 and 2 are close to sharp.

The first example demonstrates that the $\varepsilon |\ln \varepsilon|^{1/2}$ scaling in conclusion (1.8) of Theorem 2 cannot be improved. Thus, in order to obtain the sharp $O(\varepsilon)$ scaling in (1.9), it is necessary to introduce the modified Jacobians.

Lemma 17. *There exists a sequence* $\{u^\varepsilon\}_{\varepsilon\in(0,1]} \subset H^1(B_1;\mathbb{C})$ *and a constant* $C > 0$ *such that* (1.7) *and* (1.6) *are satisfied for* u *for every* ε, *and*

$$\|J(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(B_r)} \geq C\varepsilon|\ln\varepsilon|^{1/2}, \tag{7.1}$$

$$\|J'(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(B_r)} \geq C\varepsilon, \tag{7.2}$$

for every $\xi \in B_1$.

Proof. Let $u := f_\varepsilon(|x|)\frac{x}{|x|}$, where

$$f_\varepsilon(s) := \begin{cases} s/\varepsilon & \text{if } s \leq s_\varepsilon := \frac{\varepsilon}{2} + \frac{1}{2}\sqrt{\varepsilon^2(1 - 4|\ln\varepsilon|^{-1/2}} \approx \varepsilon, \\ 1 - \frac{\varepsilon}{\sqrt{|\ln\varepsilon|}s} & \text{if } s \geq s_\varepsilon. \end{cases}$$

Using the fact that $e_\varepsilon(u) = \frac{1}{2}(f_\varepsilon')^2 + \frac{1}{2r^2}f_\varepsilon^2 + \frac{1}{4\varepsilon^2}(f_\varepsilon^2 - 1)^2$ for u of the above form, it is easy to check that (1.6) is satisfied,

By the symmetry of u it is clear that $\|J(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(B_r)} = \|J(u) - \pi\delta_{-\xi}\|_{\dot W^{-1,1}(B_r)}$ for every $\xi \in B_1$, so the triangle inequality implies that

$$\begin{aligned} \|J(u) - \pi\delta_0\|_{\dot W^{-1,1}(B_r)} &\leq \frac{1}{2}\|J(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(B_r)} + \frac{1}{2}\|J(u) - \pi\delta_{-\xi}\|_{\dot W^{-1,1}(B_r)} \\ &\leq \|J(u) - \pi\delta_\xi\|_{\dot W^{-1,1}(B_r)}. \end{aligned}$$

So to verify both (1.7) and (7.1), it suffices to show that $\|J(u) - \pi\delta_0\|_{\dot W^{-1,1}(B_r)} \approx \varepsilon|\ln\varepsilon|^{1/2}$ To do this, we compute that $J(u)(x) = \frac{1}{|x|}f_\varepsilon(|x|)f_\varepsilon'(|x|)$. From this or otherwise, one checks that $\int_{B_1} J(u) = \pi(1 - o(\varepsilon))$ Thus for $\phi \in W_0^{1,\infty}(B_1)$ such that $\|\nabla\phi\|_{L^\infty} \leq 1$,

$$\begin{aligned} \int \phi(J(u) - \pi\delta_0) &= \int (\phi(x) - \phi(0))J(u)(x)\ dx + o(\varepsilon) \\ &\leq \int |x|J(u)(x)\ dx + o(\varepsilon) \\ &= 2\pi\varepsilon\sqrt{\ln\varepsilon} + o(\varepsilon) \end{aligned} \tag{7.3}$$

as $\varepsilon \to 0$, by an explicit calculation. Thus $\|J(u) - \pi\delta_0\|_{\dot W^{-1,1}(B_r)} \leq C\varepsilon\sqrt{\ln\varepsilon}$.

On the other hand,

$$\sup_{\phi\in W_0^{1,\infty}(B_1), \|\nabla\phi\|_{L^\infty}\leq 1} \int_{B_1} \phi(J(u) - \pi\delta_0) \geq \left|\int_{B_1} (1 - |x|)(J(u) - \pi\delta_0)\right|$$

and by (7.3), the right-hand side equals $2\pi\varepsilon\sqrt{\ln\varepsilon} + o(\varepsilon)$. Thus (7.1) is verified.

Similar arguments can be used to verify that (7.2) holds as well. ☐

The following lemma illustrates the sense in which Theorem 1 is close to sharp.

Lemma 18. *Suppose* $\Omega \subset \mathbb{R}^2$ *and* $\partial\Omega$ *is* C^1. *Then for every* $M > 0$ *and* $n > \frac{M}{\pi} - 1$ *there exists a constant* $C > 0$ *and a sequence of functions* $\{u^\varepsilon\}_{\varepsilon\in(0,1/2]} \subset \dot{H}^1(\Omega)$ *such that*

$$\int_\Omega e_\varepsilon(u) \le M \ln\frac{1}{\varepsilon} + C \tag{7.4}$$

and

$$\|J(u) - \pi\sum_{i=1}^n d_i\delta_{a_i}\|_{\dot{W}^{-1,1}(\Omega)} \ge \varepsilon^{1-\frac{M}{\pi(n+1)}} - C\varepsilon \tag{7.5}$$

for every $(a_1, \dots a_n) \subset \Omega^n$ *and* $(d_1, \dots, d_n) \in \mathbb{R}^n$.

Proof. Fix Ω, M, n as in the statement of the lemma. It is convenient to assume that is n odd; we will discuss at the end of the proof the modifications needed if n is even.

Let $k = (n+1)/2$, and for each $\varepsilon > 0$ sufficiently small, select points $P_1, \dots, P_k$ and $N_1, \dots, N_k$ in Ω, such that

$$|P_i - N_i| = \varepsilon^{1-\frac{M}{2(n+1)}} \qquad |P_i - P_j| \ge c \text{ if } i \ne j \tag{7.6}$$

for some c independent of ε. Then define

$$W_\varepsilon(x) = \prod_{i=1}^k w_\varepsilon(x - P_i)\overline{w}_\varepsilon(x - N_i)$$

where $\overline{w}$ denotes the complex conjugate of w, and

$$w_\varepsilon(re^{i\theta}) = \begin{cases} \frac{r}{\varepsilon}e^{i\theta} & \text{if } r \le \varepsilon, \\ e^{i\theta} & \text{otherwise.} \end{cases}$$

Then by an explicit calculation, see Bethuel, Brezis, and Hélein [2], one finds that

$$\begin{aligned} \int_\Omega e_\varepsilon(W_\varepsilon)dx \ \le \ \int_{\mathbb{R}^2} e_\varepsilon(W_\varepsilon) &= 2k\pi \ln\frac{1}{\varepsilon} + 2k\pi\ln|P_i - N_i| + O(1) \\ &= M\ln\frac{1}{\varepsilon} + O(1) \end{aligned} \tag{7.7}$$

by (7.6) and the choice of k. The point is that the energy can be expressed as a sum of logarithmic terms $\pi \ln\frac{1}{\varepsilon}$, terms from a suitable renormalized energy [2] and error terms. Due to assumptions (7.6), the error terms are small, and the contributions to the renormalized energy from all terms except those of the form $\ln|P_i - N_i|$ are bounded, uniformly in ε.

Also, using Lemma 9 and elementary estimates, one can check that

$$\|J(u) - \pi\sum_{i=1}^k(\delta_{P_i} - \delta_{N_i})\|_{\dot{W}^{-1,1}} \le 2k\varepsilon.$$

and using the interpretation of the $\dot{W}^{-1,1}$ norm as the length of a minimal connection (see Brezis, Coron, and Lieb [3]), it is easy to see that since $n = 2k-1$,

$$\|\pi \sum_{i=1}^{k}(\delta_{P_i} - \delta_{N_i}) - \pi \sum_{i=1}^{n} d_i \delta_{a_i}\|_{\dot{W}^{-1,1}} \geq \varepsilon^{1-\frac{M}{\pi(n+1)}}$$

for every $a \in \Omega^n$ and $d \in \mathbb{R}^n$. Thus the triangle inquality implies that

$$\|J(u) - \pi \sum_{i=1}^{n} d_i \delta_{a_i}\|_{\dot{W}^{-1,1}} \geq \varepsilon^{1-\frac{M}{\pi(n+1)}} - 2k\varepsilon.$$

for every such a, d.

If n is even, one can use the same argument, with at least one "dipole" P_i, N_i straddling $\partial\Omega$, and such that $\operatorname{dist}(P_i, \partial\Omega) = \operatorname{dist}(N_i, \partial\Omega) = \frac{1}{2}|P_i - N_i|$. Here we need to assume some smoothness of $\partial\Omega$. $\square$

References

[1] Alberti, G., Baldo, S., and Orlandi, G. *Variational convergence for functions of Ginzburg-Landau type*, Indiana Univ. Math. Jour. , to appear.

[2] Bethuel, F., Brezis, H. and Helein, F. *Ginzburg-Landau Vortices* Birkhäuser, Boston, (1994).

[3] Brezis, H., Coron, J.-M., and Lieb, E. *Harmonic maps with defects*, Commun Math. Phys., **107** (1986), 649-705.

[4] Brezis, H. and Nirenberg, L, *Degree theory and BMO: Part i: compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), no. 2, 197–263.

[5] Colliander, J.E. and Jerrard, R.L. *Ginzburg-Landau vortices: weak stability and Schrödinger equation dynamics* J. Anal. Math. **77** (1999), 129–205.

[6] Jerrard, R. L. *Lower bounds for generalized Ginzburg-Landau functionals*, SIAM J. Math. Anal. **30** (1999), 721–746.

[7] Jerrard, R. L. and Soner, M. *The Jacobian and the Ginzburg-Landau energy*, Calc. Var. Partial Differential Equations **14** (2002), 151–191.

[8] Jerrard, R. L. and Spirn, D. *Refined Jacobian Estimates and the Gross-Pitaevsky Vortex Dynamics*, Preprint.

[9] Lassoued, L. and Mironescu, P. *Ginzburg-Landau type energy with discontinuous constraint* J. Anal. Math. **77** (1999), 1–26.

[10] Lin, F.-H. and Xin, J. X. *On the incompressible fluid limit and the vortex motion law of the nonlinear Schrdinger equation* Comm. Math. Phys. **200** (1999), 249–274.

[11] Sandier, E. *Lower bounds for the energy of unit vector fields and applications* J. Funct. Anal. **152** (1998), 379–403.

[12] Sandier, E. and Serfaty, S. *Limiting vorticities for the Ginzburg-Landau equations* Duke Math. J. **117** (2003), 403-446.

[13] Shafrir, I. *Remarks on solutions of* $-\Delta u = (1-|u|^2)u$ *in* R^2, C. R. Acad. Sci. Paris Sr. I Math. **318** (1994), 327–331.

Department of Mathematics, University of Toronto, Toronto, CA M5S 3G3
E-mail address: `rjerrard@utoronto.ca`

School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: `spirn@math.umn.edu`