Math 8669 Homework #1, Spring 2016

1. Give examples of a finite ranked poset P such that
(a) P has the matching property but is not Sperner.
(b) P is rank unimodal but not Sperner.
(c) P is Sperner but not rank unimodal.
(d) P is Sperner and rank unimodal, but does not have the matching property.

2. Prove that if P is Sperner, and P_{max} is a maximum level, then the bipartite graphs

$$P_{max-1} \cup P_{max} \quad \text{and } P_{max+1} \cup P_{max}$$

both have complete matchings.

3. Characterize all maximum sized antichains in the Boolean algebra B_N.

4. What is the Greene-Kleitman partition for the Boolean algebra B_N?

5. Can one prove log-concavity of the coefficients of the polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ using reality of the zeros?

6. Prove that $B_n(q)$ is Sperner by verifying that it is rank unimodal and has the matching property.

7. Here is another way to verify that $P = B_N(q)$ has the matching property. For $0 \leq k \leq N$ let W_k be the $\mathbb{R}$ vector space whose basis is given by elements at level k of $B_N(q)$, so $dim(W_k) = \begin{bmatrix} N \\ k \end{bmatrix}_q$.
Let $D_k : W_k \to W_{k-1}$ and $U_k : W_k \to W_{k+1}$, $0 \leq k \leq N$, be the natural down and up linear transformations using the edges of $B_N(q)$.
(a) What is $D_{k+1}U_k - U_{k-1}D_k$ as a linear transformation on W_k?
(b) Show if $2k < n$, the map U_k is 1-1, and find $rank(U_k)$.
(c) Show that the matrix of U_k has a non-singular $\begin{bmatrix} N \\ k \end{bmatrix}_q \times \begin{bmatrix} N \\ k \end{bmatrix}_q$ submatrix, and conclude that a complete matching from P_k to P_{k+1} exists.

8. Let $\lambda_n = (n-1, n-2, \cdots, 1)$ be the "staircase" partition. Let $P_n = [\varnothing, \lambda_n]$ be the interval in Young's lattice, namely the set of all partitions μ whose Ferrers diagram fit inside λ_n, under containment of Ferrers diagrams.
(a) Show that $|P_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number.
(b) If $R_n(q)$ is the rank generating function of P_n, find a version of $C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}, n \geq 1$, for $R_n(q)$.
(c) Is P_n rank symmetric, rank unimodal*, or Sperner*?
(d) True or False?

$$\sum_{n=0}^{\infty} R_n(1/q) q^{\binom{n}{2}} t^n = \sum_{n=0}^{\infty} \frac{(-t)^n q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} \Big/ \sum_{n=0}^{\infty} \frac{(-t)^n q^{n^2-n}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

$$= \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{xq}{1 - \cfrac{xq^2}{1 - \cfrac{xq^3}{\ddots}}}}}$$

9. Let $P_n = NC(n)$ the poset of non-crossing set partitions under refinement of blocks. Recall that $|P_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number, and the k^{th} level numbers are the Narayana numbers $N_{n,k} = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$, $0 \le k \le n-1$.

(a) Verify that P_n is a rank symmetric, rank unimodal poset.

(b) Verify that P_1, P_2, P_3, P_4 have symmetric chain decompositions by exhibiting one such decomposition on each Hasse diagram.

(c) Prove that P_n has a symmetric chain decomposition.

10. The inequality that we used for log-concavity

$$e_k(x_1, \cdots, x_n)^2 \ge e_{k-1}(x_1, \cdots, x_n)e_{k+1}(x_1, \cdots, x_n), \quad 0 \le k \le n-1, \quad x_i > 0$$

is a weaker version of the *Newton inequalities*

$$\left(\frac{e_k(x_1, \cdots, x_n)}{\binom{n}{k}}\right)^2 \ge \left(\frac{e_{k-1}(x_1, \cdots, x_n)}{\binom{n}{k-1}}\right)\left(\frac{e_{k+1}(x_1, \cdots, x_n)}{\binom{n}{k+1}}\right), \quad 0 \le k \le n-1, \quad x_i > 0.$$

(a) Take $k = 1$ and $n = 3$ and show that the Newton inequalities do not follow from termwise polynomial positivity.

(b) Prove the Newton inequalities by induction on n, fixing k. First verify the case $n = k+1$ by showing a certain quadratic form is positive semidefinite. Then do the inductive case by assuming $0 < x_1 < x_2 < \cdots < x_n$ and letting

$$P(t) = \prod_{i=1}^{n}(t + x_i), \quad P'(t) = n\prod_{i=1}^{n-1}(t + x_i')$$

where $x_i < x_i' < x_{i+1}$. Use

$$(n)e_k(x_1', x_2', \cdots, x_{n-1}') = (n-k)e_k(x_1, \cdots, x_n), \quad 0 \le k \le n-1$$

in the induction.

11. Let P be finite ranked poset and suppose that $G \le Aut(P)$. Define a poset P/G whose elements are the orbits O of G on P, with order relation $O_1 \le O_2$ iff there exists $x \in O_1$, $y \in O_2$, with $x \le y$ in P. True or False: If P is Sperner, then P/G is Sperner.

12. In this problem you will prove the unimodality of the q-binomial coefficient by using an explicit formula, called the *KOH identity.*

First some notation. For an integer partition λ, let $|\lambda|$ be the sum of the parts of λ. Let λ' be the conjugate of λ, and let $m_i(\lambda)$ be the multiplicity of the part i in λ. For example, if $\lambda = 544422111$, then $|\lambda| = 24$, $\lambda' = 96441$, and $m_4(\lambda) = 3$. Finally, let

$$n(\lambda) = \sum_i (i-1)\lambda_i = \sum_j \binom{\lambda_j'}{2}.$$

It is

(KOH)
$$\begin{bmatrix} N+k \\ k \end{bmatrix}_q = \sum_{\lambda, |\lambda|=k} q^{2n(\lambda)} \prod_{i=1}^{\infty} \begin{bmatrix} (N+2)i - 2\sum_{j=1}^{i}\lambda_j' + m_i(\lambda) \\ m_i(\lambda) \end{bmatrix}_q.$$

(a) Write out (KOH) for $k = 3$ and explain why it recursively proves that $\begin{bmatrix} M \\ 3 \end{bmatrix}_q$ is a unimodal polynomial in q.

(b) Repeat (a) for a general k by showing that the individual terms in (KOH) are "centered" correctly.