Problem 8d:

Let $\tilde{R}_n(q) = q^{\binom{n}{2}} R_n(1/q)$. Then part (b) which said

$$R_n(q) = \sum_{k=1}^{n} q^{k(n-k)} R_{k-1}(q) R_{n-k}(q)$$

becomes

$$\tilde{R}_n(q) = \sum_{k=1}^{n} q^{k-1} \tilde{R}_{k-1}(q) \tilde{R}_{n-k}(q). \tag{1}$$

If we define the formal power series in t

$$A(t) = \sum_{n=0}^{\infty} \tilde{R}_n(q) t^n, \tag{2}$$

then (1) is equivalent to

$$A(t) = 1 + tA(t)A(qt). \tag{3}$$

Note that if $q = 1$ this is the Catalan generating function quadratic equation, $A(t) = 1 + tA(t)^2$.

Next, let

$$B(t) = \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-t)^n}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Let's find a q-difference equation for $B(t)$. It is, upon subtracting term by term,

$$B(t) - B(tq) = -tB(tq^2). \tag{4}$$

So

$$\frac{B(tq)}{B(t)} = 1 + t\frac{B(tq^2)}{B(tq)}\frac{B(tq)}{B(t)}$$

which says that $B(qt)/B(t)$ satisfies (3), so

$$A(t) = B(tq)/B(t)$$

is the first equation in Problem 8d.

If $F(t)$ is the continued fraction, then we have $F(t) = 1/(1 - tF(qt))$ which is $F(t) = 1 + tF(t)F(qt)$, again (3).

Note: It is possible to independently show that the continued fraction converges as a formal power series in t and is equal to $A(t)$. Consider the more general finite continued fraction ($q^{i-1} = \lambda_i$) which terminates at λ_n, $F_n(t, \lambda)$. For example

$$F_4(t,\lambda) = \cfrac{1}{1 - \cfrac{t\lambda_1}{1 - \cfrac{t\lambda_2}{1 - \cfrac{t\lambda_3}{1 - t\lambda_4}}}}$$

We have

$$F_4(t,\lambda) = \frac{1}{1 - t\lambda_1 F_3(t,\lambda^+)} = \sum_{k=0}^{\infty} (t\lambda_1 F_3(t,\lambda^+))^k,$$

where λ^+ means all of the λ indices have been increased by 1.

Let's weight the down edge of a finite Dyck path from y-coordinate i to y-coordinate $i-1$ by λ_i. Then $F_1(t,\lambda) = \sum_{k=0}^{\infty} (t\lambda_1)^k$ is the generating function for all Dyck paths which stay at or below the line $y = 1$, (only zigzags), and $F_n(t,\lambda) = \frac{1}{1 - t\lambda_1 F_{n-1}(t,\lambda^+)}$ is the generating function for all Dyck paths which stay at or below the line $y = n$. So for a fixed power of t say t^s, once n is past s, you get all such Dyck paths, and the coefficient of t^s in $F_n(t)$ stablizes as n increases. The infinite continued fraction

$$\lim_{n\to\infty} F_n(t,\lambda)$$

is the generating function for all Dyck paths with no restrictions on their heights.

By drawing a picture of a Dyck path one may see that the choice $\lambda_i = q^{i-1}$ gives the polynomials $\tilde{R}_n(q)$ as the coefficient of t^n in $F(t,\lambda) = A(t)$.