Homework #2 Mathematics 8669 Due Monday February 29, 2016

1. Verify the following identities using hypergeometric series.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x}{x+k} = 1/\binom{x+n}{n}. \tag{1}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k} = \sum_{k=1}^{n}\frac{1}{k}. \tag{2}$$

$$\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}2^{n-2k} = n+1. \tag{3}$$

$$\sum_{k=r}^{n-s}\binom{k}{r}\binom{n-k}{s} = \binom{n+1}{r+s+1}. \tag{4}$$

$$\sum_{k=0}^{r}\binom{x}{k}\binom{-x}{n-k} = \frac{n-r}{n}\binom{x-1}{r}\binom{-x}{n-r} \quad \text{(here } r \le n\text{)}. \tag{5}$$

$$\sum_{k=0}^{n/2}\binom{x}{k}\binom{x-k}{n-2k}\frac{1}{2^{2k}} = 2^{-n}\binom{2x}{n}. \tag{6}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+2r+k}{n+r}(-1)^k 2^{n-k} = (-1)^{n/2}\frac{1+(-1)^n}{2}\frac{\binom{n+r}{n/2}\binom{n+2r}{r}}{\binom{n+r}{r}}. \tag{7}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k/2}{m} = \frac{n}{m}\binom{n-m-1}{m-1}2^{n-2m}. \tag{8}$$

$$\sum_{k=0}^{n}\binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x+k}{m+n} = \binom{x}{m}\binom{y}{n}. \tag{9}$$

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}\sum_{j=1}^{k}\frac{1}{j} = \frac{(-1)^n}{n}. \tag{10}$$

$$\sum_{k=0}^{2n} \frac{\binom{2n}{k}(-1)^k}{\binom{2n+2x}{k+x}\binom{2n+2y}{k+y}} = \frac{\binom{2n}{n}\binom{2n+x+y+1}{n}}{\binom{x+n}{n}\binom{y+n}{n}\binom{2n+2x}{n+x}\binom{2n+2y}{n+y}}. \tag{11}$$

$$\sum_{k=0}^{2n}(-1)^{n-k}\binom{2n}{k}^3 = \frac{(3n)!}{n!n!n!}. \tag{12}$$

2. Expand $(1-x)^A$ in terms of powers of $x/(1-x)^2$ by Lagrange inversion. Then evaluate

$$ {}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 2a \end{matrix} \middle| \frac{-4x}{(1-x)^2}\right), \qquad {}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 2a+1 \end{matrix} \middle| \frac{-4x}{(1-x)^2}\right).$$

How is this related to the Catalan number generating function

$$\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} t^n?$$

3. Let a_1, a_2, a_3 be non-negative integers. Prove that the constant term of the Laurent polynomial

$$\prod_{1\le i\ne j\le 3}(1-x_i/x_j)^{a_i} \quad \text{is} \quad \binom{a_1+a_2+a_3}{a_1, a_2, a_3}.$$

4. Let A and B be relatively prime positive integers. What is the coefficient of z^{AB} in the power series for

$$\frac{(1-z^{A+B})^{A+B}}{(1-z^A)^A(1-z^B)^B}?$$

Do you need to use $GCD(A,B)=1$?

5. Find a product formula for the sum

$$\sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q q^{\binom{k}{2}} x^k.$$

What happens if $n\to\infty$?

6. Using weighted integer partitions, give a bijective proof of

$$(b+aq)\sum_{n=0}^{\infty}\frac{(-aq;q)_n}{(bq;q)_n}q^n = \frac{(-aq;q)_\infty}{(bq;q)_\infty} - (1-b).$$