Homework #2 Mathematics 8669 Selected solutions

1 (10). Verify the following identities using hypergeometric series.

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}\sum_{j=1}^{k}\frac{1}{j}=\frac{(-1)^n}{n}.$$

Solution: Take $\frac{d}{dC}$ of

$${}_2F_1\left(\begin{matrix}-n, & A\\ & C\end{matrix}\,\middle|\,1\right)=\frac{(C-A)_n}{(C)_n}$$

to get

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(-n)_k(A)_k}{k!(C)_k}\left(-\frac{1}{C}-\frac{1}{C+1}-\cdots-\frac{1}{C+k-1}\right)\\ &=\frac{(C-A)_n}{(C)_n}\left(-\frac{1}{C}-\frac{1}{C+1}-\cdots-\frac{1}{C+n-1}+\frac{1}{C-A}+\frac{1}{C-A+1}+\cdots+\frac{1}{C-A+n-1}\right)\end{aligned}$$

Then put $C=1$ and take the limit as $A\to n$.

2. Expand $(1-x)^A$ in terms of powers of $x/(1-x)^2$ by Lagrange inversion. Then evaluate

$${}_2F_1\left(\begin{matrix}a, & a+1/2\\ & 2a\end{matrix}\,\middle|\,\frac{-4x}{(1-x)^2}\right),\quad {}_2F_1\left(\begin{matrix}a, & a+1/2\\ & 2a+1\end{matrix}\,\middle|\,\frac{-4x}{(1-x)^2}\right).$$

How is this related to the Catalan number generating function

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}t^n?$$

Solution: Let $y=x/(1-x)^2$, and

$$(1-x)^A=\sum_{n=0}^{\infty}a_ny^n.$$

So

$$\begin{aligned}a_n=&Res_y\frac{(1-x)^A}{y^{n+1}}=Res_x\frac{(1-x)^A}{(x/(1-x)^2))^{n+1}}\frac{dy}{dx}\\ =&Res_x\frac{(1-x)^{A+2n-1}(1+x)}{x^{n+1}}\\ =&(-1)^n\left(\binom{A+2n-1}{n}-\binom{A+2n-1}{n-1}\right)\\ =&(-1)^n\frac{(A)_{2n}}{n!(A+1)_n}=(-4)^n\frac{(A/2)_n((1+A)/2)_n}{n!(A+1)_n}\end{aligned}$$

Thus

$$(1) \qquad (1-x)^A = {}_2F_1\left(\begin{matrix} A/2, & A/2+1/2 \\ & A+1 \end{matrix} \,\middle|\, \frac{-4x}{(1-x)^2}\right).$$

which answers the second question if $A=2a$.

Taking the derivative of (1) gives

$$-A(1-x)^{A-1} = -4\frac{1+x}{(1-x)^3}\frac{A/2(A/2+1/2)}{A+1}\, {}_2F_1\left(\begin{matrix} 1+A/2, & 1+A/2+1/2 \\ & A+2 \end{matrix} \,\middle|\, \frac{-4x}{(1-x)^2}\right)$$

which for $A=2a-2$ is the first requested function

$$(1-x)^{2a} = (1+x)\, {}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 2a \end{matrix} \,\middle|\, \frac{-4x}{(1-x)^2}\right).$$

The Catalan generating function is

$$C(t) = {}_2F_1\left(\begin{matrix} 1/2, & 1 \\ & 2 \end{matrix} \,\middle|\, t\right)$$

which is $A=1$ in (1). So (1) for general A tells you how to explicitly expand powers of the Catalan generating function.

3. Let a_1, a_2, a_3 be non-negative integers. Prove that the constant term of the Laurent polynomial

$$\prod_{1\le i\ne j\le 3} (1-x_i/x_j)^{a_i} \quad \text{is} \quad \binom{a_1+a_2+a_3}{a_1, a_2, a_3}.$$

Idea for Blitz-Proof: Let $F_{a_1,a_2,a_3}(x_1,x_2,x_3)$ be the Laurent polynomial on the LHS. Suppose that we show that the entire polynomial F satisfies the Pascal recurrence, not just the constant term

$$(2) \qquad F_{a_1,a_2,a_3} = F_{a_1-1,a_2,a_3} + F_{a_1,a_2-1,a_3} + F_{a_1,a_2,a_3-1}.$$

Then we are done, because we need only check the $a_1=0$ case, which is the binomial theorem.

But (2) is equivalent to

$$\begin{aligned} &(1-x_2/x_3)(1-x_2/x_1)(1-x_3/x_1)(1-x_3/x_2) \\ +&(1-x_1/x_2)(1-x_1/x_3)(1-x_3/x_1)(1-x_3/x_1) \\ +&(1-x_2/x_3)(1-x_2/x_1)(1-x_1/x_3)(1-x_1/x_2) \\ =&(1-x_1/x_2)(1-x_1/x_3)(1-x_2/x_3)(1-x_2/x_1)(1-x_3/x_1)(1-x_3/x_2) \end{aligned}$$

which is true!

4. Let A and B be relatively prime positive integers. What is the coefficient of z^{AB} in the power series for

$$\frac{(1-z^{A+B})^{A+B}}{(1-z^A)^A(1-z^B)^B}?$$

Do you need to use $GCD(A,B)=1$?

Solution: The coefficient of z^{AB} is $\binom{A+B}{B}$, and we do not need to assume that $GCD(A,B)=1$. In fact more is true, the coefficient of z^{AB} in

$$\frac{(1-\lambda\mu z^{A+B})^{A+B}}{(1-\lambda z^A)^A(1-\mu z^B)^B}$$

is

$$\binom{A+B-1}{B}\lambda^B+\binom{A+B-1}{A}\mu^A.$$

Proof if GCD(A,B)=1: Expanding in power series we have

$$\sum_{k,j,m\geq 0}\binom{A+B}{k}(-\lambda\mu z^{A+B})^k\binom{A+j-1}{j}(\lambda z^A)^j\binom{B+m-1}{m}(\mu z^B)^m$$

So the coefficent of z^{AB} has terms which satisfy $k(A+B)+jA+mB=AB$, or $(k+j)A=B(A-m-k)$. Since $GCD(A,B)=1$ the solutions are $(k+j=B, m+k=0$, so $k=m=0$, $j=B)$ and $(k+j=0, m+k=A$, so $k=j=0$, $m=A)$. These are the two terms which are given.

Sketch of Proof if GCD(A,B)=d: We need, after dividing by d, and putting

$$A'=A/d,\quad B'=B/d\quad GCD(A',B')=1,$$

$$(k+j)A'=B'(dA'-k-m).$$

The solutions are

$$k+j=wB',\quad dA'-k-m=wA'$$

for some $0\leq w\leq d$. Note that in this case the coefficient of z^{AB} includes $\lambda^{wB'}\mu^{(d-w)A'}$. We will show that for a fixed w, which is not 0 or d, this term is zero. So the only contributions are the two terms from $w=0$ ($k=j=0$ as before) and $w=d$ ($k=m=0$ as before).

Fix $w\neq 0,d$, and put $j=wB'-k$ and $m=(d-w)A'-k$. We must show that

$$\sum_{k\geq 0}\binom{A+B}{k}(-1)^k\binom{A+wB'-k-1}{wB'-k}\binom{B+(w-d)A'-k-1}{(w-d)A'-k}=0.$$

This is nearly Saalschutz's theorem, but not quite

$${}_3F_2\left(\begin{matrix}-A-B, & -wB', & -(d-w)A' \\ & 1-A-wB', & 1-B-(d-w)A'\end{matrix}\,\middle|\,1\right)=0.$$

You may write it as a sum of two terms, each evaluable by Saalschutz's theorem if you use the Pascal relation in the sum

$$\binom{A+B}{k}=\binom{A+B-1}{k}+\binom{A+B-1}{k-1}.$$

These two terms cancel, and the sum is 0.

5. Find a product formula for the sum

$$\sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q q^{\binom{k}{2}} x^k.$$

What happens if $n \to \infty$?

Solution: The identity, which is equivalent to the q-binomial theorem, is

$$\sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q q^{\binom{k}{2}} x^k = (-q/x;q)_n(-x;q)_n$$

Using for a fixed k,

$$\lim_{n\to\infty} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q = \frac{1}{(q;q)_\infty}$$

the limiting identity is

$$\frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} q^{\binom{k}{2}} x^k = (-q/x;q)_\infty(-x;q)_\infty.$$

6. Using weighted integer partitions, give a bijective proof of

$$(b+aq)\sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(bq;q)_n} q^n = \frac{(-aq;q)_\infty}{(bq;q)_\infty} - (1-b).$$

Solution: Let's slightly rewrite this as

$$\sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(bq;q)_n} \left(bq^n + aq^{n+1}\right) = \frac{(-aq;q)_\infty}{(bq;q)_\infty} - (1-b).$$

Consider the infinite products on the RHS. The numerator product is the generating function for partitions λ with distinct parts, each part weighted by a. The denominator product is the generating function for all partitions μ, each part weighted by b. So the infinte product is the generating function of all ordered pairs (λ, μ). You can think of the parts of λ as red and those of μ to be blue.

What happens if we shuffle these parts together to get a single partition $\theta = \lambda \cup \mu$?

CASE 1: θ has a unique largest part $n+1$ which is red. This part has weight aq^{n+1}, the other red parts from 1 to n may or may not appear $(-aq;q)_n$, and the blue parts must be from 1 to n, $\frac{1}{(bq;q)_n}$. This is the second term in the sum.

CASE 2: The largest part is not uniquely red. This means that there is a blue largest part, say of size n, and weight bq^n. The remaining blue parts are from 1 to n, so again $\frac{1}{(bq;q)_n}$. The red parts are distinct from 1 to n, again $(-aq;q)_n$.

CASE 2 fails only when $n = 0$ and $\theta = \varnothing$, so $bq^0 = b$ replaces 1 for the weight of the empty partition, this is the term $(b-1)$ on the RHS.