Homework #3 Mathematics 8669 Due Monday April 4, 2016

1. Find the character table of the symmetric group S_4.

2. What is the orthogonality relation for the characters of the cyclic group of order n?

3. Show that for any character χ and $g \in G$, $|\chi(g)| \leq \chi(e) =$ dimension(χ).

4. Let G be a finite group of order p^2, p a prime. By considering the possible dimensions of the irreducible representations of G, prove that G is abelian.

5. Prove that the sum of any row of the character table of G is a non-negative integer (see problem 9 for notation),

$$\sum_{i=1}^{s} \chi^L(K_i) \text{ is a non-negative integer.}$$

(Hint: Consider the character χ^C obtained by letting G at on itself by conjugation. What is $< \chi^L, \chi^C >$?)

6. The dihedral group D_n is the group of order $2n$ which is the symmetry group of the regular n-gon. It may be given in terms of reflections and rotations as the set

$$D_n = \{r^k : 0 \leq k \leq n-1\} \cup \{sr^k : 0 \leq k \leq n-1\}$$

where $r^n = e$, $s^2 = e$, $srs = r^{-1}$. Show that the set of irreducible representations of D_n consists of

(a) four 1-dimensional representations, and $(n/2 - 1)$ 2-dimensional representations for n even

(b) two 1-dimensional representations, and $(n-1)/2$ 2-dimensional representations for n odd.

You should be able to construct these characters, using induced characters from the cyclic subgroup of order n.

7. Let S_n act on the set $\{1, \cdots, n\}$ by the natural action of permutations. Let V be the n-dimensional vector space over $\mathbb{C}$ whose basis is $\{1, \cdots, n\}$. In this problem you will prove that V decomposes into two irreducibles: the identity of dimension 1, and another irreducible of dimension $n-1$. Let χ^V denote the permutation character on V.

(a) Recall that the exponential formula says that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{g \in S_n} x_1^{\#\text{1-cycles of } g} = exp\left(tx_1 + \sum_{k=2}^{\infty} t^k/k\right).$$

Show that this formula implies that the average number of fixed points of $g \in S_n$ is 1 for $n \geq 1$, and that the average of (number of fixed points)^2 is 2 for $n \geq 2$.

(b) Use part (a) to show that $\theta = \chi^V - \chi^{id}$ satisfies $< \theta, \theta >= 1$, and conclude that θ is irreducible.

8. Suppose that V and W are finite dimensional representations of G over $\mathbb{C}$, with bases $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_n\}$. Let $V \otimes W$ be the mn dimensional $\mathbb{C}$-vector space whose basis is $v_i \otimes w_j$, $1 \leq i \leq n$, $1 \leq j \leq m$. Let G act on $V \otimes W$ by

$$\sum_{i,j} c_{ij} v_i \otimes w_j \xrightarrow{\rho(g)} \sum_{i,j} c_{ij} \rho_1(g) v_i \otimes \rho_2(g) w_j.$$

(a) Check that $\rho(g)$ is a non-singular linear transfromation on $V \otimes W$, and $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$.

(b) By explicitly computing traces, show that

$$\chi^{V \otimes W}(g) = \chi^V(g)\chi^W(g).$$

9. We proved in class that the center of the group algebra has dimension equal to the number of conjugacy classes of G. In this problem you will give two different bases for this center. Let $K_1, \cdots, K_s$ be the conjugacy classes of the finite group G. Let $\{\chi^L : L \text{ is irreducible}\}$ be the irreducible characters for G.

(a) Put

$$\hat{K}_i = \sum_{g \in K_i} g \in C[G], \quad 1 \leq i \leq s.$$

Show that $g\hat{K}_i = \hat{K}_i g$ for all $g \in G$, and conclude that $\{\hat{K}_1, \cdots, \hat{K}_s\}$ is a basis for the center of $C[G]$.

(b) For an irreducible L of G put

$$e_L = \frac{dim(L)}{|G|} \sum_{g \in G} \chi^L(g^{-1}) g \in C[G].$$

Show that $g e_L = e_L g$ for all $g \in G$, and conclude that $\{e_L : L \text{ is irreducible}\}$ is a basis for the center of $C[G]$.

(c) Using the orthogonality relation for matrix elements, show that $e_K e_L = e_K \delta_{KL}$ and find $\sum_K e_K$.

(d) For each irreducible L, define a map $\phi_L : Center(\mathbb{C}[G]) \to \mathbb{C}$ by

$$\phi_L(z) = \frac{1}{dim(L)} \sum_{g \in G} z(g)\chi^L(g).$$

Show that ϕ_L is an algebra homomorphism by checking that $\phi_L(e_K) = \delta_{KL}$ and using part (c).

10. In the notation of Problem 9,

(a) prove that there exists non-negative integers α_{ij}^k such that

$$\hat{K}_i \hat{K}_j = \sum_{k=1}^{s} \alpha_{ij}^k \hat{K}_k.$$

(b) prove that there exists non-negative integers β_{LJ}^K such that

$$\chi^L(g)\chi^J(g) = \sum_K \beta_{LJ}^K \chi^K(g).$$

11. You may use the following fact from algebra: If R is a commutative ring and $x \in R$, then x is integral over $\mathbb{Z}$ if, and only if, the subring $\mathbb{Z}[x]$ of R generated by x is finitely generated, if, and only if, R contains a finitely generated Z-submodule which contains $\mathbb{Z}[x]$. Prove that $\hat{K}_i$ is integral over $\mathbb{Z}$, that is $\hat{K}_i$ satisfies a polynomial equation with integral coefficients whose leading term has coefficient 1.

12. This problem uses problems 9 and 11 to show $dim(L)$ divides $|G|$ for an irreducible L.

(a) Let $z \in Center(\mathbb{C}[G])$, so that z may be considered as a function on the conjugacy classes K_i. If $z(K_i)$ is integral over $\mathbb{Z}$ for $1 \leq i \leq s$, verify that

$$X = \sum_{g \in G} z(g)g = \sum_{i=1}^{s} z(K_i)K_i$$

is also integral over $\mathbb{Z}$.

(b) Since ϕ_L is an algebra homomorphism, conclude that $\phi_L(X)$ is also integral over $\mathbb{Z}$.

(c) Finally prove that $dim(L)$ divides $|G|$ by choosing $z(K_i) = \chi^L(g^{-1})$, and noting that

$$\phi_L(X) = \frac{|G|}{dim(L)} < X_L, X_L >= \frac{|G|}{dim(L)}$$

is integral over $\mathbb{Z}$.