Homework #3 Mathematics 8669 Selected solutions

1. Find the character table of the symmetric group S_4.

Solution: There are 5 conjugacy classes so 5 irreducible characters. We know 2 1-dimensional characters- the trivial and the sign. If we let S_4 act on $\{1, 2, 3, 4\}$, the characters values, call them $\chi(g)$ are given counting fixed points. So

$$\chi(id) = 4, \quad \chi((12)) = 2, \quad \chi((123)) = 1, \quad \chi((1234)) = 0, \quad \chi((12)(34)) = 0.$$

Subtracting the identity from this, $\phi = \chi - id$ we might test if this new class function is irreducible:

$$< \phi, \phi >= \frac{1}{24}(3 * 3 + 6 * 1 * 1 + 8 * 0 * 0 + 6 * (-1) * (-1) + 3 * (-1) * (-1))) = 1$$

so it is. We have found three irreducible characters, and $sign * \phi$ is also irreducible,

	id	(12)	(123)	(1234)	$(12)(34)$
id	1	1	1	1	1
$sign$	1	-1	1	-1	1
ϕ	3	1	0	-1	-1
$sign * \phi$	3	-1	0	1	-1

Since the sum of the squares of the dimensions of the irreducibles is 24, the last irreducible must have dimension 2. Then we use column orthogonality to fill in the last row.

	id	(12)	(123)	(1234)	$(12)(34)$
id	1	1	1	1	1
$sign$	1	-1	1	-1	1
ϕ	3	1	0	-1	-1
$sign * \phi$	3	-1	0	1	-1
ψ	2	0	-1	0	2

2. What is the orthogonality relation for the characters of the cyclic group of order n?

Solution: All of the irredecible characters are 1-dimensional. Let ρ and θ be distinct n^{th} roots of 1. Let g be a generator of G. Then two irreducible characters are

$$\chi^\rho(g^k) = \rho^k, \quad \chi^\theta(g^k) = \theta^k$$

and the orthogonality relation is

$$0 = \frac{1}{n}\sum_{k=0}^{n-1} \rho^k \overline{\theta^k} = \frac{1}{n}\sum_{k=0}^{n-1} \rho^k \theta^{-k} = \frac{1}{n}\frac{1 - (\rho/\theta)^n}{1 - \rho/\theta} = 0.$$

3. Show that for any character χ and $g \in G$, $|\chi(g)| \leq \chi(e) =$ dimension(χ).

Solution: $\chi(g)$ is the sum of $dim(V)$ eigenvalues of g. Each eigenvalue λ satisfies $\lambda^{|G|} = 1$, so is a root of unity, $|\lambda| \leq 1$.

4. Let G be a finite group of order p^2, p a prime. By considering the possible dimensions of the irreducible representations of G, prove that G is abelian.

Solution: The dimensions of the irreducible representations of G must be either 1, p or p^2, since they divide $|G|$. They also must be strictly less than p, because the sum of the squares equals p^2, and id is there. So they are all 1, and G is abelian.

5. Prove that the sum of any row of the character table of G is a non-negative integer (see problem 9 for notation),

$$\sum_{i=1}^{s} \chi^L(K_i) \text{ is a non-negative integer.}$$

Solution: Let's use the hint and find the value of the permutation character $\chi^C(g)$ obtained by letting G act on itself by conjugation. Let $g \in K_i$. Since G acts transitively on K_i, $\chi^C(g)$ is the size of the stablizer of g, which is

$$\chi^C(g) = \frac{|G|}{|K_i|}.$$

Thus

$$\begin{aligned} < \chi^L, \chi^C >_G =& \frac{1}{|G|} \sum_{i=1}^{s} |K_i| \chi^L(K_i) \chi^C(K_i) \\ =& \frac{1}{|G|} \sum_{i=1}^{s} |K_i| \chi^L(K_i) \frac{|G|}{|K_i|} \\ =& \sum_{i=1}^{s} \chi^L(K_i). \end{aligned}$$

So the row sum is the multiplicity of the ireeducible L is the permutation representation C, and is a non-negative integer.

6. **Solution:** First let's get the conjugacy classes. Since $sr^j s = r^{-j}$, this class is $K_j = \{r^j, r^{-j}\}, 1 \leq j < n/2$, and the classes $K_0 = \{e\}$, $K_{n/2} = \{r^{n/2}\}$ for n even. Because $r(sr^k)r^{-1} = sr^{k-2}$, if n is odd a single class is $J = \{sr^k : 1 \leq k \leq n\}$ and for n even there are 2 classes, each of size $n/2$, $J_1 = \{sr^{2k} : 1 \leq k \leq n/2\}$, $J_2 = \{sr^{2k+1} : 0 \leq k < n/2\}$. Here is a summary of the classes and sizes

	K_0	K_1	$\cdots$	$K_{[n/2]-1}$	$K_{n/2}$	J	J_1	J_2
neven	1	2	$\cdots$	2	1	0	$n/2$	$n/2$
nodd	1	2	$\cdots$	2	0	n	0	0

There are $n/2 + 3$ classes for n even and $(n-1)/2 + 2$ for n odd.

Next let's construct the 1-dimensional reps of D_n. Two adjacent reflections $s_1 = s$ and $s_2 = sr$ generate the entire group, with $(s_1 s_2)^n = 1$. So we can try to send s_1, s_2 independently to ± 1, this is four choices, but to preserve $(s_1 s_2)^n = 1$ if n is odd only 2 of these four choices work.

It remains to construct the 2-dimensional reps from induced reps on $H = < r >$ which has order n. Take H-coset reps of $\{e, s\}$. Let ρ be an n^{th} root of unity, and let $\chi(r^k) = \rho^k$ be the corresponding 1-dimensional character of H. The matrix for g is

$$\begin{bmatrix} \hat{\chi}(g) & \hat{\chi}(gs) \\ \hat{\chi}(sg) & \hat{\chi}(sgs) \end{bmatrix}$$

where

$$\hat{\chi}(g) = \begin{cases} \chi(g) \text{ if } g \in H, \\ 0 \text{ otherwise.} \end{cases}$$

From this we see that

$$\chi^\rho(g) = \begin{cases} \rho^k + \rho^{-k} \text{ if } g = r^k \in H, \\ 0 \text{ otherwise.} \end{cases}$$

Let's check for irreducibility

$$< \chi^\rho, \chi^\rho >_G = \frac{1}{2n} \sum_{k=0}^{n-1} (\rho^k + \rho^{-k})^2 = \begin{cases} 1 \text{ if } \rho^2 \neq 1 \\ 2 \text{ if } \rho^2 = 1. \end{cases}$$

Note that the characters values are different for different $k < n/2$.

So for n odd we take $k = 1, \cdots, (n-1)/2$ and for n even we take $k = 1, \cdots, (n-2)/2$.

7. (a) Recall that the exponential formula says that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{g \in S_n} x_1^{\#\text{1-cycles of } g} = exp\left(tx_1 + \sum_{k=2}^{\infty} t^k/k\right).$$

Show that this formula implies that the average number of fixed points of $g \in S_n$ is 1 for $n \geq 1$, and that the average of (number of fixed points)^2 is 2 for $n \geq 2$.

Solution: Let $FP_n(g) = \#$fixed points of $g \in S_n$. For the average number of fixed point we take $\frac{d}{dx_1}|_{x_1=1}$,

$$\sum_{n=0}^{\infty} E(FP_n)t^n = t\ exp\left(t + \sum_{k=2}^{\infty} t^k/k\right) = texp(-log(1-t)) = t/(1-t)$$

so $E(FP_n) = 1$ for $n \geq 1$. Taking the next derivative we get

$$\sum_{n=0}^{\infty} E(FP_n(FP_n - 1))t^n = t^2\ exp\left(t + \sum_{k=2}^{\infty} t^k/k\right) = texp(-log(1-t)) = t^2/(1-t)$$

so

$$\sum_{n=0}^{\infty} E(FP_n^2)t^n = t(1+t)/(1-t)$$

and $E(FP_n^2) = 2$ for $n \geq 2$.

(b) Use part (a) to show that $\theta = \chi^V - \chi^{id}$ satisfies $<\theta, \theta>= 1$, and conclude that θ is irreducible.

Solution: We have

$$<\theta,\theta>_{S_n} = <\chi^V, \chi^V> -2 <\chi^V, id> + <id, id>= 2-2+1 = 1 \text{ if } n \geq 2.$$

9. (a) and (b) are routine.

(c) **Solution:** The coefficient of $x \in G$ in $e_K e_L$ is, using the matrix elements,

$$\begin{aligned}(dimK)(dimL)\frac{1}{|G|^2}\sum_{g\in G}\chi^L(g^{-1})\chi^K(x^{-1}g)\\ &= \frac{(dimK)(dimL)}{|G|}\frac{1}{|G|}\sum_{g\in G}\sum_{i,s}T^K_{is}(x^{-1})T^K_{si}(g)\chi^L(g^{-1})\\ &= 0 \text{ if } K \neq L.\end{aligned}$$

If $K = L$, again expanding as matrix elements, the coefficient of $x \in G$ is

$$= \frac{(dimK)(dimL)}{|G|}\frac{1}{|G|}\sum_{g\in G}\sum_{i,s,p}T^L_{is}(x^{-1})T^L_{si}(g)T^L_{pp}(g^{-1}).$$

Since matrix elements are orthogonal, the g-sum is $\delta_{ps}\delta_{pi}$ so we get

$$\frac{(dimL)}{|G|}\sum_{x\in G}x\sum_p T_{pp}(x^{-1}) = \frac{(dimL)}{|G|}\sum_{x\in G}\chi^L(x^{-1})x = e_L.$$

(d) **Solution:** We have, if $z = e_K$,

$$\phi_L(e_K) = \frac{dimK}{dimL}\frac{1}{|G|}\sum_{g\in G}\chi^K(g^{-1})\chi^L(g) = \delta_{KL}$$

by group orthogonality of characters.