Homework #4 Mathematics 8669 Due Friday May 6, 2016

1. Find the Schur function expansions of $h_3 s_{4211}$, $e_3 s_{4211}$, $p_3 s_{4211}$, $s_{4211/21}$, and $s_{21} s_{4211}$.

2. Let the symmetric group S_7 act on all 3-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ with permutation character χ. What is $char(\chi)$ in terms of Schur functions? What are the irreducibles in this permutation representation? Is the representation multiplicity free?

3. Set up an appropriate collection of weighted lattice paths whose non-intersecting version proves the dual Jacobi-Trudi identity for skew shapes

$$s_{\lambda'/\mu'} = det(e_{\lambda_i - \mu_j - i + j})_{1 \leq i,j, \leq m}, \quad \text{where } \lambda = \lambda_1 \cdots \lambda_m.$$

4. Use the Jacobi-Trudi identity to prove the following formula for hook shape Schur functions

$$s_{n-j,1^j} = \sum_{k=0}^{j} h_{n-j+k} e_{j-k} (-1)^k.$$

5. The unsigned Stirling numbers of the first $(c(n, k))$ and second $(S(n, k))$ kinds may be defined by

$$x^n = \sum_{k=1}^{n} S(n, k) x(x - 1) \cdots (x - k + 1),$$

$$x(x + 1) \cdots (x + n - 1) = \sum_{k=1}^{n} c(n, k) x^k.$$

Recall that these numbers count permutations of $[n]$ with k cycles and set partitions of $[n]$ with k parts.
(a) Show that $c(n, k) = e_{n-k}(1, 2, \cdots, n - 1)$.
(b) Show that

$$\sum_{n=0}^{\infty} S(n, k) t^n = \frac{t^k}{(1 - t)(1 - 2t) \cdots (1 - kt)}$$

and conclude that

$$S(n, k) = h_{n-k}(1, 2, \cdots, k).$$

(c) Recall that the Stirling numbers satisfy the orthogonality relations

$$\sum_{k=\ell}^{n} S(n, k) c(k, \ell)(-1)^{k-\ell} = \sum_{k=\ell}^{n} c(n, k) S(k, \ell)(-1)^{k-\ell} = \delta_{n,\ell}.$$

Can you find and prove an orthogonality relation involving symmetric functions which generalizes the above orthogonality?

6. Prove that if λ dominates μ, and both are partitions of n, then the Kostka number $K_{\lambda,\mu} > 0$.

7. Consider the "hook" shape $\lambda = (n-j, 1^j)$.
(a) Show that if λ dominates μ, then μ must have at least $j+1$ parts.
(b) Show that

$$K_{\lambda,\mu} = \binom{(\#\text{parts of } \mu) - 1}{j}.$$

(c) Show that

$$\sum_{j=0}^{n-1} s_{(n-j,1^j)}(q-1)^j = \sum_{\mu \vdash n} q^{(\#\text{parts of } \mu)-1} m_\mu.$$

What is this identity if $q = 1$ or $q = 0$?

8. (a) Show that the total number of SSYT with content $\mu = rr$ is the coefficient of $x_1^r x_2^r$ in $\frac{1}{(1-x_1)(1-x_2)(1-x_1x_2)}$ which is $r+1$.
(b) Show that the total number of SSYT with content $\mu = rrr$ is

$$\frac{1}{16}(4r^3 + 18r^2 + 28r + 15 + (-1)^r).$$

9. Stanley Exer. 7.78(a)-(d)

10. Stanley Exer. 7.17

11. Stanley Exer. 7.52

12. Stanley Exer. 7.67

13. Suppose that we choose x_i complex such that

$$H(t) = \sum_{n=0}^{\infty} h_n t^n = \frac{1}{(1-t)(1-t^2)(1-t^3)\cdots}$$

namely $h_n = p(n)$, the number of integer partitions of n.
(a) What is e_n?
(b) What is the determinant $det(p(i-j+1))_{1\le i,j\le n}$?

14. Recall the Frobenius form for an integer partition which uses the boxes of the Ferrers diagram which lie strictly to the right and strictly below the main diagonal. For example if

$$\lambda = 8552211, \quad \text{Frobenius form of } \lambda = \begin{pmatrix} 7 & 3 & 2 \\ 6 & 3 & 0 \end{pmatrix}$$

because the main diagonal has 3 elements.

Let

$$\lambda = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r \\ \beta_1 & \beta_2 & \cdots & \beta_r \end{pmatrix}.$$

Let $F(\alpha, \beta) = s_{(\alpha+1, 1^\beta)}$ be the Schur function of a hook shape. Prove the determinant formula for Schur functions in terms of hooks

$$s_\lambda = det(F(\alpha_i, \beta_j))_{1 \le i,j \le r}.$$

15. Let $D_n = det(\frac{1}{1-x_iy_j})_{1 \le i,j \le n}$.

(a) Show that

$$D_n = a_\delta(x_1, \cdots, x_n) a_\delta(y_1, \cdots, y_n) \prod_{i,j=1}^{n} (1 - x_i y_j)^{-1}.$$

(Possible hint: Multiply the i^{th} row by $\prod_{j=1}^{n}(1-x_iy_j)$, and realize the resulting matrix as a product of two matrices, one involving x the other involving y.)

(b) Use the geometric series to prove that

$$D_n = \sum_\lambda a_{\lambda+\delta}(x) a_{\lambda+\delta}(y),$$

where λ has at most n parts.

(c) Conclude the Cauchy identity with a finite number of variables:

$$\sum_\lambda s_\lambda(x_1, \cdots, x_n) s_\lambda(y_1, \cdots, y_n) = \prod_{1 \le i,j \le n} (1 - x_i y_j)^{-1}.$$

16. (Summing along a column of the character table of S_n.) Let $g \in S_n$ have cycle type μ. Let

$$\phi(\mu) = \sum_{\lambda \vdash n} \chi^\lambda(\mu).$$

(a) Show that

$$\phi(\mu) = < \sum_{\text{all } \lambda} s_\lambda \ , \ p_\mu > .$$

(b) Show, by taking logarithms, that

$$\sum_{\text{all } \lambda} s_\lambda = \prod_{n \ odd} exp\left(\frac{p_n}{n} + \frac{p_n^2}{2n}\right) \prod_{n \ even} exp\left(\frac{p_n^2}{2n}\right).$$

(c) If $\mu = 1^{m_1} 2^{m_2} \cdots$, show that

$$\phi(\mu) = y_1(m_1) y_2(m_2) \cdots ,$$

where $y_k(m)$ is the coefficient of t^m in $exp(t+kt^2/2)$ for k odd, and the coefficient of t^m in $exp(kt^2/2)$ for k even.

(d) Show that $\phi(\mu) = 0$ if μ contains an even part with an odd multiplicity.

(e) Check the conclusion of (d) for $\mu = 4$ or 211 using the S_4 character table.