Homework #4 Mathematics 8669 Selected Solutions

2. Let the symmetric group S_7 act on all 3-element subsets of $\{1,2,3,4,5,6,7\}$ with permutation character χ. What is $char(\chi)$ in terms of Schur functions? What are the irreducibles in this permutation representation? Is the representation multiplicity free?

Solution: $char(\chi) = h_4h_3 = h_3s_4 = s_7 + s_{61} + s_{52} + s_{43}$ by the Pieri rule. We may also do this by Young's rule. It is multiplicity free.

3. Set up an appropriate collection of weighted lattice paths whose non-intersecting version proves the dual Jacobi-Trudi identity for skew shapes

$$s_{\lambda'/\mu'} = det(e_{\lambda_i-\mu_j-i+j})_{1\le i,j,\le m}, \quad \text{where } \lambda = \lambda_1\cdots\lambda_m.$$

Solution: You must be careful while setting up the lattice paths. For example, if you want each elementary function to have variables $x_1, x_2, \cdots, x_N$, and you weight each path by the sequential number of the edges which are horizontal, then each path should have N steps and a weight which is a squarefree monomial. Moreover the "tail-swapping" should preserve the weights. For example if $\lambda = 531, \mu = \varnothing$ and $N = 25$, we can consider paths from $(1,3),(2,2),(3,1)$ to $(2,26),(5,23),(8,20)$. Then each possible path has length 25, the diagonal entry will be $e_5(x_1,\cdots,x_N)e_3(x_1,\cdots,x_N)e_1(x_1,\cdots,x_N)$, and swapping tails will preserve the weights.

5. The unsigned Stirling numbers of the first $(c(n,k))$ and second $(S(n,k))$ kinds may be defined by

$$x^n = \sum_{k=1}^{n} S(n,k)x(x-1)\cdots(x-k+1),$$
$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} c(n,k)x^k.$$

Recall that these numbers count permutations of $[n]$ with k cycles and set partitions of $[n]$ with k parts.

(a) Show that $c(n,k) = e_{n-k}(1,2,\cdots,n-1)$.

(b) Show that

$$\sum_{n=0}^{\infty} S(n,k)t^n = \frac{t^k}{(1-t)(1-2t)\cdots(1-kt)}$$

and conclude that

$$S(n,k) = h_{n-k}(1,2,\cdots,k).$$

(c) Recall that the Stirling numbers satisfy the orthogonality relations

$$\sum_{k=\ell}^{n} S(n,k)c(k,\ell)(-1)^{k-\ell} = \sum_{k=\ell}^{n} c(n,k)S(k,\ell)(-1)^{k-\ell} = \delta_{n,\ell}.$$

Can you find and prove an orthogonality relation involving symmetric functions which generalizes the above orthogonality?

Solution: (a) Upon dividing both sides of the generating function by x, this is the defintion of $e_{n-k}(1, 2, \cdots, n-1)$.

(b) Let $H(k, t)$ be the right side, and label the coefficient of t^n by $SS(n, k)$. We have $(1-kt)H(k,t) = tH(k-1,t)$, so equating coefficeints of x^n gives

$$SS(n, k) - kSS(n-1, k) = SS(n-1, k-1),$$

which is the Stirling number recurrence. After checking the initial values, $SS(n, k) = S(n, k)$.

(c) Consider for $\ell \le n-1$, the polynomial of degree $n-1-\ell$.

$$\frac{(1-tx_1)(1-tx_2)\cdots(1-tx_{n-1})}{(1-tx_1)(1-tx_2)\cdots(1-tx_\ell)}$$

The coefficient of $t^{n-\ell}$ in this polynomial is 0, so

$$\sum_{k=0}^{n-\ell} e_{n-k-\ell}(x_1, \cdots, x_{n-1})(-1)^{n-k-\ell} h_k(x_1, \cdots, x_\ell) = 0, \quad \ell < n.$$

which is

$$\sum_{k=\ell}^{n} e_{n-k}(x_1, \cdots, x_{n-1})(-1)^{n-k} h_{k-\ell}(x_1, \cdots, x_\ell) = 0, \quad \ell < n$$

and the choice of $x_i = i$ gives

$$\sum_{k=\ell}^{n} c(n, k)(-1)^{n-k} S(k, \ell) = 0, \quad \ell < n.$$

6. Prove that if λ dominates μ, and both are partitions of n, then the Kostka number $K_{\lambda,\mu} > 0$.

Solution: Let's prove the following: if λ dominates μ and $K_{\lambda,\mu} > 0$, and μ covers μ_1 in dominance, then $K_{\lambda,\mu_1} > 0$. Then we will be done by noting that $K_{\lambda,\lambda} = 1$ and taking a saturated chain from λ to μ,

$$1 = K_{\lambda,\lambda}, \quad K_{\lambda,\mu_1} > 0, \quad K_{\lambda,\mu_2} > 0, \quad \cdots, \quad K_{\lambda,\mu} > 0.$$

So what are the covers in dominance? Just sliding a single corner box down to its first available cell. This may be accomplished by either subtracting one from a part μ_k and adding 1 to μ_{k+1} when $\mu_{k+1} \le \mu_k - 2$, or by sliding the cell past some parts which are equal to $\mu_k - 1$, and then inserting it into the lext row. In the second case as we are sliding the cell, we are doing an adjacent transposition of the content vector, so we know from the Bender-Knuth swiching rule that positivity is preserved.

Thus we need only check the case of sliding a box from a row to the next row, namely reduce the number of k by 1 and increase the number of $k+1$'s by 1. So it suffices to find a k that may be switched to a $k+1$. Pair the k's and the $k+1$'s as in Bender-Knuth, there are at least 2 unpaired k's. Find the rightmost such k in a row, and change it to a $k+1$.

7. Consider the "hook" shape $\lambda = (n-j, 1^j)$.

(a) Show that if λ dominates μ, then μ must have at least $j+1$ parts.

(b) Show that

$$K_{\lambda,\mu} = \binom{(\#\text{parts of } \mu) - 1}{j}.$$

(c) Show that

$$\sum_{j=0}^{n-1} s_{(n-j,1^j)}(q-1)^j = \sum_{\mu \vdash n} q^{(\#\text{parts of } \mu)-1} m_\mu.$$

Solution: (c) Many people made this problem too hard. It follows from (b),(a) and the binomal theorem.

$$\begin{aligned}
\sum_{j=0}^{n-1} s_{(n-j,1^j)}(q-1)^j &= \sum_{j=0}^{n-1} \sum_{\mu \vdash n} \binom{(\#\text{parts of } \mu) - 1}{j} (q-1)^j m_\mu \\
&= \sum_{\mu \vdash n} m_\mu \sum_{j=0}^{n-1} \binom{(\#\text{parts of } \mu) - 1}{j} (q-1)^j \\
&= \sum_{\mu \vdash n} q^{(\#\text{parts of } \mu)-1} m_\mu.
\end{aligned}$$

8. (a) Show that the total number of SSYT with content $\mu = rr$ is the coefficient of $x_1^r x_2^r$ in $\frac{1}{(1-x_1)(1-x_2)(1-x_1x_2)}$ which is $r+1$.

(b) Show that the total number of SSYT with content $\mu = rrr$ is

$$\frac{1}{16}(4r^3 + 18r^2 + 28r + 15 + (-1)^r).$$

Solution: (a) The total number of SSYT of any fixed content μ is the coefficient of m_μ in

$$\sum_\lambda s_\lambda = \prod_i (1-x_i)^{-1} \prod_{i<j} (1-x_i x_j)^{-1}.$$

which for $\mu = rr$ gives the coefficient of $x_1^r x_2^r$ in $\frac{1}{(1-x_1)(1-x_2)(1-x_1x_2)}$.

(b) The next case is the coefficient of $x_1^r x_2^r x_3^r$ in

$$\frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}.$$

This is the sum of all coefficients of $x_1^a x_2^b x_3^c$ for $0 \le a, b, c \le r$ in

$$\frac{1}{(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)} = \sum_{A\ge 0}\sum_{B\ge 0}\sum_{C\ge 0}(x_1x_2)^A(x_1x_3)^B(x_2x_3)^C.$$

We must count the number of solutions (A, B, C) to

$$0 \le A+B \le r, \quad 0 \le A+C \le r, \quad 0 \le B+C \le r.$$

For any such (A, B), the number of allowed C is $min(r-A, r-B)+1$. So we must find

$$\sum_{A=0}^{r}\sum_{B=0}^{A-r}(min(r-A, r-B)+1).$$

We can split this sum into two parts $A \le B$ and $A > B$ to eliminate the minimum function. Then each part can be summed doing basic calculus (indefinite sums of first powers, second powers) to obtain the answer as a third degree polynomial. The parity condition on r appears because the even case has a middle term. (In fact the computer can do this for you. If you cannot do this, you should see me.)

13. Suppose that we choose x_i complex such that

$$H(t) = \sum_{n=0}^{\infty} h_n t^n = \frac{1}{(1-t)(1-t^2)(1-t^3)\cdots}$$

namely $h_n = p(n)$, the number of integer partitions of n.
(a) What is e_n?
(b) What is the determinant $det(p(i-j+1))_{1\le i,j\le n}$?

Solution: (a) Euler's Pentagonal Number Theorem says that

$$1/H(t) = (1-t)(1-t^2)\cdots = 1 + \sum_{k=1}^{\infty}(-1)^k\left(t^{k(3k-1)/2} + t^{k(3k+1)/2}\right)$$

Since $E(t) = 1/H(-t)$, we see that $e_n = 0$ unless $n = k(3k \pm 1)/2$, in which case $e_n = (-1)^k(-1)^{k(3k\pm 1)/2}$.
(b) This is the Jacobi-Trudi determinant of $s_{1^n} = e_n$, so the answer is $0, \pm 1$ depending upon n as in part (a).

16. (Summing along a column of the character table of S_n.) Let $g \in S_n$ have cycle type μ. Let

$$\phi(\mu) = \sum_{\lambda \vdash n} \chi^\lambda(\mu).$$

(a) Show that

$$\phi(\mu) = <\sum_{\text{all } \lambda} s_\lambda \ , \ p_\mu > .$$

(b) Show, by taking logarithms, that

$$\sum_{\text{all } \lambda} s_\lambda = \prod_{n \ odd} exp\left(\frac{p_n}{n} + \frac{p_n^2}{2n}\right) \prod_{n \ even} exp\left(\frac{p_n^2}{2n}\right).$$

(c) If $\mu = 1^{m_1}2^{m_2}\cdots$, show that

$$\phi(\mu) = y_1(m_1)y_2(m_2)\cdots,$$

where $y_k(m)$ is the coefficient of t^m in $exp(t+kt^2/2)$ for k odd, and the coefficient of t^m in $exp(kt^2/2)$ for k even.

(d) Show that $\phi(\mu) = 0$ if μ contains an even part with an odd multiplicity.

(e) Check the conclusion of (d) for $\mu = 4$ or 211 using the S_4 character table.

Solution: (a) Let's use

$$s_\lambda = \sum_\nu \chi^\lambda(\nu)\frac{p_\nu}{z_\nu}$$

$$< \sum_{\text{all } \lambda} s_\lambda \ , \ p_\mu >=< \sum_{\text{all } \lambda}\sum_\nu \chi^\lambda(\nu)\frac{p_\nu}{z_\nu}, \ p_\mu >= \sum_{\lambda \vdash n} \chi^\lambda(\mu)$$

using the power sum orthogonality $< p_\mu, p_\nu >= \delta_{\mu,\nu} z_\mu$.

(b)

$$\begin{aligned}\sum_\lambda s_\lambda =& \prod_i (1-x_i)^{-1} \prod_{i<j}(1-x_ix_j)^{-1} \\ =& exp\left(\sum_{n=1}^{\infty} \frac{p_n(x_1,x_2,\cdots,)}{n} + \frac{p_n(x_1x_2,x_1x_2,x_1x_4,\cdots,x_2x_3,\cdots)}{n}\right)\end{aligned}$$

However since

$$p_n(x_ix_j) = \frac{1}{2}\left(p_n(x_1,x_2,\cdots,)^2 - p_{2n}(x_1,x_2,\cdots,)\right)$$

this is

$$\sum_{\text{all } \lambda} s_\lambda = \prod_{n \ odd} exp\left(\frac{p_n}{n} + \frac{p_n^2}{2n}\right) \prod_{n \ even} exp\left(\frac{p_n^2}{2n}\right).$$

(c) Use parts (a) and (b). We must find

$$\phi(\mu) =< \prod_{n \ odd} exp\left(\frac{p_n}{n} + \frac{p_n^2}{2n}\right) \prod_{n \ even} exp\left(\frac{p_n^2}{2n}\right), p_\mu > .$$

Using the orthogonality of the power sums, we can evaluate this part by part. For each odd part n^m of μ we need the coefficient of p_n^m in $exp\left(\frac{p_n}{n} + \frac{p_n^2}{2n}\right)$, while the even parts need $exp\left(\frac{p_n^2}{2n}\right)$. The z_μ factor rescales these generating functions.

(d) If μ has an even part n wth odd multiplicity, since $exp\left(\frac{p_n^2}{2n}\right)$ has no odd powers of p_n, we have $\phi(\mu) = 0$.
(e) Let's find the column of the character table for cycle type $\mu = 4$. By Murnaghan-Nakayama, only hooks give a non-zero answer, and they are

$$\chi^4(4) = 1, \quad \chi^{31}(4) = -1, \quad \chi^{211}(4) = 1, \quad \chi^{1111}(4) = -1$$

so the sum is zero as predicted.
Let's find the column of the character table for cycle type $\mu = 211$. By Murnaghan-Nakayama

$$\chi^4(211) = 1, \quad \chi^{31}(211) = 1, \quad \chi^{22}(211) = 0, \quad \chi^{211}(211) = -1, \quad \chi^{1111}(211) = -1$$

so the sum is zero as predicted.