More Math 8669 Homework #1 Solutions, Spring 2016

2. Prove that if P is Sperner, and P_{max} is a maximum level, then the bipartite graphs

$$P_{max-1} \cup P_{max} \quad \text{and } P_{max+1} \cup P_{max}$$

both have complete matchings.

Solution: Suppose, by contradiction, that there is no complete match from $P_{max-1} \to P_{max}$. Then by Hall's theorem there exists a subset $S \subset P_{max-1}$ whose relatives $R(S) \subset P_{max}$ satisfy $|S| > |R(S)|$. Then $A = S \cup (P_{max} - R(S))$ is an antichain of size larger than P_{max}, which is a contradiction.

3. Characterize all maximum sized antichains in the Boolean algebra B_N.

Solution: Claim: The maximum sized antichains are precisely the maximum sized level sets, and no others.

As in lecture, the LYM property for B_N implies that a maximum sized antichain must lie inside the maximum levels. So for N even this is unique. Let's assume $N = 2m + 1$ is odd, and prove that a maximum sized antichain A could not be in both levels, $A = A_1 \cup A_2$, $\varnothing \neq A_1 \subset B_N(m)$, $\varnothing \neq A_2 \subset B_N(m+1)$ is impossible.

Note that the bipartite graph $G = B_N(m) \cup B_N(m+1)$ is regular of degree $m+1$. Let $R(A_1) \subset B_N(m+1)$ be the relatives of A_1. Because we know that a complete match exists in G, by Hall's condition $|A_1| \leq |R(A_1)|$. But since $A_2 \subset B_N(m+1) - R(A_1)$ and $|A_1| + |A_2| = \binom{2m+1}{m}$, we have $|A_1| = |R(A_1)|$, so each of the $(m+1)|A_1|$ edges from A_1 go to $R(A_1)$, and each of the $(m+1)|R(A_1)|$ edges from $R(A_1)$ do in fact go to A_1. The same reasoning applies to A_2 and $R(A_2)$. So the bipartite graph G is disconnected, which is a contradiction.

7. Here is another way to verify that $P = B_N(q)$ has the matching property. For $0 \leq k \leq N$ let W_k be the $\mathbb{R}$ vector space whose basis is given by elements at level k of $B_N(q)$, so $dim(W_k) = \begin{bmatrix} N \\ k \end{bmatrix}_q$.
Let $D_k : W_k \to W_{k-1}$ and $U_k : W_k \to W_{k+1}$, $0 \leq k \leq N$, be the natural down and up linear transformations using the edges of $B_N(q)$.
(a) What is $D_{k+1}U_k - U_{k-1}D_k$ as a linear transformation on W_k
(b) Show if $2k < n$, the map U_k is 1-1, and find $rank(U_k)$.

Solution: From (a) $D_{k+1}U_k = U_{k-1}D_k + c_k I$, where $c_k > 0$. As a amtrix $U_{k-1} = D_k^T$, so $U_{k-1}D_k$ is positvie semidefinite, therefore $D_{k+1}U_k$ is positive definite, so invertible, This implies that $ker(U_k) = \vec{0}$ and U_k is injective and $rank(U_k) = \begin{bmatrix} N \\ k \end{bmatrix}_q$.

(c) Show that the matrix of U_k has a non-singular $\begin{bmatrix} N \\ k \end{bmatrix}_q \times \begin{bmatrix} N \\ k \end{bmatrix}_q$ submatrix, and conclude that a complete matching from P_k to P_{k+1} exists.

Solution: Any $m \times n$ matrix A with $rank(A) = m$ has an $m \times m$ non-singular matrix B, by choosing m linearly independent columns. Here we have

$$det(B) = \sum_{\pi \in S_m} sign(\pi) \prod_{i=1}^{m} B_{i\pi(i)},$$

and $det(B) \neq 0$ implies that $B_{i\pi(i)} \neq 0$ for all i for some $\pi \in S_m$.

Applying this to part (b), the permutation π gives the matching.

9. Let $P_n = NC(n)$ the poset of non-crossing set partitions under refinement of blocks. Recall that $|P_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number, and the k^{th} level numbers are the Narayana numbers $N_{n,k} = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$, $0 \leq k \leq n-1$.
(c) Prove that P_n has a symmetric chain decomposition.

Solution:
Let's do this by induction on n, the first few cases were done in part (b). Since $rank(P_n) = n-1$, we need saturated chains whose bottom and top ranks add to $n-1$.

The main idea is to consider the block containing 1. Suppose the next smallest element in 1's block is a $k \geq 3$. Then the non-crossing partitions which contain a block $(1k...)$ split into two posets: those non-crossing set partitions of $\{2, 3, \ldots k-1\}$ and those for $\{k+1, k+2, \ldots n, 1k\}$, where $1k$ is melded megapoint, $P_{k-2} \times P_{n-k+1}$. The smallest element here has two blocks ($rank = 1$), while the largest has $n-1$ blocks ($rank = n-2$), so these inductive chains are centered correctly.

Finally we deal with the two remaining cases: 1 in a block by itself or 12 in a block. These are each just P_{n-1}, so their union is $P_{n-1} \times C_1$, where C_1 is a chain of length. P_{n-1} has symmetric chains by induction, and so the product does too.

10. The inequality that we used for log-concavity

$$e_k(x_1, \cdots, x_n)^2 \geq e_{k-1}(x_1, \cdots, x_n)e_{k+1}(x_1, \cdots, x_n), \quad 0 \leq k \leq n-1, \quad x_i > 0$$

is a weaker version of the *Newton inequalities*

$$\left(\frac{e_k(x_1, \cdots, x_n)}{\binom{n}{k}}\right)^2 \geq \left(\frac{e_{k-1}(x_1, \cdots, x_n)}{\binom{n}{k-1}}\right)\left(\frac{e_{k+1}(x_1, \cdots, x_n)}{\binom{n}{k+1}}\right), \quad 0 \leq k \leq n-1, \quad x_i > 0.$$

(b) Prove the Newton inequalities by induction on n, fixing k. First verify the case $n = k+1$ by showing a certain quadratic form is positive semidefinite. Then do the inductive case by assuming $0 < x_1 < x_2 < \cdots < x_n$ and letting

$$P(t) = \prod_{i=1}^{n}(t + x_i), \quad P'(t) = n\prod_{i=1}^{n-1}(t + x'_i)$$

where $x_i < x'_i < x_{i+1}$. Use

$$(n)e_k(x'_1, x'_2, \cdots, x'_{n-1}) = (n-k)e_k(x_1, \cdots, x_n), \quad 0 \leq k \leq n-1$$

in the induction.

Solution: First let's take care of the case $n = k+1$. Dividing both sides of the desired inequality by $(x_1x_2\cdots x_n)^2$, and putting $y_i = 1/x_i$, we need

$$(y_1 + y_2 + \cdots + y_{k+1})^2 \geq \frac{2(k+1)}{k}\sum_{1\leq i<j\leq k+1} y_iy_j,$$

or

$$Q(y) = \sum_{i=1}^{k+1} y_i^2 - \frac{2}{k}\sum_{1\leq i<j\leq k+1} y_iy_j \geq 0.$$

Let A be the $(k+1)\times(k+1)$ real symmetric matrix whose diagonal entries are 1 and whose off-diagonal entries are $-1/k$. Then we need $Q(y) = y^TAy \geq 0$ for $y > 0$. But we can check this by noting that the matrix A is positive semidefinite: the eigenvalues of A are $1 + 1/k$ with multiplicity k and 0 with multiplicity 1.

Next we prove the Newton inequalities by induction on n, the base case of $n = k+1$ has just been proven. Since the zeros of $P(t)$ are distinct, Rolle's theorem implies that the zeros of $P'(t)$ must interlace with the zeros of $P(t)$, so we can write

$$P'(t) = n\prod_{i=1}^{n-1}(t+x_i'), \quad x_i < x_i' < x_{i+1}, \quad 1 \le i \le n-1.$$

Finding the coefficient of t^{n-1-k} in $P'(t)$ gives

$$(n)e_k(x_1', x_2', \cdots, x_{n-1}') = (n-k)e_k(x_1, \cdots, x_n) \quad 0 \le k \le n-1.$$

So by induction

$$\begin{aligned}\left(\frac{e_k(x_1,\cdots,x_n)}{\binom{n}{k}}\right)^2 &= \left(\frac{e_k(x_1',\cdots,x_{n-1}')}{\binom{n-1}{k}}\right)^2 \\ &\ge \left(\frac{e_{k-1}(x_1',\cdots,x_{n-1}')}{\binom{n-1}{k-1}}\right)\left(\frac{e_{k+1}(x_1',\cdots,x_{n-1}')}{\binom{n-1}{k+1}}\right) \\ &= \left(\frac{e_{k-1}(x_1,\cdots,x_n)}{\binom{n}{k-1}}\right)\left(\frac{e_{k+1}(x_1,\cdots,x_n)}{\binom{n}{k+1}}\right).\end{aligned}$$

12. In this problem you will prove the unimodality of the q-binomial coefficient by finding an explicit formula, called the *KOH identity.*

First some notation. For an integer partition λ, let $|\lambda|$ be the sum of the parts of λ. Let λ' be the conjugate of λ, and let $m_i(\lambda)$ be the multiplicity of the part i in λ. For example, if $\lambda = 544422111$, then $|\lambda| = 24$, $\lambda' = 96441$, and $m_4(\lambda) = 3$. Finally, let

$$n(\lambda) = \sum_i (i-1)\lambda_i = \sum_j \binom{\lambda_j'}{2}.$$

It is

(KOH)
$$\begin{bmatrix} N+k \\ k \end{bmatrix}_q = \sum_{\lambda, |\lambda|=k} q^{2n(\lambda)} \prod_{i=1}^{\infty} \begin{bmatrix} (N+2)i - 2\sum_{j=1}^{i}\lambda_j' + m_i(\lambda) \\ m_i(\lambda) \end{bmatrix}_q.$$

(a) Write out (KOH) for $k=3$ and explain why it recursively proves that $\begin{bmatrix} M \\ 3 \end{bmatrix}_q$ is a unimodal polynomial in q.

Solution: Since $k=3$ there are 3 partitions in the sum on the right side $\lambda = 3, 21, 111$. The (KOH) identity becomes

(1)
$$\begin{bmatrix} N+3 \\ 3 \end{bmatrix}_q = \begin{bmatrix} 3N+1 \\ 1 \end{bmatrix}_q + q^2\begin{bmatrix} N-1 \\ 1 \end{bmatrix}_q \begin{bmatrix} 2N-1 \\ 1 \end{bmatrix}_q + q^6\begin{bmatrix} N-1 \\ 3 \end{bmatrix}_q.$$

Now suppose we try to prove that $\begin{bmatrix} M \\ 3 \end{bmatrix}_q$ is unimodal by induction on M. If we can show that each of the three terms in (1) is unimodal and centered at the same center as $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}_q$, which is $3N/2$, we are done. Since the second term is a product of symmetric unimodal polynomials, it is certainly symmetric and unimodal, as are the first and last (by induction) terms.

(1) $\begin{bmatrix} 3N+1 \\ 1 \end{bmatrix}_q$: smallest term q^0, largest term q^{3N}, $0+3N=3N$ works.

(2) $q^2\begin{bmatrix} N-1 \\ 1 \end{bmatrix}_q \begin{bmatrix} 2N-1 \\ 1 \end{bmatrix}_q$: smallest term q^2, largest term $q^{2+(N-2)+(2N-2)}$, $2+3N-2=3N$ works.

(3) $q^6 \begin{bmatrix} N-1 \\ 3 \end{bmatrix}_q$: smallest term q^6, largest term $q^{6+3(N-4)}$, $6+3N-6=3N$ works.

(b) Repeat (a) for a general k by showing that the individual terms in (KOH) are "centered" correctly.

Solution: The induction goes through as before, we must check the centering condition for each term. This is

$$2n(\lambda) + \left(2n(\lambda) + \sum_{i=1}^{\infty} m_i(\lambda)((N+2)i - 2\sum_{j=1}^{i} \lambda'_j) \right) = kN.$$

Since

$$\sum_{i=1}^{\infty} m_i(\lambda) i = k$$

we must show that

$$2n(\lambda) + k = \sum_{i=1}^{\infty} m_i(\lambda) \sum_{j=1}^{i} \lambda'_j. \tag{2}$$

Here is an example how this is proven, the general case is the same.

Let $\lambda = 322111$, so $k = 10$, $n(\lambda) = 18$. Let compute $n(\lambda) + n(\lambda) + k$ pictorially:

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & \\ 2 & 2 & \\ 3 & & \\ 4 & & \\ 5 & & \end{array} \qquad \begin{array}{ccc} 5 & 2 & 0 \\ 4 & 1 & \\ 3 & 0 & \\ 2 & & \\ 1 & & \\ 0 & & \end{array} \qquad \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & 1 & \\ 1 & & \\ 1 & & \\ 1 & & \end{array}$$

Adding these we find

$$\begin{array}{ccc} 6 & 3 & 1 \\ 6 & 3 & \\ 6 & 3 & \\ 6 & & \\ 6 & & \\ 6 & & \end{array}$$

which is the right side of (2).