Math 8680 HW #2 Spring 2004

1. The q-Charlier polynomials may be defined by the 3-term recurrence

$$\begin{aligned} C_{n+1}(x;a,q) =& (x - aq^n - [n]_q)C_n(x;a,q) - a[n]_q C_{n-1}(x;a,q), \\ C_0(x;a,q) =& 1, \quad C_{-1}(x;a,q) = 0. \end{aligned}$$

(a) Establish the generating function

$$\sum_{n=0}^{\infty} C_n(x;a,q)\frac{t^n}{(q;q)_n} = \frac{(at, -t/(1-q);q)_\infty}{(t(x-1/1-q);q)_\infty}$$

where

$$\begin{aligned} [n]_q =& (1-q^n)/(1-q) \\ (A,B,C,\cdots;q)_\infty =& (A;q)_\infty (B;q)_\infty (C;q)_\infty \cdots, \\ (a;q)_n =& \prod_{i=0}^{n-1}(1-aq^i). \end{aligned}$$

(b) Show that

$$C_n(x;a,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} q^{\binom{n-k}{2}} \prod_{i=0}^{k-1}(x - [i]_q).$$

(c) Recall that the unsigned Stirling numbers of the first kind $|s(n,k)|$ count the number of permutations in S_n with exactly k cycles, and

$$\begin{aligned} x(x-1)\cdots(x-n+1) =& \sum_{k=1}^{n} |s(n,k)| x^k (-1)^{n-k} \\ =& \sum_{\sigma \in S_n} x^{\#cycles(\sigma)} (-1)^{n-\#cycles(\sigma)}. \end{aligned} \qquad (1)$$

By defining an appropriate q-version of (1) give a combinatorial interpretation for $C_n(x;a,q)$.

(d) The moments of the Charlier polynomials are $\mu_n = \sum_{k=1}^{n} S(n,k)a^k$. Show that the moments of the q-Charlier polynomials are

$$\mu_n(q) = \sum_{k=1}^{n} S_q(n,k)a^k,$$

for an appropriately defined set of q-Stirling number of the second kind which satisfy

$$S_q(n,k) = S_q(n-1,k-1) + [k]_q S_q(n-1,k).$$

(Possible Hint: Find an orthogonality relation for q-Stirling numbers of the first kind (which you have in (c)) and the second kind (defined above) which implies orthogonality of $C_n(x;a,q)$ to $C_0(x;a,q)$. Or use part (e).)

(e) Recall that an RG-function is a word w such if $i+1$ occurs in w, then i must occur to the left of $i+1$ in w. For exmaple, 112321341 is an RG-function, but 1122423 is not. Also recall that there is a bijection between RG-functions of length n whose entries are exactly $1, 2\cdots, k$ denoted $RG(n, k)$ and set partitions of $[n]$ with exactly k parts. (For example $134|28|5|67 \to 12113442$.)

Using weighted Motzkin paths, show that

$$\mu_n(q) = \sum_{k=1}^{n} \sum_{w \in RG(n,k)} a^k q^{rs(w)} = \sum_{k=1}^{n} S_q(n, k) a^k,$$

where $rs(w) =$ "$right - smaller(w)$". $rs(w)$ is computed in the following way: for each entry $w_i \in w$ find the cardinality of $\{j : j < w_i, j \text{ occurs to the right of } w_i\}$. Then add all values to find $rs(w)$. For example, if $w = 1213114221$, then $rs(w) = 0+1+0+2+0+0+3+1+1$.

(f) Show that the RG-statistic "lb=left-bigger" is equidstributed with rs.

(g) Write down the continued fraction which is the moment generating function.

(h) By considering the Al-Salam-Cralitz polynomials which appear in Chapter 18 of our text, find an explicit representing measure for $C_n(x; a, q)$ when $a > 0$, $0 < q < 1$. What happens if $q \to 1$?

(i)(*) What q-version of non-crossing set partitions gives a nice q-Catalan as moments?