**Math 8680 Homework #1 February 11, 2002**

1. Provide a combinatorial proof of the FACT

$$\sum_{j=0}^{C} \begin{bmatrix} A \\ j \end{bmatrix}_q \begin{bmatrix} B \\ C-j \end{bmatrix}_q q^{j(B-C+j)} = \begin{bmatrix} A+B \\ C \end{bmatrix}_q.$$

2. Recall that we proved that the generating function for partitions inside an $k \times r \times n$ box is

$$A_k(r,n) = det\left( q^{\binom{i-j}{2}} \begin{bmatrix} n+r \\ r-i+j \end{bmatrix}_q \right)_{1 \le i,j \le k}.$$

Evaluate this determinant inductively on $k$ using the Dodgson condensation method.

3. Use the condensation method to prove

$$det\left( \binom{a+i}{b+j} \right)_{1 \le i,j \le n} = \frac{H(a+n+1)H(n)H(a-b)H(b+1)}{H(a+1)H(a-b+n)H(b+n+1)},$$

where

$$H(A) = 1!2! \cdots (A-1)!.$$

4. Recall that the Jacobi-Trudi identity for Schur functions

$$s_\lambda(x_1, \cdots, x_n) = det\left(h_{\lambda_i+j-i}(x_1, \cdots, x_n)\right)_{1 \le i,j \le k}$$

($\lambda$ has at most $k$ parts) may be proven using non-intersecting lattice paths. What Schur function identity results if we use the Pfaffian theorem for the same non-intersecting lattice paths?

5. Prove that the Vandermonde $\lambda$-determinant is

$$\left( det_\lambda(x_j^{n-i}) \right)_{1 \le i,j \le n} = \prod_{1 \le i < j \le n} (x_i + \lambda x_j).$$

6. Let $C_n(x)$ be the generating function for all complete matchings of $\{1, 2, \cdots, 2n\}$ by the crossing number. So $C_1(x) = 1$, $C_2(x) = 2 + x$, $C_3(x) = 5 + 6x + 3x^2 + x^3$. Prove that the constant term of $C_n(x)$ is the nth Catalan number $C_n$, and

$$C_n(x) = (1-x)^{-n} \sum_{j=-n}^{n} \binom{2n}{n+j} (-1)^j x^{\binom{j}{2}}.$$

(This may be difficult.) What is the polynomial for "nestings", rather than crossings?

7. For any partition $\lambda$ with at most $2n$ parts let

$$f_\lambda(x,y) = \sum_{i=0}^{2n-1} (-1)^i x^{2n-1-i+\lambda_{i+1}} y^{i+\lambda_{2n-i}}.$$

(a) Show that $Pf(f_\varnothing(x_i,x_j))_{1 \le i,j \le 2n} = \pm \prod_{1 \le i < j < 2n} (x_i - x_j)$ and determine the sign.

(b) Show that

$$s_\lambda(x_1, \cdots, x_{2n}) = \frac{Pf(f_\lambda(x_i,x_j))_{1 \le i,j \le 2n}}{Pf(f_\varnothing(x_i,x_j))_{1 \le i,j \le 2n}}.$$