MOMENTS OF ASKEY-WILSON POLYNOMIALS

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ABSTRACT. New formulas for the nth moment $\mu_n(a, b, c, d; q)$ of the Askey-Wilson polynomials are given. These are derived using analytic techniques, and by considering three combinatorial models for the moments: Motzkin paths, matchings, and staircase tableaux. A related positivity theorem is given and another one is conjectured.

1. INTRODUCTION

The monic Askey-Wilson polynomials $P_n = P_n(x; a, b, c, d; q)$ are polynomials in x of degree n which depend upon five parameters a, b, c, d, and q. They may be defined by the three-term recurrence $P_{n+1} = (x - b_n)P_n - \lambda_n P_{n-1}$ with $P_{-1} = 0$ and $P_0 = 1$ for $b_n = \frac{1}{2}(a + a^{-1} - (A_n + C_n))$ and $\lambda_n = \frac{1}{4}A_{n-1}C_n$, where

$$A_n = \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})},$$

$$C_n = \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}.$$

We refer to [11] for the standard basic hypergeometric notation and for information about the Askey-Wilson polynomials.

In view of the above three-term recurrence relation, the Askey-Wilson polynomials are orthogonal polynomials. An explicit absolutely continuous measure [4, Theorem 2.2], may be given for these polynomials. (We assume here that $\max\{|a|, |b|, |c|, |d|\} < 1$.) It is supported on $x \in [-1, 1]$, and with $x = \cos\theta$, the measure is

$$w(\cos\theta, a, b, c, d; q) = \frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}.$$

The measure has total mass given by the *Askey-Wilson integral*,

$$(1) \qquad I_0(a, b, c, d) = \frac{(q)_\infty}{2\pi} \int_0^\pi w(\cos\theta, a, b, c, d; q)d\theta = \frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}.$$

The purpose of this paper is to study the nth moment $\mu_n(a, b, c, d; q)$ of the measure $w(x; a, b, c, d; q)$ for the Askey-Wilson polynomials

$$\mu_n(a, b, c, d; q) = C \int_{-1}^1 x^n w(x; a, b, c, d; q)\frac{dx}{\sqrt{1 - x^2}}.$$

for some normalization constant C. With the normalization of $\mu_0(a, b, c, d; q) = 1$, (the explicit C may be found from (1)), the nth moment is known to be a rational function of a, b, c, d and q. We shall give new explicit expressions for $\mu_n(a, b, c, d; q)$ and study three combinatorial models for $\mu_n(a, b, c, d; q)$. One unusual feature of these results is the mixture of binomial and q-binomial terms in the explicit formulas. We shall see why this occurs, both analytically and combinatorially.

The simplest expression for $\mu_n(a,b,c,d;q)$ is a double sum (see [9, Theorem 1.12])

$$(2) \qquad \mu_n(a,b,c,d;q) = \frac{1}{2^n} \sum_{m=0}^{n} \frac{(ab,ac,ad;q)_m}{(abcd;q)_m} q^m \sum_{j=0}^{m} \frac{q^{-j^2} a^{-2j} (aq^j + q^{-j}/a)^n}{(q,q^{1-2j}/a^2;q)_j (q,q^{2j+1}a^2;q)_{m-j}}.$$

However this expression is not obviously symmetric in a, b, c, and d, even though the Askey-Wilson polynomials $P_n(x;a,b,c,d;q)$ and the moments $\mu_n(a,b,c,d;q)$ are symmetric. Nor does it exhibit the correct poles of $\mu_n(a,b,c,d;q)$ as a rational function. For $d = 0$ the moments are polynomials in a, b, c, and q. We give new expressions for the moments $\mu_n(a,b,c,d;q)$, which are symmetric and polynomial when $d = 0$, see Theorem 2.3. We also give a symmetric version for all a,b,c,d in Theorem 2.10, although the polynomial dependence in q is not clear. We give new expressions for the moments $\mu_n(a,b,c,d;q)$ in the special case $b = -a$, $d = -c$, see Theorem 2.7 and Theorem 2.8. We prove a new positivity theorem in Corollary 5.9, and conjecture another one in Conjecture 1.

Our second goal is to combinatorially study the moments $\mu_n(a,b,c,d;q)$ as functions of a, b, c, and d. We use three combinatorial models for this purpose. The moments for any set of orthogonal polynomials may be given as weighted Motzkin paths [29]. In this case $\mu_n(a,b,c,d;q)$ is the generating function for Motzkin paths with weights which are rational functions of a,b,c,d and q. We use this setup and a generalization of an idea of D. Kim [22] to combinatorially prove Corollary 2.4 and Corollary 2.5 in Section 3. A special combinatorial model for the Askey-Wilson integral, which evaluated the normalization constant C, was given in [17]. We modify this model appropriately to give a combinatorial model for some non-normalized moments in Theorem 5.1. We also give new explicit rational expressions for $\mu_n(a,b,c,d;q)$ so that $(abcd)_n \mu_n(a,b,c,d;q)$ are clearly polynomials in a, b, c, d and q, see Theorem 5.2 and Theorem 5.8.

A third combinatorial model is given by Corteel and Williams [10], who give a combinatorial interpretation for the polynomial $2^n(abcd)_n \mu_n(a,b,c,d;q)$ using a rational transformation over the complex numbers of the parameters a, b, c, and d to parameters α, β, γ, and δ. With this transformation, $2^n(abcd)_n \mu_n(a,b,c,d;q)$ is a polynomial in α, β, γ, and δ with positive integer coefficients, which has a combinatorial meaning. (See Section 4). Using their ideas, explicit coefficients of certain terms in $2^n(abcd)_n \mu_n(a,b,c,d;q)$ as Catalan numbers are given in Theorems 4.2 and 4.6.

We conclude in Section 6 with a summary of the special cases of the Askey-Wilson polynomials whose moments had been considered combinatorially.

We give the first combinatorial proof of the formula of Corteel et al. for the moments of q-Laguerre polynomials that is essentially equivalent Corollary 2.5, see Section 6. Josuat-Vergès [19] gave a different combinatorial proof of Corollary 2.4, but our proof is the first combinatorial proof of Corollary 2.5.

2. Askey-Wilson moments

In this section we consider the moments $\mu_n(a,b,c,d;q)$ as functions of the parameters a, b, c, d and q. Our goal is to give new explicit formulas for these moments, using simple series and integral evaluations. We shall prove these specific results for the moments:

- Theorem 2.2 for the general case,
- Theorem 2.3 and its Corollary 2.4 for a symmetric polynomial version when $d = 0$,
- Theorem 2.10, a general symmetric polynomial version,
- Corollary 2.5 and Corollary 2.6, as special cases of Corollary 2.4,
- Theorem 2.8 and Theorem 2.7 for alternative formulas in the symmetric case, and
- the positivity results Proposition 5.5 and Corollary 2.5.

We use (2) and the method of proof of (2) which appears in [9] to prove these results. Positivity Conjecture 1 is also given here.

First we note the polynomial behavior of the moments. An explicit combinatorial version of this proposition, although with complex weights, is given in Proposition 4.1.

Proposition 2.1. $2^n(abcd;q)_n \mu_n(a,b,c,d;q)$ *is a polynomial in* a,b,c,d,q *with integer coefficients.*

Proof. By (2), $2^n(abcd;q)_n\mu_n(a,b,c,d;q)$ is a polynomial in b, c, and d with possibly rational function coefficients in a and q. By symmetry in a, b, c, and d, it is also a polynomial in a, thus a polynomial in a, b, c and d, with coefficients which are rational function of q. The poles of these rational functions must lie on either $|q| = 1$ or $q = 0$. On the other hand $\mu_n(a,b,c,d;q)$ is a polynomial in the three-term recurrence relation coefficients b_n and λ_n of Section 1, with integral coefficients. Since b_n and λ_n have no poles at these locations, $2^n(abcd;q)_n\mu_n(a,b,c,d;q)$ is also a polynomial in q, with integral coefficients. $\square$

Using (2) we prove the first result, which exhibits $2^n(abcd;q)_n\mu_n(a,b,c,d;q)$ as a symmetric polynomial in b, c, and d.

Theorem 2.2. *The Askey-Wilson moments are*

$$2^n\mu_n(a,b,c,d;q) = \sum_{m=0}^{n} \frac{(ab,ac,ad;q)_m}{(abcd;q)_m}(-q)^m \sum_{s=0}^{n+1}\left(\binom{n}{s} - \binom{n}{s-1}\right)$$
$$\times \sum_{p=0}^{n-2s-m} a^{-n+2s+2p}\begin{bmatrix} m+p \\ m \end{bmatrix}_q \begin{bmatrix} n-2s-p \\ m \end{bmatrix}_q q^{(-n+2s+p)m+\binom{m}{2}}.$$

Proof of Theorem 2.2. We shall show that (2) implies Theorem 2.2.

Apply the binomial and q-binomial theorems to find that the j-sum of (2) equals

$$\sum_{j=0}^{m} q^{-j^2}a^{-2j} \cdot (-1)^j q^{2j^2-j-\binom{j}{2}}\begin{bmatrix} m \\ j \end{bmatrix}_q \frac{q^{-jn}a^{-n}(1+a^2q^{2j})^n(1-a^2q^{2j})}{(a^2q^j;q)_{m+1}(q;q)_m}$$
$$= \sum_{j=0}^{m} q^{-jn}a^{-n}(-1)^j q^{\binom{j}{2}}\begin{bmatrix} m \\ j \end{bmatrix}_q \sum_{s=0}^{n+1}\left(\binom{n}{s} - \binom{n}{s-1}\right)a^{2s}q^{js}\sum_{p=0}^{\infty}\begin{bmatrix} m+p \\ m \end{bmatrix}_q \frac{a^{2p}q^{jp}}{(q;q)_m}.$$

Thus the j-sum is summable by the q-binomial theorem to obtain

$$2^n\mu_n(a,b,c,d;q) = \sum_{m=0}^{n} \frac{(ab,ac,ad;q)_m}{(q,abcd;q)_m}q^m \sum_{s=0}^{n+1}\left(\binom{n}{s} - \binom{n}{s-1}\right)$$
$$\times \sum_{p=0}^{\infty} a^{-n+2s+2p}\begin{bmatrix} m+p \\ m \end{bmatrix}_q (q^{-n+2s+p};q)_m.$$

We extend the p and s sums from $-\infty$ to ∞ by declaring these extended binomial and q-binomial coefficients to be zero. Replacing s by $n+1-s$, and then p by $p-n-1+2s$ for the second term of the difference gives

$$2^n\mu_n(a,b,c,d;q) = \sum_{m=0}^{n} \frac{(ab,ac,ad;q)_m}{(q,abcd;q)_m}q^m \sum_{s=0}^{n+1}\binom{n}{s}$$
$$\times \sum_{p=-\infty}^{\infty} a^{-n+2s+2p}\left(\begin{bmatrix} m+p \\ m \end{bmatrix}_q (q^{-n+2s+p};q)_m - \begin{bmatrix} m+p-n-1+2s \\ m \end{bmatrix}_q (q^{p+1};q)_m\right).$$

The first term is zero unless $p \geq 0$ and $(-n+2s+p \leq -m$ or $-n+2s+p \geq 1)$. The second term is zero unless $p-n-1+2s \geq 0$ and $(p+1 \leq -m$ or $p+1 \geq 1)$. If $p \geq 0$ and $-n+2s+p \geq 1$, then the first and the second terms are nonzero and equal. Thus the difference is zero unless $0 \leq p \leq n-2s-m$ for the first term and $0 \leq p-n-1+2s$ and $p+1 \leq -m$ for the second term. Undoing the change of summation variables for the second term then gives Theorem 2.2. $\square$

When $d = 0$, we can obtain an explicit formula for the moments, which is a polynomial in each parameter a, b, c, and q.

Theorem 2.3. *The Askey-Wilson moments for $d = 0$ are*

$$2^n \mu_n(a, b, c, 0; q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right)$$

$$\times \sum_{u+v+w+2t=k} a^u b^v c^w (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ v \end{bmatrix}_q \begin{bmatrix} v+w+t \\ w \end{bmatrix}_q \begin{bmatrix} w+u+t \\ u \end{bmatrix}_q,$$

where the second sum is over all integers $0 \le u, v, w \le k$ and $-k \le t \le k/2$ satisfying $u + v + w + 2t = k$.

Proof of Theorem 2.3. We show that Theorem 2.3 follows from Theorem 2.2.

In Theorem 2.2 expand the terms $(ab; q)_m$ and $(ac; q)_m$ as polynomial in a, b and c by the q-binomial theorem to obtain

$$2^n \mu_n(a, b, c, 0; q) = \sum_{s=0}^{n+1} \left(\binom{n}{s} - \binom{n}{s-1} \right) \sum_{p=0}^{n-2s} a^{-n+2s+2p} \cdot X,$$

where

$$X = \sum_{u,v \ge 0} (-1)^{u+v} q^{\binom{u}{2} + \binom{v}{2}} \frac{a^{u+v} b^u c^v}{(q; q)_u (q; q)_v} \sum_{m=0}^{n-2s-p} \frac{(q^{p+1}, q^{-n+2s+p}; q)_m q^m}{(q; q)_{m-u} (q; q)_{m-v}}.$$

The m-sum is summable by the q-Vandermonde identity [11, II.6] and we get

$$X = \sum_{u,v \ge 0} (-1)^{n-2s-p-u-v} q^{\binom{n-2s-p-u-v-1}{2}} a^{u+v} b^u c^v \begin{bmatrix} n-2s-p \\ v \end{bmatrix}_q \begin{bmatrix} p+u \\ u \end{bmatrix}_q \begin{bmatrix} p+v \\ n-2s-p-u \end{bmatrix}_q.$$

By replacing p and s with new summation variables $t = n - 2s - p - u - v$, $w = -n + 2s + 2p + u + v$, and $k = n - 2s$, we obtain Theorem 2.3. $\square$

The special case $c = 0$ of Theorem 2.3 is equivalent to a result of Josuat-Vergès [19, Theorem 6.1.1].

Corollary 2.4. *The Askey-Wilson moments for $c = d = 0$ are*

$$2^n \mu_n(a, b, 0, 0; q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u, v, t \end{bmatrix}_q,$$

where the second sum is over all nonnegative integers u, v, t satisfying $u + v + 2t = k$.

In the next section we give a combinatorial proof of Corollary 2.4. Finding a combinatorial proof of Theorem 2.3 is still open.

Problem 1. Find a combinatorial proof of Theorem 2.3.

There are two interesting special cases of Corollary 2.4 which are the next two corollaries.

Corollary 2.5. *We have*

$$2^n \mu_n(a, q/a, 0, 0; q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) (q/a)^k \sum_{i=0}^{k} a^{2i} q^{i(k-i-1)}.$$

Corollary 2.6. *We have $\mu_{2n+1}(a, -a, 0, 0; q) = 0$ and*

$$4^n \mu_{2n}(a, -a, 0, 0; q) = \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) \sum_{i=0}^{k} (-1)^i q^{\binom{i+1}{2}} (q; q^2)_{k-i} a^{2k-2i} \begin{bmatrix} 2k-i \\ i \end{bmatrix}_q.$$

Proof of Corollary 2.5. Put $b = q/a$ in Corollary 2.4, and substitute $v = k - 2t - u$, and then $t + u = i$. The resulting t-sum is evaluable by the q-Vandermonde sum [11, II.6]. $\square$

Corollary 2.5 explicitly proves that $2^n \mu_n(a, q/a, 0, 0; q)$ is a Laurent polynomial in a, with coefficients that are polynomials in q, with positive integer coefficients. Corollary 5.9 its analogue for $\mu_n(a, q/a, c, q/c; q)$.

Proof of Corollary 2.6. We can rewrite (3) as

$$2^n\mu_n(a,-a,0,0;q) = \sum_{k=0}^{n}\left(\binom{n}{\frac{n-k}{2}}-\binom{n}{\frac{n-k}{2}-1}\right)\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^iq^{\binom{i+1}{2}}\begin{bmatrix}k-i\\k-2i\end{bmatrix}_q a^{k-2i}\sum_{j=0}^{k-2i}(-1)^j\begin{bmatrix}k-2i\\j\end{bmatrix}_q.$$

Then we are done by the Gaussian formula [2, Theorem 10, p.71]

$$\sum_{i=0}^{n}(-1)^i\begin{bmatrix}n\\i\end{bmatrix}_q = \begin{cases} 0, & \text{if } n \text{ is odd,}\\ (q;q^2)_{n/2}, & \text{if } n \text{ is even.}\end{cases}$$

□

Another special case of the Askey-Wilson polynomials has a different expression for the moments. Consider $b = -a$ and $d = -c$, so that the Askey-Wilson measure is symmetric about the y-axis. In this case $b_n = 0$, so the odd moments are zero, and the $2n$th moment has a shorter alternative expression.

Theorem 2.7. *The non-zero Askey-Wilson moments for $b = -a$ and $d = -c$ are*

$$4^n\mu_{2n}(a,-a,c,-c;q) = \sum_{m=0}^{n}\frac{(-a^2;q)_{2m}(a^2c^2;q^2)_m}{(qa^2c^2;q^2)_m}q^{2m}\sum_{j=0}^{m}\frac{a^{-4j}q^{-2j^2}(aq^j+a^{-1}q^{-j})^{2n}}{(q^2,a^4q^{2+4j};q^2)_{m-j}(q^2,a^{-4}q^{2-4j};q^2)_j}.$$

Proof. We use the idea of the proof of [9, Proposition 3.1]. Let $y = 2x^2 - 1 = \cos(2\theta)$. We need to find the coefficients A_m in the expansion

$$\left(\frac{y+1}{2}\right)^n = x^{2n} = \sum_{m=0}^{n}A_m(a^2e^{2i\theta},a^2e^{-2i\theta};q^2)_m.$$

This may be done using the q-Taylor theorem, see [15, Theorem 1.1]. The resulting Askey-Wilson integral is evaluated as in [9, Lemma 3.2]. We do not give the details. □

Theorem 2.7 can be simplified in exactly the same way that (2) implied Theorem 2.2 to obtain Theorem 2.8. Again the details are not given.

Theorem 2.8. *The non-zero Askey-Wilson moments for $b = -a$ and $d = -c$ are*

$$4^n\mu_{2n}(a,-a,c,-c;q) = \sum_{m=0}^{n}\frac{(-a^2;q)_{2m}(a^2c^2;q^2)_m}{(qa^2c^2;q^2)_m}(-q^2)^m$$

$$\times\sum_{s=0}^{2n+2}\left(\binom{2n+1}{s}-\binom{2n+1}{s-1}\right)\sum_{p=0}^{n-m-s}a^{-2n+4p+2s}\begin{bmatrix}m+p\\m\end{bmatrix}_{q^2}\begin{bmatrix}n-p-s\\m\end{bmatrix}_{q^2}q^{-2m(n-p-s)+m(m-1)}.$$

There is a corresponding positivity conjecture for the moments $\mu_{2n}(a,-a,c,-c;q)$. These moments are not polynomials, but Proposition 2.1 implies that $4^n(a^2c^2;q)_{2n}\mu_{2n}(a,-a,c,-c;q)$ is a polynomial in a, c, and q. Half of the apparent poles of $\mu_{2n}(a,-a,c,-c;q)$ do not occur.

Proposition 2.9. *The Askey-Wilson moments*

$$\tau_{2n}(a^2,c^2) = 4^n(qa^2c^2;q^2)_n\mu_{2n}(a,-a,c,-c;q)/(1-q)^n$$

are polynomials in a^2, c^2 and q with integer coefficients. Moreover the sum of the coefficients in $\tau_{2n}(a^2,c^2)$ is $2^{2n}(2n-1)(2n-3)\cdots1$.

Proof. After rescaling the Askey-Wilson polynomials, we have $\tau_{2n}(a^2,c^2) = (qa^2c^2;q^2)_n\nu_{2n}$, where ν_{2n} are the $2n$th moments for the orthogonal polynomials defined by

$$p_{n+1}(x) = xp_n(x) - \lambda_np_{n-1}(x),$$

where

$$\lambda_n = \frac{(1+a^2q^{n-1})(1+c^2q^{n-1})(1-a^2c^2q^{n-2})}{(1-a^2c^2q^{2n-3})(1-a^2c^2q^{2n-1})}[n]_q.$$

From Viennot's theory [29], ν_{2n} is the generating function for Dyck paths from $(0,0)$ to $(2n,0)$ with weights given by a product of $\lambda'_i s$. So

$$\nu_{2n} = \frac{Q(a^2, c^2, q)}{\prod_{i=1}^{n}(1 - a^2 c^2 q^{2i-1})^{n+1-i}}$$

for some polynomial Q with integer coefficients. By Theorem 2.7, ν_{2n} has simple poles, as a function of c^2, possibly only at $(qa^2c^2; q^2)_n = 0$. So $(qa^2c^2; q^2)_n \nu_{2n}$ is a polynomial in a^2, c^2 and q, with integer coefficients.

For the final assertion, if $q = 1$, $\lambda_n = n(1+a^2)(1+c^2)/(1-a^2c^2)$, which is the three term recurrence coefficient for a rescaled Hermite polynomial. The $2n^{\text{th}}$ moment for the Hermite polynomials is $(2n-1)(2n-3)\cdots 1$. □

Conjecture 1. The coefficients of $\tau_{2n}(a^2, c^2)$ are non-negative integers.

If $q = 0$, one may show that $\tau_{2n}(a^2, c^2)$ is a non-negative polynomial by a combinatorial method. The sum of the coefficients is 2^{2n}. It is a generating function for certain non-crossing complete matchings.

Although simple, (2) does not clearly demonstrate the symmetry or polynomiality of $\mu_n(a, b, c, d; q)$ in all four parameters a, b, c and d. We next give such a formula, which generalizes Theorem 2.2.

Let A be an arbitrary parameter. Let

$$_8W_7(m) = {}_8W_7(A^2/q; A/a, A/b, A/c, A/d, q^{-m}; q; abcdq^m).$$

Note that $(aA, bA, cA, dA; q)_m \; {}_8W_7(m)$ is a symmetric polynomial in a, b, c, d:

$$(3) \qquad (aA, bA, cA, dA; q)_m \; {}_8W_7(m) = \sum_{j=0}^{m} \frac{(A^2/q; q)_j}{(A^2 q^m; q)_j} \frac{1 - A^2 q^{2j-1}}{1 - A^2/q} \begin{bmatrix} m \\ j \end{bmatrix}_q (-1)^j q^{\binom{j}{2}}$$

$$\times (abcd)^j (A/a, A/b, A/c, A/d; q)_j (Aaq^j, Abq^j, Acq^j, Adq^j; q)_{m-j}.$$

Using Watson's transformation [11, (III.17)] of an $_8W_7$ to a $_4\phi_3$, the following apparent rational function of A and q is in fact a polynomial in each of the parameters: a, b, c, d, A, and q.

$$\frac{(aA, bA, cA, dA; q)_m}{(A^2; q)_m} \; {}_8W_7(m) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q (cd)^j (A/c, A/d; q)_j (ab; q)_j$$

$$\times (Aaq^j, Abq^j, cd; q)_{m-j}.$$

The next result gives a symmetric polynomial version for $2^n(abcd; q)_n \mu_n(a, b, c, d; q)$. Theorem 2.10 is independent of A.

Theorem 2.10.

$$2^n \mu_n(a, b, c, d; q) = \sum_{m=0}^{n} \frac{(aA, bA, cA, dA; q)_m}{(A^2, abcd; q)_m} (-q)^m \; {}_8W_7(m) \sum_{s=0}^{n+1} \left(\binom{n}{s} - \binom{n}{s-1} \right)$$

$$\times \sum_{p=0}^{n-2s-m} A^{-n+2s+2p} \begin{bmatrix} m+p \\ m \end{bmatrix}_q \begin{bmatrix} n-2s-p \\ m \end{bmatrix}_q q^{m(-n+2s+p)+\binom{m}{2}}.$$

Proof. We again use the q-Taylor theorem and follow the proof of [9, Proposition 3.1]. We expand x^n in terms of the basis $(Ae^{i\theta}, Ae^{-i\theta}; q)_k$. The resulting integral is a special case of the Nasrallah-Rahman integral [11, (6.3.9), p. 158]. An analogue of (2) is obtained. As before, the same steps with the binomial and q-binomial theorems yield the stated result. □

If $A = a$, then Theorem 2.10 becomes Theorem 2.2. Theorem 2.10 has one defect: not all of the powers of q are positive due to the $q^{m(-n+2s+p)}$ term. The individual terms are Laurent polynomials in q.

If $A^2 = q$, the p-sum in Theorem 2.10 is evaluable by the q-Vandermonde sum [11, II.6].

Corollary 2.11. *If $A^2 = q$,*

$$2^n \mu_n(a,b,c,d;q) = \sum_{m=0}^{n} \frac{(aA, bA, cA, dA; q)_m}{(A^2, abcd; q)_m} (-q)^m {}_8W_7(m) \sum_{s=0}^{n+1} \left(\binom{n}{s} - \binom{n}{s-1} \right)$$
$$\times A^{-n+2s} \begin{bmatrix} n+m+1-2s \\ 2m+1 \end{bmatrix}_q q^{-nm+2sm+\binom{m}{2}}.$$

An explicit polynomial version of Theorem 2.3, which is unwieldy, may be given by expanding each of the eight factors in (3) by the q-binomial theorem. The resulting formula would appear to be too complex to be useful.

3. Weighted Motzkin paths

In this section we give combinatorial proofs of Corollary 2.4 and Corollary 2.5, the explicit polynomial versions of the moments with 2 parameters. The combinatorial model we shall use is Motzkin paths, based on the three-term recurrence formula,

A sketch of the proof is as follows. We first interpret the moment $2^n \mu_n(a,b,0,0;q)$ as a weighted sum of Motzkin paths. Then using Penaud's decomposition [26] we can decompose a weighted Motzkin path into a pair of paths: a Dyck prefix and another weighted Moztkin path. The Dyck prefixes give a difference of binomial coefficients. We map the new weighted Moztkin path to a new object: doubly striped skew shapes. These objects are a generalization of striped skew shapes introduced by D. Kim [22] in order to prove the moment formula for Al-Salam-Carlitz polynomials. We then find a sign-reversing involution on doubly striped skew shapes and show that the fixed points have a weighted sum equal to a q-trinomial coefficient. Thus we have found the differences of binomial coeffcients and the q-trinomial which appear in Corollary 2.4. For Corollary 2.5, we find a further cancellation on the doubly striped skew shapes which leaves only one fixed point for given size.

A *Motzkin path* is a lattice path in $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $(n,0)$ consisting of up steps $(1,1)$, down steps $(1,-1)$, and horizontal steps $(1,0)$. We say that the *level* of a step is i if it is an up step or a down step between the lines $y = i-1$ and $y = i$, or it is a horizontal step on the line $y = i$. A *weighted Motzkin path* is a Motzkin path in which each step has a certain weight. The *weight* $\mathrm{wt}(p)$ of a weighted Motzkin path p is the product of the weights of all steps. Note that the level of an up step or a down step is at least 1 and the level of a horizontal step may be 0.

Let $\mathrm{Mot}_n(a,b)$ denote the set of weighted Motzkin paths of length n such that

- the weight of an up step of level i is either q^i or -1,
- the weight of a down step of level i is either abq^{i-1} or -1,
- the weight of a horizontal step of level i is either aq^i or bq^i.

Then by Viennot's theory [29] we have

$$(4) \qquad 2^n \mu_n(a,b,0,0;q) = \sum_{P \in \mathrm{Mot}_n(a,b)} \mathrm{wt}(P).$$

We define $\mathrm{Mot}_n^*(a,b)$ to be the set of weighted Motzkin paths in $\mathrm{Mot}_n(a,b)$ such that there is no peak of weight 1, that is, an up step of weight -1 immediately followed by a down step of weight -1.

By the same idea that is a variation of Penaud's decomposition as in [18, Proposition 5.1], we have

$$(5) \qquad \sum_{P \in \mathrm{Mot}_n(a,b)} \mathrm{wt}(P) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) \sum_{P \in \mathrm{Mot}_k^*(a,b)} \mathrm{wt}(P).$$

By (4) and (5), Corollary 2.4 and Corollary 2.5 can be restated as follows.
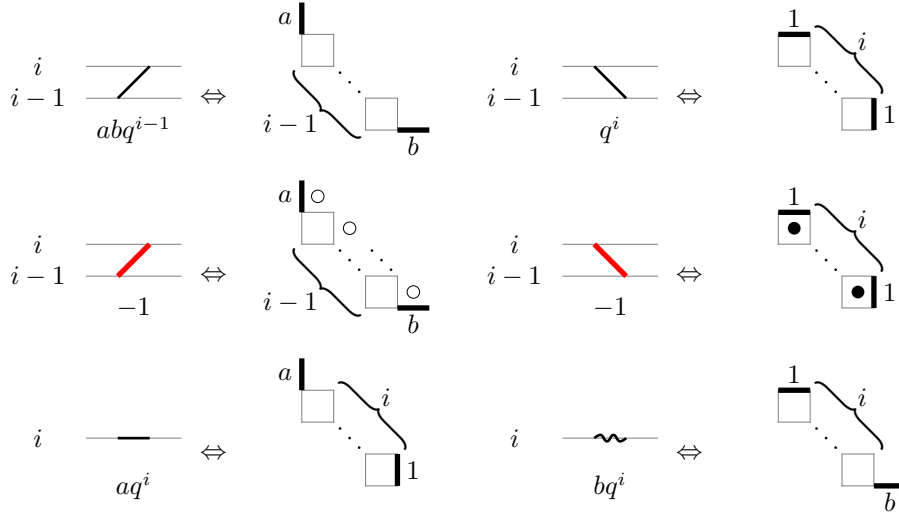
FIGURE 1. Converting a weighted Motzkin path to a doubly striped skew shape.

Theorem 3.1. *We have*

$$\sum_{P\in\mathrm{Mot}_k^*(a,b)} \mathrm{wt}(P) = \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u,v,t \end{bmatrix}_q,$$

$$\sum_{P\in\mathrm{Mot}_k^*(a,q/a)} \mathrm{wt}(P) = (q/a)^k \sum_{i=0}^{k} a^{2i} q^{i(k-i-1)}.$$

We note that the second identity in Theorem 3.1 is equivalent to Proposition 5 in [8] which is used to prove the formula (19) for the moments of q-Laguerre polynomials. Corteel et al.'s proof of [8] is combinatorial except for the proof of Proposition 5 in [8]. In this section we prove the above theorem combinatorially, thus providing the first combinatorial proof of (19).

In order to give combinatorial proofs of the two formulas in Theorem 3.1 we introduce doubly striped skew shapes. These objects are a generalization of striped skew shapes introduced by D. Kim [22].

A *doubly striped skew shape* of size $m \times n$ is a quadruple (λ, μ, W, B) of partitions $\mu \subset \lambda \subset (n^m)$ and a set W of white stripes and a set B of black stripes with $W \cap B = \emptyset$. Here, a *white stripe* is a diagonal set S of λ/μ such that λ/μ contains neither the cell to the left of the leftmost cells of S nor the cell below the rightmost cell of S, where a diagonal set means a set of cells in row $r+i$ and column $s+i$ for $i = 1, 2, \ldots, p$ for some integers r, s, p. Similarly, a *black stripe* is a diagonal set S of λ/μ such that λ/μ contains neither the cell above the leftmost cell of S and the cell to the right of the rightmost cell S. We will call a cell in a white stripe (resp. black stripe) a *white dot* (resp. *black dot*).

Let $\mathrm{DSS}(m,n)$ denote the set of doubly striped skew shapes of size $m \times n$. We define the *weight* of $(\lambda, \mu, W, B) \in \mathrm{DSS}(m,n)$ to be

$$(6) \qquad \mathrm{wt}_{a,b}(\lambda, \mu, W, B) = a^m b^n (-1)^{|W|+|B|} q^{|\lambda/\mu|-\|W\|-\|B\|} (q/ab)^{|W|},$$

where $\|W\|$ and $\|B\|$ are the total numbers of white dots and black dots respectively.

There is a simple correspondence between $\mathrm{Mot}_k^*(a,b)$ and doubly striped skew shapes.

Lemma 3.2. *We have*

$$\sum_{P\in\mathrm{Mot}_k^*(a,b)} \mathrm{wt}(P) = \sum_{i=0}^{k} \sum_{S\in\mathrm{DSS}(i,k-i)} \mathrm{wt}_{a,b}(S).$$

FIGURE 2. An example of the bijection ρ. The up steps of weight y and the down steps of weight 1 are colored red.

Proof. It is enough to show that there is a weight-preserving bijection $\rho : \mathrm{Mot}_k^*(a, b) \to \bigcup_{i=0}^k \mathrm{DSS}(i, k-i)$. Let $P \in \mathrm{Mot}_k^*(a, b)$. We will construct an upper path and a lower path which determine two partitions λ and μ respectively. The upper path and the lower path start at the origin. For each step of P, we add one step to the two lattice paths as follows. If the step is an up step of weight abq^{i-1}, we add a north step of weight a to the upper path and an east step of weight b to the lower path. If the step is an up step of weight -1, we add a north step of weight a to the upper path and an east step of weight b to the lower path, and we make a white stripe between these two steps, see Figure 1. Similarly we add one step to the upper and lower paths for other type of step in P as shown in Figure 1. Then we define $\rho(P)$ to be the resulting diagram. See Figure 2 for an example of ρ. It is easy to see that ρ is a weight-preserving bijection. $\qquad\square$

By Lemma 3.2, Theorem 3.1 is equivalent to the following proposition.

Proposition 3.3. *For $k \geq 0$, we have*

(7)
$$\sum_{i=0}^k \sum_{S \in \mathrm{DSS}(i, k-i)} \mathrm{wt}_{a,b}(S) = \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u, v, t \end{bmatrix}_q,$$

(8)
$$\sum_{i=0}^k \sum_{S \in \mathrm{DSS}(i, k-i)} \mathrm{wt}_{a,q/a}(S) = (q/a)^k \sum_{i=0}^k a^{2i} q^{i(k-i-1)}.$$

In order to prove Proposition 3.3 we need D. Kim's sign-reversing involution. We now recall his involution. By a sign-reversing involution on $\mathrm{DSS}(m, n)$ we mean an involution $f : \mathrm{DSS}(m, n) \to \mathrm{DSS}(m, n)$ such that if $S \in \mathrm{DSS}(m, n)$ is not a fixed point, i.e. $f(S) \neq S$, then $\mathrm{wt}(f(S)) = -\mathrm{wt}(S)$.

Let $\mathrm{DSS}(m, n; \alpha, \beta, \gamma, \delta)$ denote the set of $(\lambda, \mu, W, B) \in \mathrm{DSS}(m, n)$ such that $\lambda = \alpha$, $\mu = \beta$, $W = \gamma$, and $B = \delta$, where α can be "$-$" and in this case there is no restriction on λ, and it is similar for β, γ, and δ. For example, $\mathrm{DSS}(m, n; \lambda, -, \emptyset, -)$ is the set of $(\lambda, \mu, \emptyset, B) \in \mathrm{DSS}(m, n)$ for all possible μ and B.

The following is proved by D. Kim [22, Theorem 3.2]. Since we will use his proof we include it here.

Lemma 3.4. *For a partition $\lambda \subset (n^m)$, we have*

$$\sum_{S \in \mathrm{DSS}(m,n;\lambda,-,\emptyset,-)} \mathrm{wt}_{a,b}(S) = \sum_{S \in \mathrm{DSS}(m,n;\lambda,\emptyset,\emptyset,\emptyset)} \mathrm{wt}_{a,b}(S) = a^m b^n q^{|\lambda|}.$$

Proof. We construct a sign-reversing involution $\phi : \mathrm{DSS}(m, n; \lambda, -, \emptyset, -) \to \mathrm{DSS}(m, n; \lambda, -, \emptyset, -)$ as follows.

Let $(\lambda, \mu, \emptyset, B) \in \mathrm{DSS}(m, n; \lambda, -, \emptyset, -)$. Suppose μ has r nonempty rows and the lowest row containing a black dot is row s. If there is no black stripe, then we let $s = 0$. If $r = s = 0$, equivalently $\mu = B = \emptyset$, we define $(\lambda, \mu, \emptyset, B)$ to be a fixed point. Otherwise, we define $\phi(\lambda, \mu, \emptyset, B)$ as follows.

Case 1: If $s \geq r$, move the dots in the stripe containing a dot in row s (there must be a unique such stripe) all the way to the left and fill the cells containing these dots.

Case 2: If $s < r$, we remove the leftmost cells of the last t rows in μ and build a black stripe of size t ending in row r for the unique integer t for which this operation is possible.
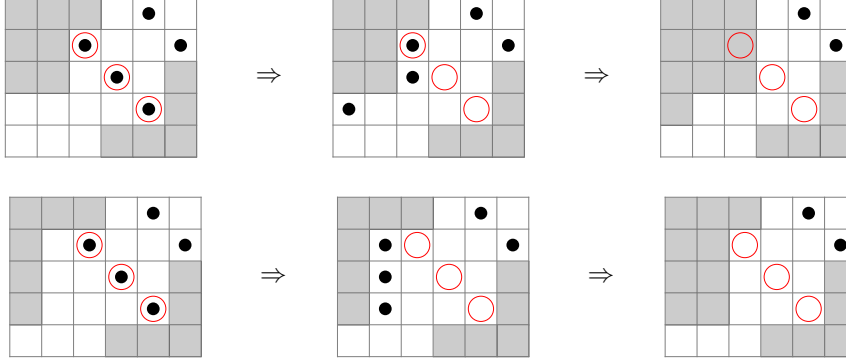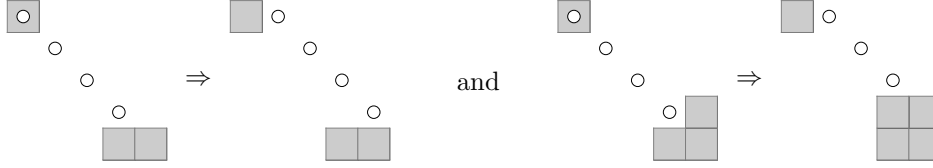
FIGURE 3. Two examples of the involution ϕ.

FIGURE 4. Slide rules

See Figure 3 for some examples of this map. It is easy to check that ϕ is a sign-reversing involution on $\mathrm{DSS}(m,n;\lambda,-,\emptyset,-)$ with only one fixed point $(\lambda,\emptyset,\emptyset,\emptyset)$, which implies the desired identity. $\qquad\square$

The involution ϕ of D. Kim cannot be directly applied to doubly striped skew shapes with white stripes. However, in the proof of the next lemma, we show that there is a simple way to extend his involution when we have white stripes.

Lemma 3.5. *We have*

$$\sum_{S\in\mathrm{DSS}(m,n)}\mathrm{wt}_{a,b}(S)=\sum_{S\in\mathrm{DSS}(m,n;-,\emptyset,-,\emptyset)}\mathrm{wt}_{a,b}(S).$$

Proof. Let $(\lambda,\mu,W,B)\in\mathrm{DSS}(m,n)$. Ignoring the white stripes, we apply the involution ϕ. Then the newly added cells may overlap with white stripes. In this case we slide each overlapped white stripe one step to the right. If the rightmost dot in the stripe cannot be moved, then we delete this dot and the cell containing it from λ and slide the white strip. See Figure 4. We define $\psi(\lambda,\mu,W,B)$ to be the resulting doubly striped skew shape.

One can easily see that ψ is a sign-reversing involution on $\mathrm{DSS}(m,n)$ with fixed point set $\mathrm{DSS}(m,n;-,\emptyset,-,\emptyset)$. Thus we get the desired identity. $\qquad\square$

Now we are ready to prove Proposition 3.3.

Proof of (7). By Lemma 3.5 we have

$$\sum_{i=0}^{k}\sum_{S\in\mathrm{DSS}(i,k-i)}\mathrm{wt}_{a,b}(S)=\sum_{i=0}^{k}\sum_{S\in\mathrm{DSS}(i,k-i;-,\emptyset,-,\emptyset)}\mathrm{wt}_{a,b}(S)$$

$$=\sum_{u+v+2t=k}a^{u}b^{v}(-1)^{t}q^{t}\sum_{(\lambda,\emptyset,W,\emptyset)\in\mathrm{DSS}(u+t,v+t),|W|=t}q^{|\lambda|-\|W\|}.$$

It remains to show that for fixed nonnegative integers u,v,t we have

(9) $$\sum_{(\lambda,\emptyset,W,\emptyset)\in\mathrm{DSS}(u+t,v+t),|W|=t}q^{|\lambda|-\|W\|}=q^{\binom{t}{2}}\begin{bmatrix}u+v+t\\u,v,t\end{bmatrix}_{q}.$$
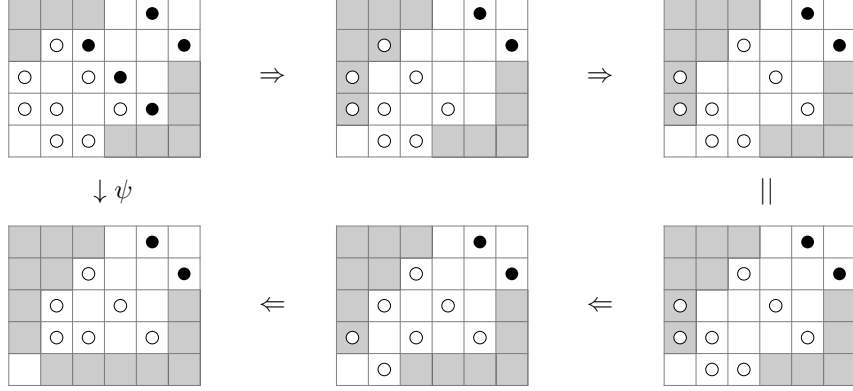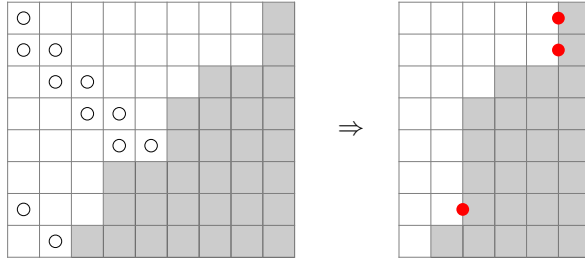
FIGURE 5. An example of the involution ψ.

FIGURE 6. An example of the intermediate step of the map from $\mathrm{DSS}(u+t, v+t)$ to $S(0^u, 1^t, 2^v)$. The word corresponding to this example is 2021000202211.

Suppose $(\lambda, \emptyset, W, \emptyset) \in \mathrm{DSS}(u + t, v + t)$ and $|W| = t$. For each white stripe we mark the row containing the leftmost white dot and chop off the column containing the rightmost white dot, see Figure 6. Then the resulting partition is contained in a $(u + t) \times v$ rectangle. Consider this partition as a lattice path from $(0, 0)$ to $(v, u+t)$ with three steps: an east step, a north step, and a marked north step. Here a marked north step corresponds to a marked row of the partition. Then we define the word $w = w_1 w_2 \ldots w_{u+v+t}$ by $w_i = 0, 1$, or 2 if the ith step is a north step, a marked north step, or an east step respectively. Then w is a permutation of u 0's, t 1's and v 2's. It is not hard to see that $(\lambda, \emptyset, W, \emptyset) \mapsto w$ is a bijection from $\mathrm{DSS}(u + t, v + t)$ to the set $S(0^u, 1^t, 2^v)$ of permutations of u 0's, t 1's and v 2's such that

$$|\lambda| - \|W\| = \mathrm{inv}(w) + \binom{t}{2},$$

where $\mathrm{inv}(w)$ is the number of pairs (i, j) of integers such that $i < j$ and $w_i > w_j$. Since

$$\sum_{w \in S(0^u, 1^t, 2^v)} q^{\mathrm{inv}(w)} = \begin{bmatrix} u + v + t \\ u, v, t \end{bmatrix}_q,$$

we get (9).

$\square$

Proof of (8). For $S \in \mathrm{DSS}(m, n)$, let S' be the doubly striped skew shape obtained by rotating S by an angle of $180°$ and exchanging blacks stripes and white stripes. The map $S \mapsto S'$ is clearly a bijection. If $S = (\lambda, \mu, W, B) \in \mathrm{DSS}(m, n)$, then $S' = (\lambda', \mu', W', B') \in \mathrm{DSS}(m, n)$ satisfies $\lambda' = (n^m) - \mu$, $\mu' = (n^m) - \lambda$, $|W'| = |B|$, $|B'| = |W|$, $\|W'\| = \|B\|$, and $\|B'\| = \|W\|$. Thus $\mathrm{wt}_{a,b}(S) = \mathrm{wt}_{a,b}(S')(q/ab)^{|W|-|B|}$. In particular, we have $\mathrm{wt}_{a,q/a}(S) = \mathrm{wt}_{a,q/a}(S')$.

FIGURE 7. A staircase tableau and the labeling of its empty cells.

By Lemma 3.5, the map $S \mapsto S'$ and Lemma 3.4, we have

$$
\sum_{i=0}^{k} \sum_{S \in \mathrm{DSS}(i,k-i)} \mathrm{wt}_{a,q/a}(S) = \sum_{i=0}^{k} \sum_{S \in \mathrm{DSS}(i,k-i;-,\emptyset,-,\emptyset)} \mathrm{wt}_{a,q/a}(S)
$$

$$
= \sum_{i=0}^{k} \sum_{S' \in \mathrm{DSS}(i,k-i;((k-i)^i),-,\emptyset,-)} \mathrm{wt}_{a,q/a}(S')
$$

$$
= \sum_{i=0}^{k} \sum_{S' \in \mathrm{DSS}(i,k-i;((k-i)^i),\emptyset,\emptyset,\emptyset)} \mathrm{wt}_{a,q/a}(S') = \sum_{i=0}^{k} a^i (q/a)^{k-i} q^{i(k-i)},
$$

which finishes the proof. □

4. STAIRCASE TABLEAUX

In this section we review the second combinatorial model for the moments of the Askey-Wilson polynomials, called staircase tableaux. The staircase tableaux were first introduced in [10] and further studied in [9]. Using the staircase tableaux we shall find the coefficient of the highest term in $2^n (abcd; q)_n \mu_n(a, b, c, d; q)$ in Theorem 4.2.

A *staircase tableau of size n* is a filling of the Young diagram of the staircase partition $(n, n - 1, \ldots, 1)$ with $\alpha, \beta, \gamma, \delta$ such that

- every diagonal cell is nonempty,
- all cells above an α or γ in the same column are empty,
- all cells to the left of a β or δ in the same row are empty.

Here a *diagonal cell* is a cell in the ith row and $(n + 1 - i)$th column for some $i \in \{1, 2, \ldots, n\}$.

We denote by $\mathcal{T}(n)$ the set of staircase tableaux of size n.

Each empty cell s of $T \in \mathcal{T}(n)$ is labeled uniquely as follows. Here, for brevity let right(s) be the first nonempty cell to the right of s in the same row, and below(s) the first nonempty cell below s in the same column.

- If right(s) has a β, then s is labeled with u.
- If right(s) has a δ, then s is labeled with q.
- If right(s) has an α or γ, and below(s) has an α or δ, then s is labeled with u.
- If right(s) has an α or γ, and below(s) has an β or γ, then s is labeled with q.

See Figure 7 for an example of a staircase tableau and the labeling of its empty cells.

For $T \in \mathcal{T}(n)$, we define b(T) to be the number of α's and δ's on the diagonal cells, and $A(T)$, $B(T)$, $C(T)$, $D(T)$, $E(T)$ to be the number of α's, β's, γ's, δ's, empty cells labeled with q in T respectively. For example, if T is the staircase tableau in Figure 7, we have b(T) = 3, $A(T) = 2$, $B(T) = 3$, $C(T) = 3$, $D(T) = 3$, and $E(T) = 11$.

Let

$$
Z_n(y; \alpha, \beta, \gamma, \delta; q) = \sum_{T \in \mathcal{T}(n)} y^{\mathrm{b}(T)} \alpha^{A(T)} \beta^{B(T)} \gamma^{C(T)} \delta^{D(T)} q^{E(T)}.
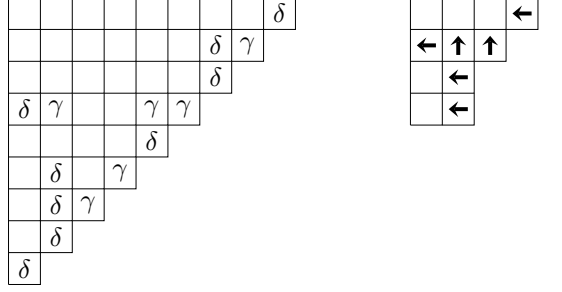$$

FIGURE 8. A staircase tableau without α's and β's and the corresponding alternative tableau.

Corteel et al. [9] showed that

$$(10) \qquad \mu_n(a,b,c,d;q) = \frac{(1-q)^n}{2^n i^n \prod_{j=0}^{n-1}(\alpha\beta - \gamma\delta q^j)} Z_n(-1;\alpha,\beta,\gamma,\delta;q),$$

where

$$\alpha = \frac{1-q}{(1+ai)(1+ci)}, \quad \beta = \frac{1-q}{(1-bi)(1-di)}, \quad \gamma = \frac{ac(1-q)}{(1+ai)(1+ci)}, \quad \delta = \frac{bd(1-q)}{(1-bi)(1-di)}.$$

Since

$$\alpha\beta - \gamma\delta q^j = \frac{(1-q)^2(1-abcdq^j)}{(1+ai)(1+ci)(1-bi)(1-di)}$$

we can rewrite (10) as the next proposition.

Proposition 4.1. *The Askey-Wilson moments satisfy*

$$2^n(abcd;q)_n\mu_n(a,b,c,d;q) = i^{-n} \sum_{T\in\mathcal{T}(n)} (-1)^{b(T)}(1-q)^{A(T)+B(T)+C(T)+D(T)-n}q^{E(T)}$$

$$\times (ac)^{C(T)}(bd)^{D(T)}\big((1+ai)(1+ci)\big)^{n-A(T)-C(T)}\big((1-bi)(1-di)\big)^{n-B(T)-D(T)}.$$

The highest degree term appearing in Proposition 4.1 is $a^n b^n c^n d^n q^{\binom{n}{2}}$. This term can be obtained when $A(T)+B(T)+C(T)+D(T)+E(T) = \binom{n+1}{2}$ and $A(T) = B(T) = 0$, and the coefficient is

$$i^{-n}(-1)^{b(T)}(-1)^{C(T)+D(T)-n}(-1)^{n-C(T)}(-1)^{n-D(T)} = i^n(-1)^{b(T)}.$$

Thus

$$(11) \qquad \left[a^n b^n c^n d^n q^{\binom{n}{2}}\right] 2^n(abcd;q)_n\mu_n(a,b,c,d;q) = \sum_{T\in\mathcal{CT}(n)} i^n(-1)^{b(T)},$$

where $\mathcal{CT}(n)$ is the set of $T \in \mathcal{T}(n)$ with $A(T) = B(T) = 0$ and $C(T)+D(T)+E(T) = \binom{n+1}{2}$. We will see below that the tableaux in $\mathcal{CT}(n)$ are effectively the same as Catalan tableaux in [30].

For the remainder of this section we will prove the following theorem.

Theorem 4.2. *We have*

$$\left[a^n b^n c^n d^n q^{\binom{n}{2}}\right] 2^n(abcd;q)_n\mu_n(a,b,c,d;q) = \operatorname{Cat}\left(\frac{n}{2}\right),$$

where $\operatorname{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$ *if n is a nonnegative integer, and* $\operatorname{Cat}(n) = 0$ *otherwise.*

In order to prove the above theorem we recall alternative tableaux, which are in simple bijection with permutation tableaux, see [25, 31].

An *alternative tableau* is a filling of a Young diagram with possibly empty rows and columns with up arrows and left arrows such that there is no arrow pointing to another arrow. The *size* of an alternative tableau is the sum of the number of rows and columns. A cell is called a *free cell* if

there is no arrow pointing to this cell. An alternative tableau without free cells is called a *Catalan tableau*. Viennot [32] showed that the number of Catalan tableaux of size n is be equal to $\mathrm{Cat}(n)$.

Suppose $T \in \mathcal{T}(n)$ satisfies $A(T) = B(T) = 0$. Then nonempty cells of T have only γ and δ. We replace every γ with an up arrow and every δ with a left arrow. Then by definition every diagonal cell contains an arrow and there is no arrow pointing to another arrow. For each diagonal cell we remove the column (resp. row) containing it if it has an up arrow (resp. left arrow). Then the resulting object is an alternative tableau. If $\mathrm{b}(T) = k$ then the corresponding alternative tableau has $n - k$ rows and k columns, see Figure 8. If moreover T satisfies $C(T) + D(T) + E(T) = \binom{n+1}{2}$, thus $T \in \mathcal{CT}(n)$, then the corresponding alternative tableau has no free cells, i.e., it is a Catalan tableau. Thus we obtain the following lemma.

Lemma 4.3. *The number of $T \in \mathcal{CT}(n)$ with $\mathrm{b}(T) = k$ is equal to the number of Catalan tableaux of size n with $n - k$ rows.*

The following lemma is proved by Burstein [6, Theorem 3.1] using the permutation tableaux description.

Lemma 4.4. *There is a bijection between the set of Catalan tableaux of size n with k rows and the set of noncrossing partitions of $\{1, 2, \ldots, n + 1\}$ with k blocks.*

Since the number of noncrossing partitions of $\{1, 2, \ldots, n\}$ with k blocks is the Narayana number $\mathrm{Nara}(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$, by Lemmas 4.3 and 4.4 we obtain

$$(12) \qquad \sum_{T \in \mathcal{CT}(n)} i^n (-1)^{\mathrm{b}(T)} = i^n \sum_{k=0}^{n} (-1)^k \mathrm{Nara}(n+1, n+1-k).$$

There is a formula for the alternating sum of Narayana numbers.

Lemma 4.5. [5, Proposition 2.2 corrected]

$$\sum_{k=1}^{n} (-1)^k \mathrm{Nara}(n, k) = (-1)^{\frac{n+1}{2}} \mathrm{Cat}\left(\frac{n-1}{2}\right).$$

Now we can prove Theorem 4.2.

Proof of Theorem 4.2. By (11) and (12) we have

$$\left[a^n b^n c^n d^n q^{\binom{n}{2}} \right] 2^n (abcd; q)_n \mu_n(a, b, c, d; q) = i^n \sum_{k=1}^{n+1} (-1)^{n+1-k} \mathrm{Nara}(n+1, k).$$

Then we are done by Lemma 4.5. $\square$

By analyzing the Catalan tableaux one can prove the following. We will omit the proofs.

Theorem 4.6. *We have*

$$\left[a^{n-1} b^n c^n d^n q^{\binom{n}{2}} \right] 2^n (abcd; q)_n \mu_n(a, b, c, d; q) = -\mathrm{Cat}\left(\frac{n+1}{2}\right),$$

$$\left[a^{n-1} b^{n-1} c^n d^n q^{\binom{n}{2}} \right] 2^n (abcd; q)_n \mu_n(a, b, c, d; q) = \mathrm{Cat}\left(\frac{n+2}{2}\right) - \mathrm{Cat}\left(\frac{n}{2}\right).$$

5. Matchings

In this section we provide the third combinatorial approach to compute $\mu_n(a, b, c, d; q)$, using matchings and the q-Hermite polynomials as in [17]. We give a combinatorial expression for the non-normalized moments in Theorem 5.1. We derive new expressions for $\mu_n(a, b, c, d; q)$ as a rational function in Theorem 5.2 and Theorem 5.8, and a new positivity result in Corollary 5.9.

The approach of [17] was to realize the Askey-Wilson integral (1) as a generating function for an integral of a product of four q-Hermite polynomials

$$I_0(t_1, t_2, t_3, t_4) = \frac{(q)_\infty}{2\pi} \sum_{n_1, n_2, n_3, n_4 \geq 0} \int_0^\pi \prod_{i=1}^{4} \frac{H_{n_i}(\cos\theta | q) t_i^{n_i}}{(q)_{n_i}} v(\cos\theta | q) d\theta.$$

See [17] for the definitions of $H_n(\cos\theta; q)$ and $v(\cos\theta|q)$.

Put

$$
\begin{aligned}
I_n(t_1, t_2, t_3, t_4) :={} & \frac{(q)_\infty}{2\pi} \int_0^\pi (\cos\theta)^n w(\cos\theta, t_1, t_2, t_3, t_4; q) d\theta \\
={} & \frac{(q)_\infty}{2\pi} \sum_{n_1, n_2, n_3, n_4 \geq 0} \int_0^\pi (\cos\theta)^n \prod_{i=1}^4 \frac{H_{n_i}(\cos\theta|q) t_i^{n_i}}{(q)_{n_i}} v(\cos\theta|q) d\theta,
\end{aligned}
$$

(13)

so that

(14)
$$
I_n(a, b, c, d) = \mu_n(a, b, c, d; q) I_0(a, b, c, d).
$$

Upon rescaling, the Askey-Wilson integral (1) followed from a combinatorial evaluation of the integral of a product of four q-Hermite polynomials, see (17).

The *rescaled q-Hermite polynomials* $\widetilde{H}_n(x|q)$ (see [17]) are defined in terms of the q-Hermite polynomials by

$$
\widetilde{H}_n(x|q) = \frac{H_n(\frac{\sqrt{1-q}}{2} x|q)}{(1-q)^{n/2}}.
$$

Let

(15)
$$
\widetilde{f}_n(n_1, n_2, n_3, n_4; q) = \frac{(q)_\infty}{2\pi} \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} x^n \widetilde{H}_{n_1}(x|q) \widetilde{H}_{n_2}(x|q) \widetilde{H}_{n_3}(x|q) \widetilde{H}_{n_4}(x|q) \widetilde{v}(x|q) dx,
$$

where

$$
\widetilde{v}(x|q) = \frac{v(\frac{\sqrt{1-q}}{2} x|q) \cdot \frac{\sqrt{1-q}}{2}}{\sqrt{1 - (1-q)x^2/4}}.
$$

By (13) and (15) we have

(16)
$$
I_n(a, b, c, d) = \left(\frac{\sqrt{1-q}}{2} \right)^n \sum_{n_1, n_2, n_3, n_4 \geq 0} \frac{\widetilde{a}^{n_1} \widetilde{b}^{n_2} \widetilde{c}^{n_3} \widetilde{d}^{n_4}}{[n_1]_q! [n_2]_q! [n_3]_q! [n_4]_q!} \widetilde{f}_n(n_1, n_2, n_3, n_4; q),
$$

where $\widetilde{a} = a/\sqrt{1-q}$, $\widetilde{b} = b/\sqrt{1-q}$, $\widetilde{c} = c/\sqrt{1-q}$, $\widetilde{d} = d/\sqrt{1-q}$. The rescaled q-Hermite polynomials have a close connection with matchings. We need the following definitions.

A *matching* is a set partition of $\{1, 2, \ldots, n\}$ in which every block has size 1 or 2. A block of size 2 is called an *edge* and a block of size 1 is called a *fixed point*. A *complete matching* is a matching without fixed points. For a matching π we define the *crossing number* $\mathrm{cr}(\pi)$ to be the number of pairs (e_1, e_2) of blocks $e_1, e_2 \in \pi$ such that $e_1 = \{i_1, j_1\}, e_2 = \{i_2, j_2\}$ with $i_1 < i_2 < j_1 < j_2$ or $e_1 = \{i_1, j_1\}, e_2 = \{j_2\}$ with $i_1 < i_2 < j_1$.

For fixed integers $n_0, n_1, n_2, \ldots, n_k$, let $S_i = \{m_{i-1} + 1, m_{i-1} + 2, \ldots, m_i\}$ for $i = 0, 1, 2, \ldots, k$, where $m_{-1} = 0$ and $m_i = n_0 + n_1 + \cdots + n_i$ for $i = 0, 1, \ldots, k$. We define $\mathcal{CM}(n_0; n_1, n_2, \ldots, n_k)$ to be the set of complete matchings π on $\bigcup_{i=0}^k S_i$ such that if an edge of π is contained in S_i, then $i = 0$.

Ismail, Stanton and Viennot [17, Theorem 3.2] showed that

(17)
$$
\widetilde{f}_0(n_1, n_2, n_3, n_4; q) = \sum_{\sigma \in \mathcal{CM}(0; n_1, n_2, n_3, n_4)} q^{\mathrm{cr}(\sigma)}.
$$

We have, using the same arguments as in [17, Theorem 3.2], an analogous result for $\widetilde{f}_n(n_1, n_2, n_3, n_4; q)$.

Theorem 5.1. *We have*

$$
\widetilde{f}_n(n_1, n_2, n_3, n_4; q) = \sum_{\sigma \in \mathcal{CM}(n; n_1, n_2, n_3, n_4)} q^{\mathrm{cr}(\sigma)}.
$$

Due to (14) and (16), one may use Theorem 5.1 to give another explicit formula for $\mu_n(a, b, c, d; q)$.

We can use Theorem 5.1 to find new explicit formulas for the moments, and also explain the mixed binomial and q-binomial coefficients. The reason is that the generating function for a crossing number of matchings always has such a formula.

Let $\mathcal{M}(n, m)$ denote the set of matchings on $\{1, 2, \ldots, n\}$ with m fixed points. Note that $\mathcal{M}(n, m) = \emptyset$ unless $n \equiv m \mod 2$. Let

$$P(n, m) = \sum_{\pi \in \mathcal{M}(n,m)} q^{\mathrm{cr}(\pi)}, \qquad \overline{P}(n, m) = (1 - q)^{(n-m)/2} P(n, m).$$

Josuat-Vergès [20, Proposition 5.1] showed the following (see also [7, Proposition 15]): if $n \equiv m$ mod 2, we have

$$(18) \qquad \overline{P}(n, m) = \sum_{k \geq 0} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) (-1)^{(k-m)/2} q^{\binom{(k-m)/2+1}{2}} \begin{bmatrix} \frac{k+m}{2} \\ \frac{k-m}{2} \end{bmatrix}_q.$$

Theorem 5.2. *We have*

$$2^n \mu_n(a, b, c, d; q) = \sum_{\alpha, \beta, \gamma, \delta \geq 0} a^\alpha b^\beta c^\gamma d^\delta \overline{P}(n, \alpha + \beta + \gamma + \delta) \begin{bmatrix} \alpha + \beta + \gamma + \delta \\ \alpha, \beta, \gamma, \delta \end{bmatrix}_q \frac{(ad)_{\beta+\gamma}(ac)_\beta(bd)_\gamma}{(abcd)_{\beta+\gamma}},$$

where $\overline{P}$ is given by (18).

Proof. Consider a complete matching $\sigma \in \mathcal{CM}(n; n_1, n_2, n_3, n_4)$. Let $S_0 = \{1, \ldots, n\}$ and $S_i = \{m_{i-1} + 1, m_{i-1} + 2, \ldots, m_i\}$ for $i = 1, 2, 3, 4$, where $m_0 = n$ and $m_i = n + n_1 + \cdots + n_i$. Let α (respectively β, γ, δ) be the number of edges between S_0 and S_1 (respectively S_2, S_3, S_4) in σ. For $i, j \in \{1, 2, 3, 4\}$, let n_{ij} be the number of edges between the blocks S_i and S_j in σ. It is straightforward to check that the sum of $q^{\mathrm{cr}(\sigma)}$ for all σ having the above numbers is equal to

$$P(n, \alpha + \beta + \gamma + \delta) \begin{bmatrix} \alpha + \beta + \gamma + \delta \\ \alpha, \beta, \gamma, \delta \end{bmatrix}_q q^{\beta n_{13} + \beta n_{14} + \gamma n_{14} + \gamma n_{24} + n_{13} n_{24}}$$

$$\times \begin{bmatrix} n_1 \\ \alpha, n_{12}, n_{13}, n_{14} \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_{21}, \beta, n_{23}, n_{24} \end{bmatrix}_q \begin{bmatrix} n_3 \\ n_{31}, n_{32}, \gamma, n_{34} \end{bmatrix}_q \begin{bmatrix} n_4 \\ n_{41}, n_{42}, n_{43}, \delta \end{bmatrix}_q$$

$$\times [\alpha]_q! [\beta]_q! [\gamma]_q! [\delta]_q! [n_{12}]_q! [n_{13}]_q! [n_{14}]_q! [n_{23}]_q! [n_{24}]_q! [n_{34}]_q!.$$

Thus by (13) we have

$$I_n = \left(\frac{\sqrt{1-q}}{2} \right)^n \sum_{\alpha, \beta, \gamma, \delta \geq 0} \widetilde{a}^\alpha \widetilde{b}^\beta \widetilde{c}^\gamma \widetilde{d}^\delta P(n, \alpha + \beta + \gamma + \delta) \begin{bmatrix} \alpha + \beta + \gamma + \delta \\ \alpha, \beta, \gamma, \delta \end{bmatrix}_q$$

$$\times \sum_{n_{12} \geq 0} \frac{\left(\widetilde{a}\widetilde{b} \right)^{n_{12}}}{[n_{12}]_q!} \sum_{n_{14} \geq 0} \frac{\left(\widetilde{a}\widetilde{d}q^{\beta+\gamma} \right)^{n_{14}}}{[n_{14}]_q!} \sum_{n_{23} \geq 0} \frac{\left(\widetilde{b}\widetilde{c} \right)^{n_{23}}}{[n_{23}]_q!} \sum_{n_{34} \geq 0} \frac{\left(\widetilde{c}\widetilde{d} \right)^{n_{34}}}{[n_{34}]_q!}$$

$$\times \sum_{n_{13}, n_{24} \geq 0} \frac{\left(\widetilde{a}\widetilde{c}q^\beta \right)^{n_{13}}}{[n_{13}]_q!} \frac{\left(\widetilde{b}\widetilde{d}q^\gamma \right)^{n_{24}}}{[n_{24}]_q!}$$

$$= \frac{1}{2^n} \sum_{\alpha, \beta, \gamma, \delta \geq 0} a^\alpha b^\beta c^\gamma d^\delta \overline{P}(n, \alpha + \beta + \gamma + \delta) \begin{bmatrix} \alpha + \beta + \gamma + \delta \\ \alpha, \beta, \gamma, \delta \end{bmatrix}_q$$

$$\times \frac{(abcdq^{\beta+\gamma})_\infty}{(ab)_\infty (adq^{\beta+\gamma})_\infty (bc)_\infty (cd)_\infty (acq^\beta)_\infty (bdq^\gamma)_\infty}.$$

Here we used the following identities:

$$\frac{(x/(1-q))^n}{[n]_q!} = \frac{x^n}{(q)_n}, \qquad \sum_{n \geq 0} \frac{x^n}{(q)_n} = (x)_n, \qquad \sum_{n,m \geq 0} \frac{q^{nm} x^n y^m}{(q)_m (q)_n} = \frac{(xy)_\infty}{(x)_\infty (y)_\infty}.$$

Since (14) is $\mu_n(a, b, c, d; q) = I_n/I_0$, we are done by (1). $\qquad \square$

Corollary 5.3.

$$2^n \mu_n(a, b, c, 0; q) = \sum_{\alpha, \beta, \gamma \geq 0} a^\alpha b^\beta c^\gamma \overline{P}(n, \alpha + \beta + \gamma) \begin{bmatrix} \alpha + \beta + \gamma \\ \alpha, \beta, \gamma \end{bmatrix}_q (ac)_\beta.$$

Proof. Since $\mu_n(a, b, c, d; q)$ is symmetric in a, b, c, d, it is sufficient to find $\mu_n(a, b, 0, d; q)$. If $c = 0$ in Theorem 5.2, then nonzero terms have $\gamma = 0$. Thus we obtain the desired identity with c and γ replaced with d and δ respectively. $\qquad\square$

Corollary 5.4.

$$2^n \mu_n(a, b, 0, 0; q) = \sum_{\alpha, \beta \geq 0} a^\alpha b^\beta \overline{P}(n, \alpha + \beta) \begin{bmatrix} \alpha + \beta \\ \alpha \end{bmatrix}_q.$$

Proof. This is an immediate consequence of Theorem 5.2 when $c = d = 0$. $\qquad\square$

Note that the moment $2^n \mu_n(a, b, 0, 0; q)$ is not a polynomial with positive coefficients. However, if we use a factor that is a power of $1 - q$, we can express this moment using polynomials with positive coefficients as follows.

Proposition 5.5. *We have*

$$2^n \mu_n(a, b, 0, 0; q) = \sum_{u, v \geq 0} a^u b^v \begin{bmatrix} u + v \\ u \end{bmatrix}_q (1 - q)^{(n-u-v)/2} \sum_{\pi \in \mathcal{M}(n, u+v)} q^{\mathrm{cr}(\pi)}.$$

Another corollary is a simplified version of the $q = 0$ result in [9, Theorem 3.7].

Corollary 5.6.

$$2^n \mu_n(a, b, c, d; 0) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) \sum_{\alpha+\beta+\gamma+\delta=k} a^\alpha b^\beta c^\gamma d^\delta \Phi(\beta, \gamma),$$

where

$$\Phi(\beta, \gamma) = \begin{cases} \frac{(1-ac)(1-bd)(1-ad)}{1-abcd} & \text{if } \beta \neq 0 \text{ and } \gamma \neq 0, \\ \frac{(1-bd)(1-ad)}{1-abcd} & \text{if } \beta = 0 \text{ and } \gamma \neq 0, \\ \frac{(1-ac)(1-ad)}{1-abcd} & \text{if } \beta \neq 0 \text{ and } \gamma = 0, \\ 1 & \text{if } \beta = \gamma = 0. \end{cases}$$

Finally, we shall use Theorem 5.2 to give an yet another formula, Theorem 5.8, for the moments. We shall need the q-analogue of Saalschutz's theorem, for which Zeilberberger [34] gave a bijective proof.

Lemma 5.7.

$$\begin{bmatrix} a + b + k \\ a \end{bmatrix}_q \begin{bmatrix} b + c + k \\ b \end{bmatrix}_q \begin{bmatrix} c + a + k \\ c \end{bmatrix}_q = \sum_{j \geq 0} q^{j(j+k)} \frac{[a + b + c + k - j]_q!}{[a - j]_q! [b - j]_q! [c - j]_q! [j]_q! [k + j]_q!}.$$

Theorem 5.8. *We have*

$$2^n \mu_n(a, b, c, d; q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) \sum_{\alpha+\beta+\gamma+\delta+2t=k} a^\alpha b^\beta c^\gamma d^\delta \frac{(ac)_\beta (bd)_\gamma}{(abcd)_{\beta+\gamma}}$$

$$\times (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} \alpha + \beta + \gamma + t \\ \alpha \end{bmatrix}_q \begin{bmatrix} \beta + \gamma + \delta + t \\ \beta, \gamma, \delta + t \end{bmatrix}_q \begin{bmatrix} \delta + \alpha + t \\ \delta \end{bmatrix}_q,$$

in the second sum, $\alpha, \beta, \gamma, \delta \geq 0$ and $-k \leq t \leq k/2$.

Proof. In Theorem 5.2, if we expand $(ad)_{\beta+\gamma}$ using the q-binomial theorem

$$(ad)_{\beta+\gamma} = \sum_{j \geq 0} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} \beta + \gamma \\ j \end{bmatrix}_q (ad)^j$$

and change the indices $\alpha \mapsto \alpha - j$ and $\delta \mapsto \delta - j$, we get

$$2^n \mu_n(a, b, c, d; q) = \sum_{\alpha, \beta, \gamma, \delta, \geq 0} a^\alpha b^\beta c^\gamma d^\delta \frac{(ac)_\beta (bd)_\gamma}{(abcd)_{\beta+\gamma}} \begin{bmatrix} \beta + \gamma \\ \beta \end{bmatrix}_q X,$$

where

$$X = \sum_{j \geq 0} (-1)^j q^{\binom{j}{2}} \overline{P}(n, \alpha + \beta + \gamma + \delta - 2j) \frac{[\alpha + \beta + \gamma + \delta - 2j]_q!}{[\alpha - j]_q![\beta + \gamma - j]_q![\delta - j]_q![j]_q!}.$$

We have

$$\overline{P}(n, \alpha + \beta + \gamma + \delta - 2j) = \sum_{k \geq 0} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) (-1)^{t+j} q^{\binom{t+j+1}{2}} \begin{bmatrix} k - t - j \\ t + j \end{bmatrix}_q,$$

where $t = (k - \alpha - \beta - \gamma - \delta)/2$. Thus

$$X = \sum_{k \geq 0} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) (-1)^t q^{\binom{t+1}{2}} \sum_{j \geq 0} q^{j(j+t)} \frac{[\alpha + \beta + \gamma + \delta + t - j]_q!}{[\alpha - j]_q![\beta + \gamma - j]_q![\delta - j]_q![j]_q![t + j]_q!}.$$

By Lemma 5.7, we can evaluate the j-sum and we get the desired formula. $\square$

In Theorem 5.8, if $c = 0$, then $\gamma = 0$ in the sum, and we obtain Theorem 2.3. We also have another corollary which includes a positivity result.

Corollary 5.9. *We have*

$$2^n \mu_n(a, b, q/a, q/b; q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) \frac{1}{[k+1]_q} \sum_{\substack{|A|+|B| \leq k \\ A+B \equiv k \mod 2}} a^A b^B q^{\frac{k-A-B}{2}},$$

where the second sum is over all integers A and B such that $|A| + |B| \leq k$ and $A + B \equiv k$ mod 2. Thus $[n+1]_q! 2^n \mu_n(a, b, q/a, q/b; q)$ is a Laurent polynomial in a and b whose coefficients are positive polynomials in q.

Proof. When $c = q/a$ and $d = q/b$, we have $\frac{(ac)_\beta (bd)_\gamma}{(abcd)_{\beta+\gamma}} \begin{bmatrix} \beta+\gamma \\ \beta \end{bmatrix}_q = \frac{1}{[\beta+\gamma+1]_q}$. Thus by Theorem 5.8, letting $A = \alpha - \gamma$ and $B = \beta - \delta$, we get

$$2^n \mu_n(a, b, q/a, q/b; q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) \sum_{\substack{|A|+B \leq k \\ -A+|B| \leq k \\ A+B \equiv k \mod 2}} a^A b^B q^{\frac{k-A-B}{2}} \cdot X(A, B),$$

where

$$X(A, B) = \sum_{\beta = \max(B,0)}^{\frac{k+A+B}{2}} \frac{1}{[\beta + \frac{k-A-B}{2} + 1]_q} \begin{bmatrix} \frac{k+A-B}{2} \\ \beta - B \end{bmatrix}_q$$

$$\times \sum_{\alpha = \max(A,0)}^{\frac{k+A-B}{2}} (-1)^t q^{\binom{t}{2}} \begin{bmatrix} \beta + \frac{k-A-B}{2} + 1 \\ \beta + \alpha - A + 1 \end{bmatrix}_q \begin{bmatrix} \frac{k-A+B}{2} + \alpha \\ \alpha \end{bmatrix}_q,$$

$t = \frac{k+A+B}{2} - \alpha - \beta$. Note that the ranges of indices are determined by the q-binomial coefficients in the sums of $X(A, B)$. We will consider the two cases: $A < 0$ and $A \geq 0$.

CASE 1: $A < 0$. One can evaluate the α-sum of $X(A, B)$ using the q-Vandermonde [11, (II.7)]

$$X(A, B) = \sum_{\beta = \max(B,0)}^{\frac{k+A+B}{2}} \frac{1}{[\beta + \frac{k-A-B}{2} + 1]_q} \begin{bmatrix} \frac{k+A-B}{2} \\ \beta - B \end{bmatrix}_q (-1)^{\frac{k+A+B}{2} - \beta} q^{\binom{\frac{k+A+B}{2} - \beta}{2}} \frac{(q^{\beta - \frac{k+A+B}{2} + 1})_{\frac{k+A-B}{2}}}{(q^{\beta - A + 2})_{\frac{k+A-B}{2}}}.$$

Since $B \leq \beta \leq \frac{k+A+B}{2}$, we have $-\frac{k+A-B}{2} \leq \beta - \frac{k+A+B}{2} \leq 0$. Thus $(q^{\beta - \frac{k+A+B}{2} + 1})_{\frac{k+A-B}{2}} = 0$ unless $\beta = \frac{k+A+B}{2}$, which can occur if and only if $k + A - B \geq 0$ and $k + A + B \geq 0$. Thus when $A < 0$ we have $X(A, B) = \frac{1}{[k+1]_q}$ if $|A| + |B| \leq k$, and $X(A, B) = 0$ otherwise.

CASE 2: $A \geq 0$. Note that in this case we have $A + |B| = |A| + |B| \leq k$. Then in the α-sum we have $A \leq \alpha \leq \frac{k+A-B}{2}$. By changing the index $\alpha \mapsto \frac{k+A-B}{2} - \alpha$, and using the other type of

q-Vandermonde [11, (II.6)], one can check that the α-sum and of $X(A, B)$ is equal to

$$(-1)^{\beta-B}q^{\binom{\beta-\frac{B}{2}+1}{2}-(k+1)\frac{k+A-B}{2}}\begin{bmatrix}k\\\frac{k+A-B}{2}\end{bmatrix}_q\frac{(q^{\beta-\frac{k+A+B}{2}+1})_{\frac{k+A-B}{2}}}{(q^{-k})_{\frac{k+A-B}{2}}},$$

which is zero unless $\beta = \frac{k+A+B}{2}$. Since $|A| + |B| \leq k$, there is $\beta = \frac{k+A+B}{2}$ in the β-sum. In fact, when $\beta = \frac{k+A+B}{2}$ one can check that the above term is 1. Thus when $A \geq 0$ we have $X(A, B) = \frac{1}{[k+1]_q}$.

In both cases, we have $X(A, B) = \frac{1}{[k+1]_q}$ when $|A| + |B| \leq k$ and $X(A, B) = 0$ otherwise. Thus we obtain the identity. $\qquad\square$

One can show that the sum of the coefficients in Corollary 5.9 is $2^n(n + 1)!$.

Conjecture 2. $2^n[n+i+j-1]_q!\mu_n(a, b, q^i/a, q^j/b; q)$ is a Laurent polynomial in a, b and polynomial in q with nonnegative coefficients.

Next we give an alternative formula for $\mu_n(a, b, -b, -a; q)$ using Theorem 5.2. Note that $\mu_n(a, b, -b, -a; q) = 0$ unless n is even. We need the following lemma, [11, (II.9)]

Lemma 5.10.

$$\sum_{i=0}^{n}(-1)^i\begin{bmatrix}n\\i\end{bmatrix}_q(x)_{n-i}(x)_i = \begin{cases}(q; q^2)_{n/2}(x^2; q^2)_{n/2}, & \text{if } n \text{ is even},\\0, & \text{if } n \text{ is odd}.\end{cases}$$

Theorem 5.11.

$$4^n\mu_{2n}(a, b, -b, -a; q) = \sum_{\alpha,\beta}a^{2\alpha}b^{2\beta}\overline{P}(2n, 2\alpha + 2\beta)\begin{bmatrix}2\alpha + 2\beta\\2\alpha\end{bmatrix}_q\frac{(q; q^2)_\alpha(q; q^2)_\beta(-a^2; q)_{2\beta}}{(qa^2b^2; q^2)_\beta}$$

Proof. Letting $c = -b$, $d = -a$, and changing the indices $\alpha + \delta \mapsto \alpha$, $\beta + \gamma \mapsto \beta$ in Theorem 5.2, we get

$$4^n\mu_{2n}(a, b, -b, -a; q) = \sum_{\alpha,\beta\geq 0}a^\alpha b^\beta\overline{P}(2n, \alpha + \beta)\begin{bmatrix}\alpha + \beta\\\alpha\end{bmatrix}_q\frac{(-a^2)_\beta}{(a^2b^2)_\beta}$$

$$\times\sum_{\gamma\geq 0}(-1)^\gamma\begin{bmatrix}\beta\\\gamma\end{bmatrix}_q(-ab)_{\beta-\gamma}(-ab)_\gamma\sum_{\delta\geq 0}(-1)^\delta\begin{bmatrix}\alpha\\\delta\end{bmatrix}_q.$$

By Lemma 5.10, we get the theorem. $\qquad\square$

If we use the techniques of this section on the q^2-Hermite polynomials $H_n(x|q^2)$, then we can find another alternative theorem, whose proof we omit.

Theorem 5.12.

$$4^n\mu_{2n}(a, b, -a, -b; q) = \sum_{m=0}^{n}\overline{P}(2n, 2m)\frac{(q; q^2)_m}{(qa^2b^2; q^2)_m}\sum_{\alpha+\beta+i=m}a^{2\alpha+2i}b^{2\beta+2i}(-q; q)_iq^{\binom{i}{2}}\begin{bmatrix}m\\\alpha, \beta, i\end{bmatrix}_{q^2}$$

6. Connections with other results

Recently the problem of finding formulas for moments of orthogonal polynomials has gained some attention. In this section we will rederive several moment formulas using Corollary 2.4 and Corollaries 2.5 and 2.6. We will use the following fact.

Lemma 6.1. *If μ_n is the nth moment of the orthogonal polynomial $P_n(x)$ defined by*

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x),$$

and μ'_n is the nth moment of the orthogonal polynomials $P'_n(x)$ defined by

$$P'_{n+1}(x) = (x - c(d + b_n))P'_n(x) - c^2\lambda_n P'_{n-1}(x),$$

then

$$\mu'_n = c^n\sum_{m=0}^{n}d^{n-m}\binom{n}{m}\mu_m.$$

6.1. **The q-Laguerre polynomials.** The q-Laguerre polynomials are defined by $b_n = [n]_q + y[n+1]_q$ and $\lambda_n = y[n]_q^2$. Let ν_n denote the nth moment of the q-Laguerre polynomials. We refer the reader to [8, Proposition 1] for many interesting interpretations of ν_n involving permutations, permutation tableaux, matrix ansatz, etc.

Josuat-Vergès [20] and Corteel et al. [8] showed that

$$(19) \quad \nu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \sum_{j=0}^{n-k} y^j \left(\binom{n}{j}\binom{n}{j+k} - \binom{n}{j-1}\binom{n}{j+k+1} \right) \sum_{i=0}^{k} (-1)^k y^i q^{i(k+1-i)}.$$

Using Lemma 6.1 we have

$$\nu_n = y^{n/2} \sum_{m=0}^{n} (y^{1/2} + y^{-1/2})^{n-m} \binom{n}{m} 2^m \mu_m(-qy^{1/2}, -y^{-1/2}, 0, 0; q).$$

By using Corollary 2.5 for $2^m \mu_m(-qy^{1/2}, -y^{-1/2}, 0, 0; q)$, the binomial theorem for $(y^{1/2}+y^{-1/2})^{n-m}$, and Vandermonde's theorem to sum the m-sum, we obtain (19).

Using the rank generating function for the totally nonnegative Grassmannian cells, Williams [33, Corollary 5.3] showed another formula for ν_n:

$$(20) \quad \nu_n = \sum_{i=0}^{n} y^i \sum_{j=0}^{i} (-1)^j [i-j]_q^n q^{i(j-i)} \left(\binom{n}{j} q^{i-j} + \binom{n}{j-1} \right).$$

We now show that the two formulas (19) and (20) are equivalent under simple manipulation. In other words, we will prove the following identity:

$$(21) \quad (1-q)^n \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^j [i-j]_q^n y^i q^{i(j-i)} \left(\binom{n}{j} q^{i-j} + \binom{n}{j-1} \right)$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{n-k} y^j \left(\binom{n}{j}\binom{n}{j+k} - \binom{n}{j-1}\binom{n}{j+k+1} \right) \sum_{i=0}^{k} (-1)^k y^i q^{i(k+1-i)}.$$

By the binomial expansion for $(1-q)^n [i-j]_q^n = (1-q^{i-j})^n$, the left hand side of (21) is equal to

$$\sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^j y^i q^{i(j-i)} \sum_{k=0}^{n} \binom{n}{k} (-q^{i-j})^k \left(\binom{n}{j} q^{i-j} + \binom{n}{j-1} \right)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{k=0}^{n+1} (-1)^{j+k} y^i q^{(i-j)(k-i)} \left(\binom{n}{j-1}\binom{n}{k} - \binom{n}{j}\binom{n}{k-1} \right).$$

The summand is anti-symmetric in k and j. Thus we can let k range from $i+1$ to $n+1$. Replacing $k \mapsto k+j+1$ and $i \mapsto i+j$ and changing the order of sums we get the right hand side of (21).

6.2. **Josuat-Vergès's formula.** Josuat-Vergès [19] defined a partition function $Z_n(a,b,y,q)$ and showed that $(1-q)^n Z_n(a,b,y,q)$ is equal to the nth moment defined by $b_n = 1 + y + (a+by)q^n$ and $\lambda_n = y(1-q^i)(1-abq^{i-1})$. He found the following formula for $Z_n(a,b,y,q)$

$$(22) \quad Z_n(a,b,y,q) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} (-y)^i q^{\binom{i+1}{2}} \begin{bmatrix} k+i \\ i \end{bmatrix}_q$$

$$\times \sum_{j=0}^{n-k-2i} y^j \left(\binom{n}{j}\binom{n}{k+2i+j} - \binom{n}{j-1}\binom{n}{k+2i+j+1} \right) \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q a^r (yb)^{k-r}.$$

By the Rescaling Lemma 6.1, we get

$$(1-q)^n Z_n(a,b,y,q) = y^{n/2} \sum_{m=0}^{n} (y^{1/2} + y^{-1/2})^{n-m} \binom{n}{m} 2^m \mu_m(ay^{-1/2}, by^{1/2}, 0, 0; q).$$

Thus we get in the same manner as (19)

$$
(1-q)^n Z_n(a,b,y,q) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} y^j \left(\binom{n}{j}\binom{n}{j+k} - \binom{n}{j-1}\binom{n}{j+k+1} \right)
$$
$$
\times \sum_{u+v+2t=k} a^u b^v y^{v+t}(-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u,v,t \end{bmatrix}_q.
$$

It is easy to check that the above formula is equivalent to (22).

6.3. **Josuat-Vergès and Rubey's formula.** Josuat-Vergès and Rubey [21] considered the nth moment $R_n(a,b,c,d;q)$ of the orthogonal polynomials defined by $b_n = d + (a+b)q^n$ and $\lambda_n = (1-q^n)(c-abq^{n-1})$ and showed that

$$
(23) \qquad R_n(a,b,c,d;q) = \sum_{k=0}^{n} \frac{k+1}{n+1} \sum_{\ell=0}^{n-k} \binom{n+1}{\ell}\binom{\ell}{2\ell-n+k} c^{n-k-\ell} d^{2\ell-n+k} M_k^*(a,b,c;q),
$$

where

$$
\sum_{k\geq 0} M_k^*(a,b,c;q) z^k = \frac{1}{1-at} {}_2\phi_1\left[\begin{matrix} cb^{-1}qz, q \\ aqz \end{matrix} \,\middle|\, q, bz \right].
$$

Using (23) Josuat-Vergès and Rubey [21] derive moment formulas for various orthogonal polynomials.

By the Rescaling Lemma 6.1 we have

$$
R_n(a,b,c,d;q) = c^{n/2} \sum_{m=0}^{n} (dc^{-1/2})^{n-m} \binom{n}{m} \mu_m(ac^{-1/2}, bc^{-1/2}, 0, 0; q).
$$

Thus we get
(24)
$$
R_n(a,b,c,d;q) = \sum_{k=0}^{n} \sum_{m=k}^{n} c^{\frac{m-k}{2}} d^{n-m} \binom{n}{m} \left(\binom{m}{\frac{m-k}{2}} - \binom{m}{\frac{m-k}{2}-1} \right) \sum_{u+v+2t=k} a^u b^v c^t (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u,v,t \end{bmatrix}_q.
$$

It is easy to check using q-binomial theorems that (24) is equivalent to (23).

6.4. **The (t,q)-Euler numbers.** Kim [24] defined the (t,q)-Euler numbers $E_n(t,q)$ by

$$
\sum_{n\geq 0} E_n(t,q) z^n = \cfrac{1}{1 - \cfrac{[1]_q[1]_{t,q}z}{1 - \cfrac{[2]_q[2]_{t,q}z}{\cdots}}},
$$

and showed that

$$
E_n(t,q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) t^k q^{k(k+1)} T_k(t^{-1}, q^{-1}),
$$

where $[n]_{t,q} = (1-tq^n)/(1-q)$ and

$$
(25) \qquad T_k(t,q) = \sum_{j=0}^{k} \sum_{i=0}^{j} (-1)^{j+i} t^{2i} q^{j^2+i^2+i} \begin{bmatrix} k-j \\ i \end{bmatrix}_{q^2} \left(\begin{bmatrix} k-i \\ j-i \end{bmatrix}_{q^2} + t \begin{bmatrix} k-i-1 \\ j-i-1 \end{bmatrix}_{q^2} \right).
$$

Note that we have $(1-q)^{2n} E_n(t,q) = 2^{2n} \mu_{2n}(\sqrt{-qt}, -\sqrt{-qt}, 0, 0; q)$. Thus by Corollary 2.6 implies

(26)
$$
E_n(t,q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{n}{n-k-1} \right) \sum_{i=0}^{k} (-1)^k q^{\binom{i+1}{2}} (q;q^2)_{k-i} (qt)^{k-i} \begin{bmatrix} 2k-i \\ i \end{bmatrix}_q.
$$

We now show that the above two formulas for $E_n(t, q)$ are equivalent. It is sufficient to show that

$$(27) \qquad (-1)^k q^{k^2} T_k(t, q^{-1}) = \sum_{i=0}^{k} q^{\binom{i}{2}} (q; q^2)_{k-i} t^i \begin{bmatrix} 2k - i \\ i \end{bmatrix}_q.$$

In (25) if we replace $j \mapsto k - j$, interchange the i-sum and the j-sum, and replace $j \mapsto j + i$, then the j-sum is summable by the q-binomial theorem. Thus the left hand side of (27) is equal to

$$\sum_{i=0}^{\lfloor k/2 \rfloor} t^{2i} q^{2i^2 - i} (q^{2i+1}; q^2)_{k-2i} \begin{bmatrix} k - i \\ i \end{bmatrix}_{q^2} + \sum_{i=0}^{\lfloor k/2 \rfloor} t^{2i+1} q^{2i^2 + i} (q^{2i+3}; q^2)_{k-2i-1} \begin{bmatrix} k - i - 1 \\ i \end{bmatrix}_{q^2}$$

$$= \sum_{i=0}^{\lfloor k/2 \rfloor} t^{2i} q^{2i^2 - i} (q; q^2)_{k-2i} \begin{bmatrix} 2k - 2i \\ 2i \end{bmatrix}_q + \sum_{i=0}^{\lfloor k/2 \rfloor} t^{2i+1} q^{2i^2 + i} (q; q^2)_{k-2i-1} \begin{bmatrix} 2k - 2i - 1 \\ 2i + 1 \end{bmatrix}_q,$$

which is equal to the right hand side of (27).

Zeng [35] showed that

$$(28) \qquad E_n(t, q) = t^{-n} \sum_{m=0}^{n} \sum_{i=0}^{m} (-1)^{n-i} \frac{q^{2m - 2in + i^2 - n - i} [2m]_{t,q}! [2i + 1]_{t,q}^{2n}}{[2i]_q!! [2m - 2i]_q!! \prod_{k=0, k \neq i}^{m} [2k + 2i + 2]_{t^2, q}},$$

where $[2m]_{t,q}! = \prod_{i=1}^{2m} [i]_{t,q}$ and $[2i]_q!! = \prod_{k=1}^{i} [2k]_q$.

It is straightforward to check that Zeng's formula is equivalent to Theorem 2.7 when $a = \sqrt{-qt}$ and $c = 0$.

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