Addition Theorems for the q-Exponential Function

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Abstract

We give a new proof of the Suslov addition theorem for the q-exponential function on a q-quadratic grid and the q-analogue of the expansion of a plane wave in spherical harmonics. We also prove another addition theorem for the q-exponential function. The addition theorem and the q-plane wave formula are used to evaluate some definite integrals and establish certain power series identities.

1. Introduction. Suslov [12] has given a commutative q-analogue of the exponential addition theorem $e^{x+y} = e^x e^y$. Specifically, for $|q| < 1$ let

$$(1.1) \qquad \mathcal{E}_q(\cos\theta, \cos\phi; \alpha) := \frac{(\alpha^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} (-e^{i(\phi+\theta)} q^{(1-n)/2}, -e^{i(\phi-\theta)} q^{(1-n)/2}; q)_n \times \frac{(\alpha e^{-i\phi})^n}{(q;q)_n} q^{n^2/4},$$

where the q-shifted factorials are defined as in [7]. If

$$(1.2) \qquad \mathcal{E}_q(x; \alpha) = \mathcal{E}_q(x, 0; \alpha).$$

then $\mathcal{E}_q(0; \alpha) = 1$, and $\lim_{q\to 1} \mathcal{E}_q(x; (1-q)\alpha/2) = \exp(\alpha x)$. The notation for $\mathcal{E}_q$ adopted here is the same as in [12] and is different from the original notation in [11]. Suslov [12] proved the following addition formula for the $\mathcal{E}_q$ function

$$(1.3) \qquad \mathcal{E}_q(x, y; \alpha) = \mathcal{E}_q(x; \alpha)\mathcal{E}_q(y; \alpha),$$

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which is the commutative q-analogue of $\exp(\alpha(x+y)) = \exp(\alpha x)\ \exp(\alpha y)$.

The Suslov addition theorem is an important contribution to the growing area of q-series. This work grew out of attempts to understand Suslov's result. We found a very simple proof of (1.3). The key idea is to consider the expansion of $\mathcal{E}_q(x;\alpha)$ in terms of Rogers q-Hermite polynomials, $H_n(x|q)$, then apply known connection coefficient theorems, such as (1.8). Our proof appears in §3.

The function $\mathcal{E}_q(x;a)$ is a q-analogue of $e^{\alpha x}$, but unlike $e^{\alpha x}$, $\mathcal{E}_q(x;a)$ is not symmetric in x and α. The variables x and α seem to play different roles, so one would expect the function $\mathcal{E}_q(x;a)$ to have two different addition theorems. The Suslov addition theorem is in the x variable. In §3 we establish the following addition theorem,

$$(1.4)\qquad \begin{aligned}&(q\alpha^2, q\beta^2; q^2)_\infty \mathcal{E}_q(x;\alpha)\mathcal{E}_q(x;\beta)\\ &= \sum_{n=0}^{\infty} q^{n^2/4}\alpha^n\, H_n(x|q)(-\alpha\beta q^{(n+1)/2};q)_\infty \frac{(-q^{(1-n)/2}\beta/\alpha;q)_n}{(q;q)_n},\end{aligned}$$

in the α variable. As $q\to 1$ (1.4), with α and β replaced by $\alpha(1-q)/2$ and $\beta(1-q)/2$ respectively, reduces to

$$e^{\alpha x}e^{\beta x} = \sum_{n=0}^{\infty} \frac{\alpha^n x^n (1+\beta/\alpha)^n}{n!}.$$

This technique, namely applying connection coefficient relations, also applies to other theorems. An important classical expansion formula is the expansion of the plane wave in spherical harmonics, see (6.5). Ismail and Zhang [11] gave a q-analogue of this expansion. In the present normalization their formula is

$$(1.5)\qquad \begin{aligned}\mathcal{E}_q(x; i\alpha/2) \;=\; &\frac{(2/\alpha)^\nu (q;q)_\infty}{(-q\alpha^2/4;q^2)_\infty (q^{\nu+1};q)_\infty} \sum_{n=0}^{\infty} \frac{(1-q^{n+\nu})}{(1-q^\nu)} q^{n^2/4} i^n\\ &\times J^{(2)}_{\nu+n}(\alpha;q)\ C_n(x;q^\nu|q),\end{aligned}$$

where $J^{(2)}_{\nu+n}$ are q-Bessel functions [8], [7] and $\{C_n(x;\beta|q)\}$ are the continuous q-ultraspherical polynomials [2], [7], to be defined below. We shall refer to (1.5) as the q-plane wave expansion. Different proofs of (1.5) were given in [5], [6], [10], and [9]. The proof by Floreanini and Vinet [5] is group theoretic and is of independent interest. In §2 we give a new proof of (1.5) based on Rogers' formula (1.8). For a proof of the plane wave expansion and its connections to the addition theorem of Bessel functions see [13, Chapter 11].

Recall that the q-Bessel function $J^{(2)}_\nu(x;q)$ and the continuous q-ultraspherical polynomials are defined by

$$(1.6)\qquad J^{(2)}_\nu(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{(q,q^{\nu+1};q)_n} q^{n(\nu+n)},$$

and

$$(1.7)\qquad C_n(\cos\theta;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}}\, e^{i(n-2k)\theta},$$

respectively.

Rogers' connection coefficient formula (see [2] or [7, (7.6.14)]) for the continuous q-ultraspherical polynomials is

$$(1.8)\qquad C_n(x;\gamma|q) \ = \ \sum_{k=0}^{[n/2]} \frac{\beta^k(\gamma/\beta;q)_k(\gamma;q)_{n-k}}{(q;q)_k(q\beta;q)_{n-k}} \frac{(1-\beta q^{n-2k})}{(1-\beta)} C_{n-2k}(x;\beta|q).$$

The generating function for the C_n's is

$$(1.9)\qquad \sum_{n=0}^{\infty} C_n(\cos\theta;\beta|q)t^n = \frac{(\beta te^{i\theta}, \beta te^{-i\theta};q)_\infty}{(te^{i\theta}, te^{-i\theta};q)_\infty}.$$

The continuous q-Hermite polynomials are defined by

$$(1.10)\qquad H_n(x|q) := C_n(x;0|q)(q;q)_n,$$

and have the generating function

$$(1.11)\qquad \sum_{n=0}^{\infty} H_n(\cos\theta|q)\frac{t^n}{(q;q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta};q)_\infty}.$$

Al-Salam-Chihara polynomials [1], [9] have the generating function

$$(1.12)\qquad \sum_{n=0}^{\infty} \frac{p_n(\cos\theta;a,b|q)}{(q;q)_n}(t/a)^n = \frac{(at,bt;q)_\infty}{(te^{i\theta}, te^{-i\theta};q)_\infty}.$$

In §4 we give an expansion of $\mathcal{E}_q(x;\alpha)$ in a series of the special Al-Salam-Chihara polynomials $p_n(x;a,-a|q)$. The addition theorem (1.4) implies the evaluation of several definite integrals and provides several series identities. Section 5 contains a small sample of such results. In particular (5.4) generalizes a result of Bustoz and Suslov [3], which is the backbone of their theory of the q-Fourier integral on a q-quadratic grid. Our proof is much simpler than the proof of the special case in [3].

This paper is essentially self-contained and in §2 we develop all the preliminaries needed in §3. In the subsequent sections we state all of the necessary formulas. They are mostly orthogonality relations of certain q-orthogonal polynomials.

2. The q-plane wave expansion. In this section we prove (1.5) in two steps: first expand $\mathcal{E}_q(x;\alpha)$ in terms of the q-Hermite polynomials (Lemma 2.1), then use (1.8) to expand in terms of the q-ultraspherical polynomials. The proof also relies on the q-Kummer (Bailey-Daum) sum [7, (II.9)]

$$
{}_2\phi_1(a,b;aq/b;q,-q/b) = \frac{(-q;q)_\infty (aq, aq^2/b^2;q^2)_\infty}{(-q/b, aq/b;q)_\infty}. \tag{2.1}
$$

Lemma 2.1 *We have*

$$
(q\alpha^2;q^2)_\infty \mathcal{E}_q(x;\alpha) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}\alpha^n}{(q;q)_n} H_n(x|q). \tag{2.2}
$$

It is not hard to see that Lemma 2.1 is the special case $\beta = q^\nu = 0$ of (1.5).

Proof of Lemma 2.1. With $x = \cos\theta$ apply

$$
(-iq^{(1-n)/2}e^{i\theta}, -iq^{(1-n)/2}e^{-i\theta};q)_n = \frac{(-iq^{(1-n)/2}e^{i\theta}, -iq^{(1-n)/2}e^{-i\theta};q)_\infty}{(-iq^{(1+n)/2}e^{i\theta}, -iq^{(1+n)/2}e^{-i\theta};q)_\infty}
$$

then expand the right-hand side in q-ultraspherical polynomials using (1.9) to get

$$
\frac{(q\alpha^2;q^2)_\infty}{(\alpha^2;q^2)_\infty}\mathcal{E}_q(x;\alpha) = \sum_{n\ge k\ge 0} q^{n^2/4}\frac{(-i\alpha)^n}{(q;q)_n} C_k(x;q^{-n}|q)(-iq^{(n+1)/2})^k. \tag{2.3}
$$

Now expand $C_k(x;q^{-n}|q)$ in the basis $H_k(x|q)$ ((1.8) with $\gamma = q^{-n}$, $\beta = 0$) to see that the right-hand side R of (2.3) is

$$
R = \sum_{k=0}^{\infty} \frac{H_k(x|q)}{(q;q)_k} \sum_{n=k}^{\infty} \frac{(q^{-n};q)_k}{(q;q)_n}\alpha^n(-i)^{n+k}q^{n^2/4+k(n+1)/2} \sum_{j=0}^{n-k} \frac{(q^{-n+k};q)_j}{(q;q)_j} q^{j(j+1)/2}.
$$

The j sum can be evaluated by the limiting case $b \to \infty$ of (2.1) and is zero unless $n - k = 2m$ is even, in which case it sums to $(-q;q)_\infty (q^{1-2m};q^2)_\infty$ which simplifies to $(q^{1-2m};q^2)_m$. Thus we find

$$
R = \sum_{k=0}^{\infty} \frac{H_k(x|q)}{(q;q)_k} q^{k^2/4}\alpha^k \sum_{m=0}^{\infty} \frac{\alpha^{2m}}{(q^2;q^2)_m}.
$$

The m sum is summable from the q-binomial theorem [7, (II.3)], proving Lemma 2.1.

Remark. It is important to note that the series in Lemma 2.1 analytically continues the left-hand side to an entire function in x and in α.

Proof of the q-plane wave expansion. Use Lemma 2.1, and expand the q-Hermite polynomials in terms of the q-ultraspherical polynomials ((1.8) with $\gamma = 0$, $\beta = q^\nu$). The result is

$$(2.4)\qquad (q\alpha^2; q^2)_\infty \, \mathcal{E}_q(x;\alpha) = \sum_{n=0}^{\infty} \frac{(1-\beta q^n)}{(1-\beta)} \alpha^n q^{n^2/4} C_n(x;\beta|q) \times \sum_{k=0}^{\infty} \frac{\alpha^{2k}\beta^k}{(q;q)_k(\beta q;q)_{n+k}} q^{k(n+k)}.$$

The k-sum contributes the q-Bessel function and the infinite products to (1.5).

Observe that the formal interchange of q and q^{-1} amounts to interchanging the formal power series expansions in z of $(z;q)_\infty$ and $1/(zq^{-1};q^{-1})_\infty$. Furthermore for $|q| \neq 1$, it readily follows from (1.1) and (1.2) that

$$(2.5)\qquad \mathcal{E}_q(x;\alpha) = \mathcal{E}_{q^{-1}}(x; -\alpha\sqrt{q}).$$

Thus, we would expect Lemma 2.1 to be equivalent to the following corollary.

Corollary 2.2 *We have*

$$(2.6)\qquad \frac{1}{(\alpha^2;q^2)_\infty} \mathcal{E}_q(x;\alpha) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}\alpha^n}{(q;q)_n} H_n(x|q^{-1}).$$

Proof of Corollary 2.2. Take the $\beta \to \infty$ limit of (1.7),

$$(2.7)\qquad \lim_{\beta\to\infty} \beta^{-n} C_n(x;\beta|q) = \frac{q^{n(n-1)/2}(-1)^n}{(q;q)_n} H_n(x|q^{-1}),$$

and let $\beta \to \infty$ in (2.4) to see that $(q\alpha^2;q^2)_\infty \mathcal{E}_q(x;\alpha)$ is

$$\sum_{n=0}^{\infty} \alpha^n q^{n^2/4} \frac{H_n(x|q^{-1})}{(q;q)_n} (\alpha^2;q)_\infty.$$

Corollary 2.2 immediately follows from this.

Since Corollary 2.2 is crucial to our first proof of (1.3), it is of interest to note that it also follows directly from Lemma 2.1 without resorting to the q-plane wave expansion. This is so since (1.8) with $\gamma = 0$ and $\beta \to \infty$ give, in view of (1.10) and (2.7),

$$(2.8)\qquad \frac{H_n(x|q)}{(q;q)_n} = \sum_{k=0}^{[n/2]} \frac{(-1)^k q^{k(3k-2n-1)/2}}{(q;q)_k(q;q)_{n-2k}} H_{n-2k}(x|q^{-1}).$$

Using (2.8) we see that the left-hand side of (2.2) is a double sum and after interchanging the sums it becomes $(\alpha^2;q)_\infty$ times the right-hand side of (2.6) which establishes Corollary 2.2.

3. The addition theorems. In this section we give two proofs of the addition theorem (1.3). The first proof uses the same technique as the previous section. The second proof uses separation of variables and circumvents many of the technical difficulties in Suslov's original proof, [12].

First Proof of (1.3). Let

$$A := \frac{(q\alpha^2;q^2)_\infty}{(\alpha^2;q^2)_\infty}\mathcal{E}_q(\cos\theta,\cos\phi;\alpha). \tag{3.1}$$

Thus, as in (2.3) we have

$$A = \sum_{n,k=0}^{\infty} q^{n^2/4}\frac{(\alpha e^{-i\phi})^n}{(q;q)_n}C_k(\cos\theta;q^{-n}|q)(-e^{i\phi}q^{(n+1)/2})^k.$$

Now expand the C_n's in terms of the H_n's using (1.8) to establish

$$\begin{aligned} A \;=\; & \sum_{n,k,j=0}^{\infty} q^{n^2/4}q^{k(n+1)/2}(-1)^k\alpha^n e^{i\phi(k-n)}q^{-nj+j(j-1)/2} \\ & \times\frac{(-1)^j(q^{-n};q)_{k-j}}{(q;q)_n(q;q)_j}\frac{H_{k-2j}(x|q)}{(q;q)_{k-2j}}. \end{aligned} \tag{3.2}$$

The restrictions on j,k,n are $k\ge 2j$ and $n\ge k-j$. We set $k=2j+l$ and $n=N+l$ to find

$$A = \sum_{l=0}^{\infty}\frac{q^{l^2/4}\alpha^l}{(q;q)_l}H_l(\cos\theta|q)\sum_{N=0}^{\infty}\frac{q^{N^2/4}\alpha^N}{(q;q)_N}\sum_{j=0}^{N}\begin{bmatrix}N\\ j\end{bmatrix}_{q^{-1}}e^{i\phi(2j-N)}.$$

The j-sum is $H_N(cos\,\phi|q^{-1})$, so Lemma 2.1 and Corollary 2.3 complete the proof.

The second proof of (1.3) relies on the symmetry of $\mathcal{E}(x,y;\alpha)$ in x and y. To see the symmetry first observe that

$$\begin{aligned} & e^{-2in\phi}(-e^{i(\theta+\phi)}q^{(1-2n)/2},-e^{i(\phi-\theta)}q^{(1-2n)/2};q)_{2n} \\ &\quad = e^{-2in\phi}(-e^{i(\theta+\phi)}q^{(1-2n)/2},-e^{i(\phi-\theta)}q^{(1-2n)/2};q)_n \\ &\quad\quad \times(-e^{i(\theta+\phi)}q^{1/2},-e^{i(\phi-\theta)}q^{1/2};q)_n \\ &\quad = q^{-n^2}(-q^{1/2}e^{i(\theta+\phi)},-q^{1/2}e^{i(\phi-\theta)},-q^{1/2}e^{i(\theta-\phi)},-q^{1/2}e^{-i(\theta+\phi)};q)_n. \end{aligned}$$

The above expression is clearly symmetric in θ and ϕ. Similarly we find

$$
\begin{aligned}
& e^{-i(2n+1)\phi}(-e^{i(\theta+\phi)}q^{-n}, -e^{i(\phi-\theta)}q^{-n}; q)_{2n+1} \\
&\quad = e^{-i(2n+1)\phi}(1+e^{i(\theta+\phi)})(1+e^{i(\phi-\theta)}) \\
&\qquad \times(-q^{-n}e^{i(\theta+\phi)}, -q^{-n}e^{i(\phi-\theta)}, -qe^{-i(\phi+\theta)}, -qe^{i(\theta-\phi)}; q)_n \\
&\quad = 4q^{-n(n+1)}\cos((\theta+\phi)/2)\cos((\phi-\theta)/2) \\
&\qquad \times(-qe^{i(\theta+\phi)}, -qe^{i(\theta-\phi)}, -qe^{i(\phi-\theta)}, -qe^{-i(\theta+\phi)}; q)_n.
\end{aligned}
$$

The last expression is also symmetric in θ and ϕ. Therefore the terms in the series (1.1) defining $\mathcal{E}_q(\cos\theta, \cos\phi; \alpha)$ are symmetric in θ and ϕ, hence $\mathcal{E}_q(x, y; \alpha)$ is symmetric in x and y.

Second Proof of (1.3). In this proof we use (3.2) and the symmetry of $\mathcal{E}(x, y; \alpha)$ in x and y.

We rewrite (3.2) as

$$
A = \sum_{j,l,m=0}^{\infty} \frac{q^{(j-m)^2/4}\alpha^{j+m}e^{i(j-m)\phi}}{(q;q)_m(q;q)_j}\frac{q^{l^2/4}\alpha^l}{(q;q)_l}H_l(\cos\theta|q).
$$

Lemma 2.1 implies that the l sum is $(q\alpha^2;q^2)_\infty\,\mathcal{E}_q(\cos\theta;\alpha)$. Thus $\mathcal{E}_q(x,y;\alpha)$ is the product of $\mathcal{E}_q(x;\alpha)$ and a function independent of x. This and the symmetry of $\mathcal{E}_q(x,y;\alpha)$ in x and y give a separation of variables of the form

$$
\mathcal{E}_q(x,y;\alpha) = \mathcal{E}_q(x;\alpha)\mathcal{E}_q(y;\alpha)g(\alpha), \tag{3.3}
$$

where g is a function of α and is independent of x or y. To find $g(\alpha)$, we set $x = y = 0$. The generating function (1.11) implies

$$
\sum_{n=0}^{\infty} H_n(0|q)\frac{t^n}{(q;q)_n} = \frac{1}{(-t^2;q^2)_\infty},
$$

and we find

$$
H_{2n+1}(0|q) = 0, \quad H_{2n}(0|q) = (-1)^n(q;q^2)_n. \tag{3.4}
$$

Therefore Lemma 2.1 implies

$$
(q\alpha^2;q^2)_\infty\,\mathcal{E}_q(0;\alpha) = \sum_{n=0}^{\infty}(-a^2)^n q^{n^2}\frac{(q;q^2)_n}{(q;q)_{2n}} = (q\alpha^2;q^2)_\infty,
$$

which is $\mathcal{E}_q(0;\alpha) = 1$. So (3.3) implies that $g(\alpha) = 1$ and the second proof is complete.

Our proof of (1.4) uses the linearization formula [2], [7]

$$\frac{H_m(x|q)H_n(x|q)}{(q;q)_m(q;q)_n} = \sum_{k=0}^{\min(m,n)} \frac{H_{m+n-2k}(x|q)}{(q;q)_k(q;q)_{m-k}(q;q)_{n-k}}. \tag{3.5}$$

Proof of (1.4). From Lemma 2.1 and (3.5) we get

$$\begin{aligned}
&(q\alpha^2, q\beta^2; q^2)_\infty \mathcal{E}_q(x;\alpha)\mathcal{E}_q(x;\beta) \\
&= \sum_{m,n=0}^{\infty} q^{(m^2+n^2)/4}\alpha^m\beta^n \sum_{k=0}^{\min(m,n)} \frac{H_{m+n-2k}(x|q)}{(q;q)_k(q;q)_{m-k}(q;q)_{n-k}} \\
&= \sum_{m,n=0}^{\infty} q^{(m^2+n^2)/4}\alpha^m\beta^n(-\alpha\beta q^{(m+n+1)/2};q)_\infty \frac{H_{m+n}(x|q)}{(q;q)_m(q;q)_n} \\
&= \sum_{N=0}^{\infty} (-\alpha\beta q^{(N+1)/2};q)_\infty \frac{\alpha^N H_N(x|q)}{(q;q)_N} q^{N^2/4} {}_1\phi_0(q^{-N}; -; q, -q^{(N+1)/2}\beta/\alpha),
\end{aligned}$$

which simplifies to (1.4).

4. Another example. Any set of polynomials $p_n(x)$, whose the connection coefficients from $H_n(x|q)$ are explicitly given, will have an $\mathcal{E}_q(x;\alpha)$ expansion analogous to the plane wave expansion. For example, the special case $b = -a$ of the Al-Salam-Chihara polynomials of (1.12) satisfies

$$\frac{H_n(x|q)}{(q;q)_n} = \sum_{j=0}^{[n/2]} \frac{a^{4j-n}}{(q^2;q^2)_j} \frac{p_{n-2j}(x;a,-a|q)}{(q;q)_{n-2j}}. \tag{4.1}$$

Formula (4.1) follows easily from the generating function of $p_n(x;a,-a|q)$. If (4.1) is used in Lemma 2.1, the following theorem results.

Theorem 4.1 *The function $\mathcal{E}_q(x;\alpha)$ has the expansion*

$$(q\alpha^2;q^2)_\infty\, \mathcal{E}_q(x;\alpha) = \sum_{n=0}^{\infty} (-\alpha^2 a^2 q^{n+1};q^2)_\infty \alpha^n a^{-n} q^{n^2/4} \frac{p_n(x;a,-a|q)}{(q;q)_n}. \tag{4.2}$$

The case $a = 0$ of Theorem 4.1, when $a^{-n}p_n(x;a,-a|q) = H_n(x|q)$, is Lemma 2.1. The case $a = \sqrt{q}$ of Theorem 4.1, when $p_n(x;\sqrt{q},-\sqrt{q}|q) = q^{n/2}H_n(x|q^2)$, is [9, (4.3)].

5. Some integrals and sums. In this section we reinterpret Theorem 4.1 as integral evaluations. We need the orthogonality relations for continuous q-ultraspherical polynomials $\{C_n(x;\beta|q)\}$ [7, (7.4.15)]

$$\begin{aligned}(5.1)\qquad &\int_0^\pi C_m(\cos\theta;\beta|q)C_n(\cos\theta;\beta|q)\frac{(e^{2i\theta},e^{2i\theta};q)_\infty}{(\beta e^{2i\theta},\beta e^{-2i\theta};q)_\infty}\,d\theta\\ &=\frac{2\pi(\beta,q\beta;q)_\infty}{(q,\beta^2;q)_\infty}\frac{(\beta^2;q)_n(1-\beta)}{(q;q)_n(1-\beta q^n)}\,\delta_{m,n},\end{aligned}$$

for $|\beta|<1$. The case $\beta=0$ of (5.1) is the orthogonality relation for q-Hermite polynomials

$$(5.2)\qquad \int_0^\pi H_m(\cos\theta|q)H_n(\cos\theta|q)(e^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta=\frac{2\pi(q;q)_n}{(q;q)_\infty}\,\delta_{m,n},$$

while the Al-Salam-Chihara polynomials satisfy

$$\begin{aligned}(5.3)\qquad &\int_0^\pi p_m(\cos\theta;a,b|q)p_n(\cos\theta;a,b|q)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta};q)_\infty}\,d\theta\\ &=\frac{2\pi a^{2n}\,(q,ab;q)_n}{(q,ab;q)_\infty}\delta_{m,n}\end{aligned}$$

Theorem 5.1 *We have*

$$\begin{aligned}(5.4)\qquad &\int_0^\pi \mathcal{E}_q(\cos\theta;\alpha)\mathcal{E}_q(\cos\theta;\beta)(e^{2i\theta},e^{-2i\theta};q)_\infty H_n(\cos\theta|q)\,d\theta\\ &=\frac{2\pi q^{n^2/4}\alpha^n(-\alpha\beta q^{(n+1)/2};q)_\infty\,(-q^{(1-n)/2}\beta/\alpha;q)_n}{(q;q)_\infty\,(q\alpha^2,q\beta^2;q^2)_\infty},\\ (5.5)\qquad &\int_0^\pi \mathcal{E}_q(\cos\theta;\alpha)\mathcal{E}_q(\cos\theta;\beta)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}\,d\theta\\ &=\frac{2\pi(\gamma,q\gamma,-\alpha\beta q^{1/2};q)_\infty}{(q,\gamma^2;q)_\infty\,(q\alpha^2,q\beta^2;q^2)_\infty}\,{}_2\phi_2\left(\left.\begin{matrix}-q^{1/2}\alpha/\beta,-q^{1/2}\beta/\alpha\\ q\gamma,-\alpha\beta q^{1/2}\end{matrix}\,\right|\,q,-\alpha\beta\gamma q^{1/2}\right).\end{aligned}$$

Proof. Formula (5.4) immediately follows from (1.4) and (5.2). To prove (5.5), first substitute for the product of $\mathcal{E}_q$'s from (1.4). Thus the left-hand side of (5.5) is

$$\begin{aligned}(5.6)\qquad &\sum_{n=0}^\infty \frac{q^{n^2/4}\alpha^n}{(q;q)_n}(-\alpha\beta q^{(n+1)/2};q)_\infty(-q^{(1-n)/2}\beta/\alpha;q)_n\\ &\times\int_0^\pi \frac{H_n(\cos\theta|q)C_0(\cos\theta;\gamma|q)}{(q\alpha^2,q\beta^2;q^2)_\infty}\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}\,d\theta.\end{aligned}$$

¿From (1.8) and the orthogonality relation (5.1) it follows that the integral is 0 for n odd while for n even we have

$$\int_0^\pi H_{2n}(x|q)C_0(\cos\theta;\gamma|q)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}\,d\theta=\frac{\gamma^n(q;q)_{2n}}{(q,q\gamma;q)_n}\frac{2\pi(\gamma,q\gamma;q)_\infty}{(q,\gamma^2;q)_\infty}.$$

This reduces (5.6) to the right-hand side of (5.5) and the proof is complete.

Bustoz and Suslov [3] established the special case $\gamma = 1/2$ of (5.5) using divided difference operators. This special case played a key role in their theory of the q-Fourier integral. It is hoped that (5.5) will eventually lead to a one parameter extension of the Bustoz-Suslov theory.

The integral evaluation

$$\begin{aligned}(5.7)\qquad &\int_0^\pi \mathcal{E}_q(\cos\theta;\alpha)\mathcal{E}_q(\cos\theta;\beta)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}C_n(\cos\theta;\gamma|q)\,d\theta\\ &=\frac{2\pi(\gamma,\gamma q^{n+1},-\alpha\beta q^{(n+1)/2};q)_\infty(-q^{(1-n)/2}\beta/\alpha;q)_n}{(q;q)_n\,(q,\gamma^2q^n;q)_\infty\,(q\alpha^2,q\beta^2;q^2)_\infty}\alpha^n q^{n^2/4}\\ &\times{}_2\phi_2\left(\begin{matrix}-q^{(n+1)/2}\alpha/\beta,-q^{(n+1)/2}\beta/\alpha\\ q^{n+1}\gamma,-\alpha\beta q^{(n+1)/2}\end{matrix}\,\middle|\,q,-\alpha\beta\gamma q^{(n+1)/2}\right)\end{aligned}$$

extends (5.5) and can be proved in the same fashion. The expansion formula associated with (5.7) is

$$\begin{aligned}(5.8)\qquad \mathcal{E}_q(z;\alpha)\mathcal{E}_q(z;\beta) &= \sum_{n=0}^{\infty}\frac{(-\alpha\beta q^{(n+1)/2};q)_\infty\,(-q^{(1-n)/2}\beta/\alpha;q)_n}{(\gamma;q)_n\,(q\alpha^2,q\beta^2;q^2)_\infty}q^{n^2/4}C_n(z;\gamma|q)\\ &\times\alpha^n\,{}_2\phi_2\left(\begin{matrix}-q^{(n+1)/2}\alpha/\beta,-q^{(n+1)/2}\beta/\alpha\\ q^{n+1}\gamma,-\alpha\beta q^{(n+1)/2}\end{matrix}\,\middle|\,q,-\alpha\beta\gamma q^{(n+1)/2}\right).\end{aligned}$$

Observe that the ${}_2\phi_2$ in (5.8) with $\gamma = q^\nu$ and $\beta \to 0$ reduces to a multiple of $\alpha^{-\nu-n}J^{(2)}_{\nu+n}(\alpha;q)$, so this limiting case of (5.8) is equivalent to the q-plane wave expansion.

One can extend (5.5) in a direction different from (5.7) by replacing $H_n(x|q)$ by $p_n(x;a,-a|q)$. One can prove from (1.4), (4.1) and (5.3) that

$$\begin{aligned}(5.9)\qquad &\int_0^\pi \mathcal{E}_q(\cos\theta;\alpha)\mathcal{E}_q(\cos\theta;\beta)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(a^2e^{2i\theta},a^2e^{-2i\theta};q^2)_\infty}p_n(\cos\theta;a,-a|q)\,d\theta\\ &=\frac{2\pi a^n(-\alpha\beta q^{(n+1)/2};q)_\infty(-q^{(1-n)/2}\beta/\alpha;q)_n}{(q,-a^2q^n;q)_\infty\,(q\alpha^2,q\beta^2;q^2)_\infty}\alpha^n q^{n^2/4}\\ &\times{}_2\phi_2\left(\begin{matrix}-q^{(n+1)/2}\alpha/\beta,-q^{(n+1)/2}\beta/\alpha\\ -q,-\alpha\beta q^{(n+1)/2}\end{matrix}\,\middle|\,q,-\alpha\beta a^2 q^{(n+1)/2}\right).\end{aligned}$$

The evaluation (5.9) is equivalent to the orthogonal expansion

$$\begin{aligned}(5.10)\qquad &(q\alpha^2,q\beta^2;q^2)_\infty\,\mathcal{E}_q(x;\alpha)\mathcal{E}_q(x;\beta)\\ &=\sum_{n=0}^{\infty}\alpha^n a^{-n}q^{n^2/4}(-\alpha\beta q^{(n+1)/2};q)_\infty(-q^{(1-n)/2}\beta/\alpha;q)_n\frac{p_n(\cos\theta;a,-a|q)}{(q;q)_n}\\ &\times{}_2\phi_2\left(\begin{matrix}-q^{(n+1)/2}\alpha/\beta,-q^{(n+1)/2}\beta/\alpha\\ -q,-\alpha\beta q^{(n+1)/2}\end{matrix}\,\middle|\,q,-\alpha\beta a^2 q^{(n+1)/2}\right).\end{aligned}$$

We can deduce series identities from (5.5). For example if we substitute for the $\mathcal{E}_q$'s from (1.5) with $\gamma = q^\nu$ in (5.5) we arrive at the expansion

$$
\begin{aligned}
(5.11) \qquad & \sum_{n=0}^{\infty} \frac{(1-q^{\nu+n})(q^{2\nu};q)_n}{(1-q^\nu)(q;q)_n}(-1)^n q^{n^2/2} J_{\nu+n}^{(2)}(2\alpha;q) J_{\nu+n}^{(2)}(2\beta;q) \\
&= \frac{(\alpha\beta)^\nu\,(q^{\nu+1},q^{\nu+1},\alpha\beta q^{1/2};q)_\infty}{(q,q;q)_\infty}\, {}_2\phi_2\left(\left.\begin{matrix} -q^{1/2}\alpha/\beta, -q^{1/2}\beta/\alpha \\ q^{\nu+1}, \alpha\beta q^{1/2}\end{matrix}\,\right|\, q, \alpha\beta q^{\nu+1/2}\right).
\end{aligned}
$$

If α and β are replaced by $(1-q)\alpha$ and $(1-q)\beta$, and $q \to 1$, then (5.11) reduces to the multiplication formula

$$
(5.12) \qquad \sum_{n=0}^{\infty} \frac{(\nu+n)}{\nu}\frac{(2\nu)_n}{n!}(-1)^n J_{\nu+n}(2\alpha) J_{\nu+n}(2\beta) = \left(\frac{\alpha\beta}{\alpha+\beta}\right)^\nu \frac{J_\nu(2(\alpha+\beta))}{\Gamma(\nu+1)}.
$$

This is very interesting because it is the special case $\phi = 0$ of the Gegenbauer addition theorem [13, (11.4.2)].

One can also derive variations on these integral evaluations and summation identities. For example by taking products of (1.4) then applying (5.2) we find

$$
\begin{aligned}
(5.13) \qquad & \int_0^\pi \left[\prod_{k=1}^4 \mathcal{E}_q(\cos\theta;\alpha_k)\right](e^{2i\theta}, e^{-2i\theta};q)_\infty\, d\theta \\
&= \frac{2\pi}{(q;q)_\infty \prod_{k=1}^4 (q\alpha_k^2;q^2)_\infty} \sum_{n=0}^\infty q^{n^2/2}(\alpha_1\alpha_3)^n \\
&\times(-\alpha_1\alpha_2 q^{(n+1)/2}, -\alpha_3\alpha_4 q^{(n+1)/2};q)_\infty(-q^{(1-n)/2}\alpha_2/\alpha_1, -q^{(1-n)/2}\alpha_4/\alpha_3;q)_n.
\end{aligned}
$$

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