A HYPERGEOMETRIC HIERARCHY FOR THE ANDREWS EVALUATIONS

D. Stanton[1]

Abstract. Several ${}_6F_5(1)$ evaluations are given which generalize Andrews' ${}_5F_4(1)$ evaluations. All such evaluations are shown to be equivalent to transformations for a ${}_4F_3(z)$. The methodology allows for higher evaluations, for example an ${}_8F_7(1)$ is given which specializes to over 100 ${}_5F_4(1)$ results, including all of Andrews'.

1. Introduction.

In [1] George Andrews listed 20 1-balanced ${}_5F_4(1)$ evaluations, which he proved by induction using contiguous relations. He also stated 10 others, and said that many other related evaluations exists. This proliferation of evaluations has been somewhat mysterious, in that they did not fit into the hypergeometric hierarchy. Moreover, applying the WZ methodology [3] to prove them has met with only partial success. The purpose of this paper is to organize the evaluations into the hypergeometric framework by unifying their proof, and finding the more general transformations that naturally imply the evaluations.

The idea is to find several 1-balanced ${}_6F_5(1)$ evaluations, each one of which gives nine different ${}_5F_4(1)$ evaluations. 20 such results are given in Theorem 1 and the Appendix ((A1)-(A20)). In §4 (see Theorem 2) we state ${}_7F_6(1)$ and ${}_8F_7(1)$ evaluations, which prove 45 and 165 ${}_5F_4(1)$ evaluations, and 9 and 45 ${}_6F_5(1)$ evaluations respectively. All of Andrews' evaluations are corollaries of Theorem 2.

In §5 a general result is given for changing any such evaluation into a transformation for a hypergeometric series at z (Proposition 3). New transformations for hypergeometric series arise, three are stated.

We use standard notation for hypergeometric series found in [4]. Also we let

$$< z^n | F(z) >$$

denote the coefficient of z^n in a formal power series $F(z)$.

2. Preliminaries.

In [2], a several transformations were given which proved Andrews' original [1, (1.6)] ${}_5F_4(1) = 0$. One of them is

(2.1)

$$(a+n+1)_n \, {}_5F_4\left(\begin{matrix} x+1, z+1, x-z+1/2, -n, -a-n; 1 \\ 2z+2, 2x-2z+1, (-a-2n)/2, (1-a-2n)/2 \end{matrix}\right)$$

$$= (a+x+n+2)_n \, {}_5F_4\left(\begin{matrix} x+1, x-2z, 1-x+2z, -n/2, (1-n)/2; 1) \\ z+3/2, x-z+1, a+x+n+2, -1-a-x-2n \end{matrix}\right).$$

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We shall see that suitably modifying (2.1) yields all of the results. In this section we review one proof of (2.1) and motivate the method for the generalization.

Equation (2.1) followed from the transformation [2, (5.4)]

$$
(2.2)\qquad \begin{aligned} &{}_3F_2\left(\begin{matrix} x+1, z+1, x-z+1/2; 4y(1-y) \\ 2z+2, 2x-2z+1 \end{matrix}\right) = \\ &(1-y)^{-x-1}\ {}_3F_2\left(\begin{matrix} x+1, x-2z, 1-x+2z; \frac{-y^2}{4(1-y)} \\ z+3/2, x-z+1 \end{matrix}\right). \end{aligned}
$$

We recapitulate one proof of (2.1), since the same technique will be used for the main results.

If we multiply both sides of (2.2) by $(1-y)^{-a}/(1-2y)$, and put $z=4y(1-y)$, the right side is

$$
(2.3)\qquad \sum_{j=0}^{\infty} \frac{(x+1, x-2z, 1-x+2z)_j}{(1, z+3/2, x-z+1)_j} \left(\frac{-z^2}{64}\right)^j (1-y)^{-a-x-1-3k}/(1-2y).
$$

We find the coefficient of z^n in (2.3) by using

$$
(2.4)\qquad \frac{(1-y)^{-A}}{(1-2y)} = \sum_{k=0}^{\infty} \frac{(A+k+1)_k}{k!} \left(\frac{z}{4}\right)^k.
$$

The resulting coefficient is

$$
\frac{(a+x+n+2)_n}{4^n n!}\ {}_5F_4\left(\begin{matrix} x+1, x-2z, 1-x+2z, -n/2, (1-n)/2; 1 \\ z+3/2, x-z+1, -1-x-a-2n, a+x+n+2 \end{matrix}\right).
$$

The same method can be applied to the left side of (2.2), to obtain (2.1).

We record the above method in a lemma.

Lemma 1. *If $z=4y(1-y)$, then for any formal power series $F(z)=\sum_k \alpha_k z^k$,*

$$
(1)\qquad \begin{aligned} &< z^n | F(z)(1-y)^{-a}/(1-2y) >= \\ &\qquad \frac{(a+n+1)_n}{4^n n!} \sum_{k=0}^{n} \alpha_k \frac{(-n, -a-n)_k}{((-a-2n)/2, (1-a-2n)/2)_k}, \end{aligned}
$$

$$
(2)\qquad \begin{aligned} &< z^n | F(z)(1-y)^{-a} >= \\ &\qquad a\frac{(a+n+1)_{n-1}}{4^n n!} \sum_{k=0}^{n} \alpha_k \frac{(-n, -a-n)_k}{((1-a-2n)/2, (2-a-2n)/2)_k}. \end{aligned}
$$

Proof. The first part follows from (2.4), the second part from

$$
(2.5)\qquad (1-y)^{-A} = \sum_{k=0}^{\infty} \frac{A(A+k+1)_{k-1}}{k!} \left(\frac{z}{4}\right)^k.
$$

□

We modify (2.2) to a ${}_4F_3$ transformation by inserting the parameter pair $B+1, B$ on the left side. The B-generalization of (2.2) (see Lemma 2) is a ${}_6F_5$ transformation. To state the results, we modify Andrews' H-function

$$H(n,m,a_1,a_2,a_3) = \tag{2.6}$$
$$ {}_5F_4\left(\begin{matrix} -m-n, x+m+n+1+a_1, x-z+1/2, x+m+a_2, z+n+1; 1 \\ (2+x)/2, (1+x)/2, 2z+m+n+1+a_3, 2x-2z+1+a_1+a_2-a_3 \end{matrix}\right)$$

to

$$H2(n,a_1,a_2,a_3,a_4) = \tag{2.7}$$
$$ {}_6F_5\left(\begin{matrix} -n, x+n/2+1+a_1, x-z+1/2, x+a_2, z+1, B+1; 1 \\ \theta_1, \theta_1+1/2, 2z+1+a_3, 2x-2z+1+a_1+a_2-a_3-a_4, B \end{matrix}\right).$$

where $\theta_1 = (2+x+a_4-n/2)/2$.

The dependence upon x, z, and B in the H functions has been depressed, although sometimes we may append x, z to the notation. $H2$ is a linear function of $1/B$, and thus is uniquely determined by its value at two different B's. We next see that any $H2$ evaluation implies nine H evaluations.

Proposition 1. *$H2(n,a_1,a_2,a_3,a_4)$ can be specialized to obtain nine different H functions:*

$$\begin{aligned}
&H(n/2-1-a_4, n/2+1+a_4, a_1-a_4, a_2-2a_4-2, a_3-2a_4-2),\\
&H(n/2-a_4, n/2+a_4, a_1-a_4-1, a_2-2a_4-1, a_3-2a_4-2),\\
&H(n/2-a_4, n/2+a_4, a_1-a_4-1, a_2-2a_4-1, a_3-2a_4),\\
&H(n/2-a_4-1, n/2+a_4+1, a_1-a_4-1, a_2-2a_4-2, a_3-2a_4-3),\\
&H(n/2-a_4-1, n/2+a_4+1, a_1-a_4-1, a_2-2a_4-2, a_3-2a_4-2),\\
&H(n/2-a_4, n/2+a_4, a_1-a_4, a_2-2a_4, a_3-2a_4),\\
&H(n/2-a_4-1, n/2+a_4+1, a_1-a_4-1, a_2-2a_4-1, a_3-2a_4-2),\\
&H(n/2-a_4-1, n/2+a_4, a_1-a_4, a_2-2a_4-1, a_3-2a_4-1) \text{ and}\\
&H(n/2-a_4-1, n/2+a_4, a_1-a_4-1, a_2-2a_4-2, a_3-2a_4-2).
\end{aligned}$$

Proof. If $B = x+n/2+a_1+1$, the $H2$ function becomes

$$H(n/2-1-a_4, n/2+1-a_4, a_1-a_4, a_2-2a_4-2, a_3-2a_4-2, x-n/2+a_4+1, z-n/2+a_4+1).$$

The next 8 choices are given by $B = z+1, x-z+1/2, 2z+a_3$, $2x-2z+a_1+a_2-a_3-a_4, (x+1-n/2+a_4)/2, x+a_2, -n, 0$, respectively. □

3. The main theorem.

We prove Theorem 1 from a ${}_6F_5$ transformation, which is Lemma 2.

Lemma 2. *If n is a non-negative integer,*

$$(-x-3n/2-a-1)\frac{(-x-n/2-a)_{n-1}}{n!}H2(n,a,1,1,a)=$$
$$\sum_{j=0}^{n/2}\frac{(x+1,x-2z,1-x+2z)_j}{(1,z+3/2,x-z+1)_j}\frac{(-n/2-a+j+1)_{n-2j-1}}{(n-2j)!}$$
$$\times\left((1+n/B)(-3n/2-a+3j)+(x+3n/2+a+1)(n-2j)/B\right).$$

Proof. Apply $(1-y)^{-\alpha}(B+z\frac{d}{dz})/B$ to both sides of (2.2), where $z=4y(1-y)$. Lemma 1 implies that the coefficient of z^n on the left side is

$$\frac{\alpha(\alpha+n+1)_{n-1}}{n!4^n}{}_6F_5\left(\begin{matrix}x+1,x+1,x-z+1/2,B+1,-n,-\alpha-n\\2z+2,2x-2z+1,B,(1-\alpha-2n)/2,(2-\alpha-2n)/2\end{matrix}\right).$$

If $-\alpha-n=x+n/2+a+1$, then the ${}_6F_5$ becomes the stated $H2$.

Let $R(y)$ denote the right side of (2.2). Then

$$< z^n|(1-y)^{-\alpha}R(y)+\frac{z}{B}(1-y)^{-\alpha}\frac{d}{dz}(R(y)) >=$$
$$< z^n|(1-y)^{-\alpha}R(y)+\frac{z}{B}\frac{d}{dz}((1-y)^{-\alpha}R(y))-\frac{z}{B}R(y)\frac{d}{dz}((1-y)^{-\alpha}) >=$$
$$< z^n|(1-y)^{-\alpha}R(y)+\frac{z}{B}\frac{d}{dz}((1-y)^{-\alpha}R(y))-\frac{z}{B}R(y)\alpha(1-y)^{-\alpha-1}/4(1-2y) > .$$

We can routinely find the coefficient of z^n in each term using Lemma 1. Summing these terms gives the stated result. □

In the statement of Theorem 1,

$$R_n(x,z)=\frac{(1-x+2z,x-2z,1/2)_n}{(z+3/2,1+x-z,-1-x)_n}.$$

Theorem 1. *If n is a non-negative integer,*

(A1)
$$H2(2n,0,1,1,0)=R_n(x,z)\frac{(B+2n)(x+1)}{B(x+3n+1)},$$

(A2)
$$H2(2n+1,-1/2,1,1,-1/2)=R_n(x,z)\frac{(2n+1)(B-x-1-n)(x+1)}{B(x+3n+2)(x-n+1)},$$

(A3)
$$H2(2n+1,1/2,1,1,1/2)=R_n(x,z)\frac{(2n+1)(2B-x-1+n)(x+1)}{B(x+3n+3)(x+n+1)}.$$

Proof. If we set $a = 0$ in Lemma 2, and n is even, the right side factor $(-n/2 + j + 1)_{n-2j-1}$ is zero for $0 \le j < n/2$, so only the $j = n/2$ term contributes. The choices $a = \pm 1/2$ work in the same way for n odd. □

According to Proposition 1, Theorem 1 gives 27 H evaluations. 23 of them are distinct, and 11 appeared in Andrews' list of 30 in [1].

We can now easily prove many more $H2$ evaluations, by finding $H2$'s whose nine B specializations include two that are known from previous cases. We list these in the Appendix. Note that (A4) is independent of B. The number of H evaluations implied by these results is over 100.

4. Higher evaluations.

Because of the proliferation of $H2$ evaluations, one may ask if it is possible to give a 1-balanced ${}_7F_6$ evaluation, which will imply nine different $H2$ evaluations. One may also ask for a single evaluation which implies every one on Andrews list. In this section we explicitly answer both of these questions. In particular Theorem 2 evaluates a 1-balanced ${}_8F_7$ which gives Andrews' 30 ${}_5F_4$'s and over 100 others.

Let

$$\begin{aligned} &H3(n, a_1, a_2, a_3, a_4, x, z) = \\ (4.1)\qquad &{}_7F_6\left(\begin{matrix}-n, x+n/2+1+a_1, x-z+1/2, x+a_2, z+1, B+1, C+1; 1\\ \theta_1, \theta_1+1/2, 2z+1+a_3, 2x-2z+1+a_1+a_2-a_3-a_4, B, C\end{matrix}\right). \end{aligned}$$

where $\theta_1 = (3 + x + a_4 - n/2)/2$. (We delete x and z in formula involving H functions if x and z are constant throughout.)

As before, $H3$ is a linear polynomial in $1/C$. As in Proposition 1, $H3$ can be specialized in nine ways to obtain an $H2$, any two of which determine $H3$. For example, $C = z + 1$, $C = x + a_2$ give

$$\begin{aligned} &H3(n, a_1, a_2, a_3, a_4, x, z) = \frac{(C-z-1)(x+a_2)}{C(x+a_2-z-1)} H2(n, a_1, a_2+1, a_3, a_4+1, x, z) + \\ (4.2)\qquad &\frac{(C-x-a_2)(z+1)}{C(z+1-x-a_2)} H2(n, a_1-1, a_2-1, a_3-2, a_4, x+1, z+1). \end{aligned}$$

Applying (A6) and (A7) to (4.2), we see that $H3(2n + 1, 3/2, 1, 3, -1/2, x, z)$ is evaluable. A similar argument with

$$\begin{aligned} &H3(n, a_1, a_2, a_3, a_4) = \frac{(2C-2-x-a_4+n/2)(2z+a_3)}{C(4z+2a_3-2-x-a_4+n/2)} H2(n, a_1, a_2, a_3-1, a_4+1) + \\ (4.3)\qquad &\frac{(C-2z-a_3)(2+x+a_4-n/2)}{C(2+x+a_4-n/2-2z-a_3))} H2(n, a_1, a_2, a_3, a_4), \end{aligned}$$

(A3), and (A8) evaluates $H3(2n + 1, 1/2, 1, 2, -1/2, x, z)$.

We can iterate this technique to evaluate a single ${}_8F_7$ that specializes to $\binom{11}{3} = 165$ H's, including all 30 on Andrews' list. Of the 165 possible specializations, 145 are distinct. Let

$$H4(n, a_1, a_2, a_3, a_4) =$$

(4.4)

$$ {}_8F_7\left({-n, x+n/2+1+a_1, x-z+1/2, x+a_2, z+1, B+1, C+1, D+1; 1 \atop \theta_1, \theta_1+1/2, 2z+1+a_3, 2x-2z+1+a_1+a_2-a_3-a_4, B, C, D} \right).$$

where $\theta_1 = (4+x+a_4-n/2)/2$.

Theorem 2. *If n is a non-negative integer, then*

$$\begin{aligned}
&H4(2n+1, 1/2, 1, 3, -3/2) = \\
&K_1((C-z-1)(x+1)(A6)(x,z) - (C-x-1)(z+1)(A7)(x+1,z+1))+ \\
&K_2((2C-3-x+n+2)(2z+2)(A3)(x,z) - (C-2z-2)(x-n+1)(A8)(x,z)),
\end{aligned}$$

where (A6), (A7), (A3), and (A8), are given in Theorem 1 and the Appendix, and

$$K_1 = \frac{(D-2z-3)(x+n+2)}{CD(x+n-2z-1)(x-z)}, K_2 = \frac{(D-x-n-2)(2z+3)}{CD(x+n-2z-1)(x-n-4z-3)}.$$

Proof. Use

$$H4(n, a_1, a_2, a_3, a_4) = \frac{(D-2z-a_3)(x+n/2+1+a_1)}{D(x+n/2+1+a_1-2z-a_3)} H3(n, a_1+1, a_2, a_3, a_4+1)+$$

(4.5)

$$\frac{(D-x-n/2-1-a_1)(2z+a_3)}{D(2z+a_3-x-n/2-1-a_1)} H3(n, a_1, a_2, a_3-1, a_4+1)$$

and the two above evaluations of $H3$'s. □

5. Transformations.

In this section we derive ${}_4F_3$ transformations from any of the evaluations in Theorem 1 or the Appendix.

Suppose that $F(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ is a formal power series. We will use the following lemma.

Lemma 3. *If $z = 4y(1-y)$, $w = y/\sqrt{1-y}$. Then*

$$< z^n | F(z)(1-y)^{-a}/(1-2y) > = < w^n | F(z)(1-y)^{-a-3n/2}/(1-y/2)4^{-n} > .$$

Proof. We change variables from z to y to w:

$$\begin{aligned}
&< z^n | F(z)(1-y)^{-a}/(1-2y) > = < y^n | F(z)(1-y)^{-a-n-1} 4^{-n} > = \\
&< w^n | F(z)(1-y)^{-a-3n/2}/(1-y/2)4^{-n} > .
\end{aligned}$$

□

Most of the evaluations(A1)-(A20) have the form below, so we suppose that

$$\sum_{k=0}^{n} \alpha_k \frac{(-n, x+n/2+\gamma)_k}{((x-n/2+\gamma)/2, (x-n/2+\gamma+1)/2)_k} = \beta_n. \tag{5.1}$$

$$\sum_{k=0}^{n} \alpha_k \frac{(-n, x+n/2+\gamma)_k}{((1+x-n/2+\gamma)/2, (2+x-n/2+\gamma)/2)_k} = \delta_n. \tag{5.2}$$

Let

$$G(y) = F(4y(1-y))(1-y)^{x+\gamma}/(1-y/2).$$

Proposition 2. *If (4.1) and (4.2) hold, then*

$$G(y) + G(y/(y-1)) = 2\sum_{N=0}^{\infty} \beta_{2N} \frac{(1-x-\gamma-N)_{2N}}{(2N)!} (\frac{y}{\sqrt{1-y}})^{2N} \tag{1}$$

$$G(y) - G(y/(y-1)) = 2\sum_{N=0}^{\infty} \beta_{2N+1} \frac{(1/2-x-\gamma-N)_{2N+1}}{(2N+1)!} (\frac{y}{\sqrt{1-y}})^{2N+1} \tag{2}$$

$$\begin{gathered} G(y)(1-2y) + G(y/(y-1))(1+y) = \\ 2\sum_{N=0}^{\infty} \delta_{2N}(-x-3N-\gamma) \frac{(1-x-\gamma-N)_{2N-1}}{(2N)!} (\frac{y}{\sqrt{1-y}})^{2N} \end{gathered} \tag{3}$$

$$\begin{gathered} G(y)(1-2y) - G(y/(y-1))(1+y) = \\ 2\sum_{N=0}^{\infty} \delta_{2N+1}(-x-3N-\gamma-3/2) \frac{(1/2-x-\gamma-N)_{2N}}{(2N+1)!} (\frac{y}{\sqrt{1-y}})^{2N+1}. \end{gathered} \tag{4}$$

Proof. We prove the first statement, the other three are done similarly. Expand $G(y)+G(y/(y-1))$ as a power series in $w = y/\sqrt{1-y}$. Since the map $y \to y/(y-1)$ sends w to $-w$, $G(y) + G(y/(y-1))$ is an even function of w. Lemma 2 implies

$$\begin{aligned} < w^{2N}|G(y) > &=< w^{2N}|F(z)(1-y)^{x+\gamma}/(1-y/2) >= \\ &=< z^{2N}|F(z)(1-y)^{x+\gamma+3N}4^{2N}/(1-2y) > . \end{aligned}$$

The result follows from Lemma 1 and (4.1). The odd part gives the second statement, while the second part of Lemma 1 gives the final two results. □

We apply Proposition 1 to (A1), (A2), and (A4). We find

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} x+1, z+1, x-z+1/2, B+1; 4y(1-y) \\ 2z+2, 2x-2z+1, B \end{matrix}\right)(1-y)^{x+1}(1-2y)+ \\
(5.3)\quad &{}_4F_3\left(\begin{matrix} x+1, z+1, x-z+1/2, B+1; -4y/(1-y)^2) \\ 2z+2, 2x-2z+1, B \end{matrix}\right)(1-y)^{-x-1}(1+y) = \\
&2(1-y/2){}_4F_3\left(\begin{matrix} x+1, 1-x+2z, x-2z, B/2+1; -y^2/4(1-y) \\ z+3/2, 1+x-z, B/2 \end{matrix}\right),
\end{aligned}
$$

(5.4)

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} x+1, z+1, x-z+1/2, B+1; 4y(1-y) \\ 2z+2, 2x-2z+1, B \end{matrix}\right)(1-y)^{x+1/2}(1-2y)- \\
&{}_4F_3\left(\begin{matrix} x+1, z+1, x-z+1/2, B+1; -4y/(1-y)^2) \\ 2z+2, 2x-2z+1, B \end{matrix}\right)(1-y)^{-x-1/2}(1+y) = \\
&2(1-y/2)\frac{(x-B+1)y}{B\sqrt{1-y}}\ {}_4F_3\left(\begin{matrix} x+1, 1-x+2z, x-2z, x-B+2; -y^2/4(1-y) \\ z+3/2, 1+x-z, x-B+1 \end{matrix}\right),
\end{aligned}
$$

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} x+2, z+1, x-z+1/2, B+1; 4y(1-y) \\ 2z+3, 2x-2z+2, B \end{matrix}\right)(1-y)^{x+2}+ \\
(5.5)\quad &{}_4F_3\left(\begin{matrix} x+2, z+1, x-z+1/2, B+1; -4y/(1-y)^2) \\ 2z+3, 2x-2z+2, B \end{matrix}\right)(1-y)^{-x-1} = \\
&2(1-y/2)\ {}_4F_3\left(\begin{matrix} x+2, 1-x+2z, x-2z; -y^2/4(1-y) \\ z+3/2, 1+x-z \end{matrix}\right).
\end{aligned}
$$

(4.3)-(4.5) generalize the ${}_3F_2$ transformations given in [2].

We can also give ${}_6F_5(1)$ transformations (analogous to Lemma 2), by multiplying the transformations in Proposition 2 by $(1-y)^{-a}$, and expanding either as function of z of y. These results have the form ${}_6F_5(1)+{}_6F_5(1)={}_6F_5(1)$.

6. Remarks.

It is not surprising that results such as (A1)-(A20) exist in view of Andrews' results. Inserting the a pair $B+1, B$ into a ${}_5F_4$ gives a sum of two ${}_5F_4$'s, so there should be a result with two terms. The choices in (A1)-(A9) are particularly nice, appearing as one term.

Andrews' original ${}_5F_4 = 0$ follows immediately from any of (A1)-(A9) by specializing the linear factor in B to be 0.

Theorem 2 specializes to 11 of the 20 evaluations (A1)-(A20). One could give a ${}_9F_8$ evaluation (as a sum of 8 terms) generalizing Theorem 2 which implies (A1)-(A20).

Evaluations of 2, 3 and 4- balanced ${}_5F_4$'s are obtained by taking the $B, C, D \to \infty$ limits of Theorem 2 and (A1)-(A20).

One can obtain transformations of 2-balanced series, without appealing to B. For example, if the second part of Lemma 1 is applied to (2.2) one finds

(6.1)

$$\frac{(a)_{2n}}{(a+1)_n}\,{}_5F_4\left(\begin{matrix}x+1,z+1,x-z+1/2,-n,-a-n;1\\2z+2,2x-2z+1,(1-a-2n)/2,(2-a-2n)/2\end{matrix}\right)$$
$$=\frac{(a+x+1)_{2n}}{(a+x+2)_n}\,{}_6F_5\left(\begin{matrix}x+1,x-2z,1-x+2z,-n/2,(1-n)/2,(x+a+4)/3;1\\z+3/2,x-z+1,a+x+n+2,-a-x-2n,(x+a+1)/3\end{matrix}\right).$$

Appendix.

We state several other $H2$ evaluations, besides the three given in Theorem 1. (A4)-(A9) were chosen because of the simple form, (A10)-(A20) because of the simple form of the $B\to\infty$ limit, which is a 2-balanced ${}_5F_4$ evaluation.

We use

$$S_n(x,z,a,b,c)=\frac{(1-x+2z+a,x-2z+b,1/2)_n}{(z+3/2+c,1+x-z+a+b,-1-x)_n},$$

so that $R_n(x,z)=S_n(x,z,0,0,0)$.

(A4) $$H2(2n,1,2,2,0)=R_n(x,z),$$

(A5) $$H2(2n+1,3/2,2,1,1/2)=S_n(x,z-1,2,-2,1)\frac{(2n+1)(1-2B+2x-2z)}{B(2z-2x-3)},$$

(A6) $$H2(2n+1,3/2,2,3,1/2)=S_n(x,z,0,0,1)\frac{(2n+1)(B-1-z)}{B(z+2)},$$

(A7) $$H2(2n+1,1/2,0,1,-1/2)=S_n(x-2,z,-1,3,0)\frac{(2n+1)(-x+n)(x-B)}{B(x+n)_2}$$

(A8) $$H2(2n+1,1/2,1,2,-1/2)=S_n(x-1,z,0,1,0)\frac{(2n+1)(B-2z-2)}{B(2z+2n+3)}$$

(A9) $$H2(2n+1,1/2,1,1,-1/2)=S_n(x-1,z,-1,2,0)\frac{(2n+1)(1-B+2x-2z)}{2B(z-x-n-1)}$$

(A10)

$$H2(2n,1,1,2,0,x,z)=S_n(x-1,z,0,1,0)\frac{(x-n+1)(B(2z-x-n+1)+2n(z+1))}{B(x+n+1)(2z-x+n+1)}$$

(A11) $$H2(2n,1,1,1,0,x,z)=R_n(x,z)\frac{(1+x)(B(2z-x+n)+n(2z-2x-1))}{(2z-x)B(x+n+1)}$$

$$H2(2n+1,1/2,2,2,1/2,x,z)= \tag{A12}$$
$$R_n(x,z)\frac{(2n+1)(B(2z+x+3n+5)-(z+1)(2x+2n+4))}{(2z+2n+3)B(x+3n+3)}$$

$$H2(2n+1,3/2,2,2,1/2,x,z)= \tag{A13}$$
$$R_n(x,z)\frac{(2n+1)(B(3x-2z+3n+4)+(2z-2x-1)(x+n+2))}{2B(x+3n+3)(x-z+n+1)}$$

$$H2(2n+1,3/2,2,2,1/2,x,z)= \tag{A14}$$
$$R_n(x,z)\frac{(2n+1)(B(x+n+2)+z(2z+1-2x)-2x-1)}{B(2z+2n+3)(x-z+n+1)}$$

$$H2(2n+1,-1/2,0,0,-1/2,x,z)= \tag{A15}$$
$$S_n(x-1,z,-1,2,0)\frac{x(2n+1)(B(4z-x+3n+2)-2x(z+2n+1)+2nz)}{(2z+1)B(x+3n+2)(x+n)}$$

$$H2(2n+1,-1/2,0,1,-1/2,x,z)= \tag{A16}$$
$$S_n(x-1,z,0,1,0)\frac{x(2n+1)(B(4z-3x-3n)+(2x+1)(x+n)+2z(n-x))}{2(z-x)B(x+3n+2)(x+n)}$$

$$H2(2n+1,1/2,2,1,-1/2,x,z)= \tag{A17}$$
$$S_n(x,z-1,1,-1,1)\frac{(2n+1)(B(2z+2n+1)+(n+x+2)(2z-2x-1))}{(2z-2x-3)B(-x+n-1)}$$

$$H2(2n+1,1/2,2,3,-1/2,x,z)= \tag{A18}$$
$$S_n(x,z,1,-1,1)\frac{(2n+1)(B(z-x-n)+(x+n+2)(z+1))}{(z+2)B(-x+n-1)}$$

$$H2(2n,0,1,2,0,x,z)=S_n(x,z,0,-1,0)\frac{(1+x)(B(2z-x-3n+1)+4n(z+1))}{(1-x+2z)B(x+3n+1)} \tag{A19}$$

$$H2(2n,0,1,0,0,x,z)=S_n(x,z-1,1,-2,0)\frac{(1+x)(B(2z-x+3n)+n(4z-4x-2))}{(-x+2z)B(x+3n+1)} \tag{A20}$$

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