THE COMBINATORICS OF q-CHARLIER POLYNOMIALS

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Abstract. We describe various aspects of the Al-Salam-Carlitz q-Charlier polynomials. These include combinatorial descriptions of the moments, the orthogonality relation, and the linearization coefficients.

1. Introduction.

The Charlier polynomials $C_n^a(x)$ are well-known analytically [4], and have been studied combinatorially by various authors [8], [12], [16], [17], [20]. The moments for the measure of these orthogonal polynomials are

$$\mu_n = \sum_{k=1}^{n} S(n,k)a^k, \tag{1.1}$$

where $S(n,k)$ are the Stirling numbers of the second kind. The purpose of this paper is to study combinatorially an appropriate q-analogue of $C_n^a(x)$, whose moments are a q-Stirling version of (1.1). While studying these polynomials, we use statistics on set partitions which are q-Stirling distributed.

Our main result (Theorem 3) is the combinatorial proof of the linearization coefficients for these polynomials. In the $q=1$ case, the linearization coefficients are given as a polynomial in a, whose coefficients are quotients of factorials (see (4.4)). This has a simple combinatorial explanation. However, in the q-case the coefficients are not the analogous quotients of q-factorials. They are alternating sums of quotients of q-factorials, and thus a combinatorial explanation is much more difficult. From the combinatorial interpretations of the polynomials and their moments, in terms of weighted partial permutations and set partitions, we deduce a combinatorial interpretation for the linearization coefficients of a product of three q-Charlier polynomials. We then apply a weight-preserving sign-reversing involution defined in five steps. Theorem 3 is obtained by enumerating the remaining fixed points. Some of the steps of the involution are quite straight-forward, but some others are more complicated. They use more sophisticated techniques such as encoding of permutations or set partitions into *0–1 tableaux* (cf [6], [18]), which are fillings of Ferrers diagrams with 0's and 1's such that there is exactly one 1 in each column. They also use *interpolating statistics* on set partitions, as were

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX introduced by White in [22]. Indeed, the characterization of the final set of fixed points uses a bijection Ψ_S of White between interpolating statistics, making their enumeration all the more complicated.

It turns out that our q-Charlier polynomials are not what have classically been called q-Charlier; in fact they are rescaled versions of the Al-Salam-Carlitz polynomials [4, p.196]. Some comparisons to the classical q-Charlier are given in §7. Zeng [24] has also studied both families of polynomials from the associated continued fractions.

The basic combinatorial interpretation of the polynomials is given in Theorem 1. Several facts about the polynomials can be proven combinatorially. The combinatorics of set partitions, restricted growth functions and 0–1 tableaux is discussed in §3, and the statistic for the moments is given in Theorem 2. In §4, we state our main theorem, Theorem 3, giving the linearization coefficient for a product of three q-Charlier polynomials, and we set up the general combinatorial context for its demonstration. The five steps of the weight-preserving sign-reversing involution proving Theorem 3 are given in §5, and the combinatorial evaluation of the remaining fixed points is the subject of §6.

We use the standard notation for q-binomial coefficients and shifted factorials found in [11]. We will also need

$$[n]_q = \frac{1-q^n}{1-q},$$

and

$$[n]!_q = [n]_q[n-1]_q \cdots [1]_q.$$

2. The q-Charlier polynomials.

We define the q-Charlier polynomials by the three term recurrence relation

$$C_{n+1}(x,a;q) = (x - aq^n - [n]_q)C_n(x,a;q) - a[n]_q q^{n-1} C_{n-1}(x,a;q), \tag{2.1}$$

where $C_{-1}(x,a;q) = 0$ and $C_0(x,a;q) = 1$.

It is not hard to show that these polynomials are rescaled versions of the Al-Salam Carlitz polynomials [4, p.196]

$$C_n(x,a;q) = a^n U_n(\frac{x}{a} - \frac{1}{a(1-q)}, \frac{-1}{a(1-q)}). \tag{2.2}$$

Since the generating function of the $U_n(x,b)$ is known [4], we see that

$$\sum_{n=0}^{\infty} C_n(x,a;q)\frac{t^n}{(q)_n} = \frac{(at)_\infty(-\frac{t}{1-q})_\infty}{(t(x-\frac{1}{1-q}))_\infty}. \tag{2.3}$$

This gives the explicit formula

$$C_n(x,a;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} q^{\binom{n-k}{2}} \prod_{i=0}^{k-1}(x-[i]_q). \tag{2.4}$$

Clearly, we want a q-version of [16], which gives the Charlier polynomials as a generating function of weighted *partial permutations*, i.e. pairs (B,σ), where

$B \subseteq \{1, 2, \cdots, n\} = [n]$, and σ is a permutation on $[n] - B$. Thus we need only interpret the individual terms in (2.4) for a combinatorial interpretation. The inside product can be expanded in terms of the q-Stirling numbers of the first kind. We let $cyc(\sigma)$ be the number of cycles of a permutation σ and $inv(\sigma)$ be the number of inversions of σ written as a product of disjoint cycles (increasing minima, minima first in a cycle).

$$\prod_{i=0}^{k-1}(x - [i]_q) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{k-cyc(\sigma)} q^{inv(\sigma)} x^{cyc(\sigma)}.$$

For the sum over k in (2.4), we sum over all $(n-k)$ subsets $B \subseteq [n]$. Let

$$inv(B) = \sum_{b \in B}(b-1),$$

so that the generating function for these subsets is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}}.$$

We have established the following theorem.

Theorem 1. *The q-Charlier polynomials are given by*

$$\begin{aligned} C_n(x, a; q) &= \sum_{B \subseteq [n]} \sum_{\sigma \in \mathfrak{S}_{n-B}} q^{inv(\sigma)+inv(B)} (-1)^{n-cyc(\sigma)} a^{|B|} x^{cyc(\sigma)}, \\ &= \sum_{B \subseteq [n]} \sum_{\sigma \in \mathfrak{S}_{n-B}} \omega_q(B, \sigma) x^{cyc(\sigma)}. \end{aligned}$$

A combinatorial proof of the three-term recurrence relation (2.1) can be given using Theorem 1. An involution is necessary. For more details, we refer the reader to [5].

3. The moments.

An explicit measure for the q-Charlier polynomials is known, [4, p.196]. It is not hard to find the n^{th} moment of this measure explicitly. The result is a perfect q-analogue of (1.1)

$$\mu_n = \sum_{k=1}^{n} S_q(n, k) a^k, \tag{3.1}$$

where $S_q(n, k)$ is the q-Stirling number of the second kind, given by the recurrence

$$S_q(n, k) = S_q(n-1, k-1) + [k]_q S_q(n-1, k), \tag{3.2}$$

where $S_q(0, k) = \delta_{0,k}$. In fact, one sees that [13]

$$S_q(n, k) = \frac{1}{(1-q)^{n-k}} \sum_{j=0}^{n-k} \binom{n}{k+j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q (-1)^j. \tag{3.3}$$

Clearly (3.1) suggests that there is some statistic on set partitions, whose generating function is μ_n. This statistic, rs, arises from the Viennot theory of Motzkin paths associated with the three-term recurrence (2.1) [20]. We do not give the details of the construction here.

However, let us review some combinatorial facts about q-Stirling numbers. Set partitions of $[n] = \{1, 2, \ldots, n\}$ can be encoded as *restricted growth functions* (or *RG-functions*) as follow: if the blocks of π are ordered by increasing minima, the RG-function $w = w_1 w_2 \ldots w_n$ is the word such that w_i is the block where i is located. For example, if $\pi = 147|28|3|569$, $w = 123144124$. Note that set partitions on any set A can be encoded as RG-functions as long as A is a totally ordered set.

In [21], Wachs and White investigated four natural statistics on set partitions, called ls, lb, rs and rb. They are defined as follow:

$$ls(\pi) = ls(w) = \sum_{i=1}^{n} |\{j : j < w_i, j \text{ appears to the left of position } i\}|,$$

$$lb(\pi) = lb(w) = \sum_{i=1}^{n} |\{j : j > w_i, j \text{ appears to the left of position } i\}|,$$

$$rs(\pi) = rs(w) = \sum_{i=1}^{n} |\{j : j < w_i, j \text{ appears to the right of position } i\}|,$$

$$rb(\pi) = ls(w) = \sum_{i=1}^{n} |\{j : j > w_i, j \text{ appears to the right of position } i\}|.$$

Thus in the example, $ls(\pi) = 13$, $lb(\pi) = 7$, $rs(\pi) = 7$ and $rb(\pi) = 11$. They showed, using combinatorial methods, that each had the same distribution (up to a constant) on the set $RG(n, k)$ of all restricted growth functions of length n and maximum k, and that their generating function was indeed $S_q(n, k)$ for rs and lb (respectively $q^{\binom{k}{2}} S_q(n, k)$ for ls and rb).

We also use another encoding of set partitions in terms of *0–1 tableaux*. A 0–1 tableau is a pair $\varphi = (\lambda, f)$ where $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a partition of an integer $m = |\lambda|$ and $f = (f_{ij})_{1 \leq j \leq \lambda_i}$ is a "filling" of the corresponding Ferrers diagram of shape λ with 0's and 1's such that there is exactly one 1 in each column. 0–1 tableaux were introduced by Leroux in [18] to establish a q-log concavity result conjectured by Butler [3] for Stirling numbers of the second kind.

There is a natural correspondence between set partitions π of $[n]$ with k blocks and 0–1 tableaux with $n - k$ columns of length less than or equal to k. Simply write the RG-function $w = w_1 w_2 \ldots w_n$ associated to π as a $k \times n$ matrix, with a 1 in position (i, j) if $w_j = i$, and 0 elsewhere. The resulting matrix is row-reduced echelon, of rank k, with exactly one 1 in each column. A 0–1 tableau (in the third quadrant) is then obtained by removing all the pivot columns and the 0's that lie on the left of a 1 on a pivot column. Figure 1 illustrates these manipulations for $\pi = 1247|39(12)|568(11)|(10)$.

We define two statistics on 0–1 tableaux φ: first, the *inversion number*, $inv(\varphi)$, which is equal to the number of 0's below a 1 in φ; and the *non-inversion number*, $nin(\varphi)$, which is equal to the number of 0's above a 1 in φ. For example, for φ in Figure 1, $inv(\varphi) = 7$ and $nin(\varphi) = 8$. Note that an easy involution on the columns of 0–1 tableaux sends the inversion number to the non-inversion number and vice-versa. We call this map the *symmetry involution*.

It is not hard to see that the inversion number (respectively non-inversion number) on 0–1 tableaux corresponds to the statistic lb (resp. $ls - \binom{k}{2}$) on set partitions.

Similarly, permutations σ of $[n]$ in k cycles can be encoded as 0–1 tableaux with $n-k$ columns of distinct lengths less than or equal to $n-1$. The correspondence is defined by recurrence on n. Suppose σ is written as a standard product of cycles. If $n = 1$, then $\sigma = (1)$ corresponds to the empty 0–1 tableau $\varphi = \varnothing$. Otherwise, let $\sigma \in \mathfrak{S}_{n+1}$ and let φ denote the 0–1 tableau associated to the permutation σ in which $(n+1)$ has been erased. There are two cases. If $(n+1)$ is the minimum of a cycle in σ, then σ corresponds to φ. If $(n+1)$ is not the minimum of a cycle, then it appears in σ at a certain position i, $2 \le i \le n+1$. The permutation σ then corresponds to the 0–1 tableau φ plus a column of length n with a 1 in the $(i-1)$-th position (from top to bottom). For example, $\sigma = (1,3,4,7,2)(5,6)(8)$ corresponds to the following 0–1 tableau. It is not hard to see that under this transformation, the inversion number on 0–1 tableaux corresponds to the inversion number on permutations, as defined in §2. Thus, their generating functions are the q-Stirling numbers of the first kind $c_q(n,k)$.

In [6], de Médicis and Leroux investigated q and p,q-Stirling numbers from the point of view of the unified 0–1 tableau approach. In particular, they proved combinatorially or algebraically a number of identities involving q-Stirling numbers.

For the combinatorial interpretation of the moments of the q-Charlier polynomials in terms of set partitions π, we need two statistics. The number of blocks $\#blocks(\pi)$ is one, and the other statistic is $rs(\pi)$.

Theorem 2. *The n^{th} moment for the q-Charlier polynomials is given by*

$$\mu_n = \sum_{\pi \in P(n)} a^{\#blocks(\pi)} q^{rs(\pi)}.$$

As we mentioned, many other q-Stirling distributed statistics have been found [21]. It is surprising that the Viennot theory naturally gives a so-called "hard" statistic (rs), not an easy one (e.g. lb, [21]). Other variations on the rs-statistic can be given from the Motzkin paths, although the lb-statistic is not among them. It can be derived from the Motzkin paths associated with the "odd" polynomials for (2.1).

4. The orthogonality relation and the linearization of products.

Let L be the linear functional on polynomials that corresponds to integrating with respect to the measure for the Charlier polynomials. The orthogonality relation is

$$L(C_n^a(x)C_m^a(x)) = a^n n!\ \delta_{m,n}. \tag{4.1}$$

The q-version of (4.1) is

$$L_q(C_n(x,a;q)C_m(x,a;q)) = a^n q^{\binom{n}{2}} [n]!_q \delta_{m,n}. \tag{4.2}$$

Since the polynomials $C_n(x,a;q)$ and L_q have combinatorial definitions from Theorems 1 and 2, it is possible to restate (4.2) as a combinatorial problem. We will give an involution which then proves (4.2) in this framework.

A more general question is to find $L(C^a_{n_1}(x)C^a_{n_2}(x)\cdots C^a_{n_k}(x))$ for any k. A solution is equivalent to finding the coefficients a_{n_k} in the expansion

$$C^a_{n_1}(x)C^a_{n_2}(x)\cdots C^a_{n_{k-1}}(x) = \sum_{n_k} a_{n_k} C^a_{n_k}(\ x).$$

This had been done bijectively for some classes of Sheffer orthogonal polynomials in [5], [7], [9], [10]. Moreover, in the q-case of Hermite polynomials, some remarkable consequences have been found [15].

For the Charlier polynomials, it is easy to see that

(4.3)
$$\sum_{n_1,\cdots,n_k=0}^{\infty} L(C^a_{n_1}(x)C^a_{n_2}(x)\cdots C^a_{n_k}(x))\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_k^{n_k}}{n_k!} = e^{a(e_2(t_1,\cdots,t_k)+\cdots+e_k(t_1,\cdots,t_k))},$$

where e_i is the elementary symmetric function of degree i, [19]. In this case $L(C^a_{n_1}C^a_{n_2}\cdots C^a_{n_k})$ is a polynomial in a with positive integer coefficients; a combinatorial interpretation of this coefficient has been given ([12] and [23]). For $k=3$, (4.3) is equivalent to

(4.4)
$$L(C^a_{n_1}(x)C^a_{n_2}(x)C^a_{n_3}(x)) = \sum_{l=0}^{\lfloor (n_1+n_2-n_3)/2\rfloor} \frac{a^{n_3+l}n_1!n_2!n_3!}{l!(n_3-n_2+l)!(n_3-n_1+l)!(n_1+n_2-n_3-2l)!}.$$

One can hope that $L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x))$ is simply a weighted version, with an appropriate statistic, of the $q=1$ case. However this is false. For example,

$$L_q(C_2(x)C_2(x)C_1(x)) = q(q^2+2q+1)a^2 + q(q^3+q^2-q-1)a^3.$$

Nonetheless, we have an exact formula for $L_q(C_{n_1}(x,a,q)C_{n_2}(x,a,q)C_{n_3}(x,a,q))$, which is equivalent to one of Al-Salam-Verma [1].

Theorem 3. *Let $n_3 \geq n_1 \geq n_2 \geq 0$. Then*

$$L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x)) = \sum_{l=0}^{n_1+n_2-n_3}\sum_{j=0}^{l} a^{n_3+l}q^K(q-1)^{l-j}\frac{[n_1-j]!_q}{[n_1-l]!_q}\begin{bmatrix} n_2 \\ l-j\end{bmatrix}_q$$

(4.5)
$$[n_3]!_q \begin{bmatrix} n_1 \\ j\end{bmatrix}_q \begin{bmatrix} n_2-l+j \\ n_3-n_1+j\end{bmatrix}_q \frac{[j]!_q[n_1-j]!_q}{[n_3-n_2+l]!_q}\begin{bmatrix} n_1+n_2-n_3-l \\ j\end{bmatrix}_q,$$

where

$$K = \binom{l-j}{2}+\binom{n_1}{2}+j(-n_3-j+1)+\binom{j}{2}+\binom{n_2-l+j}{2} + (n_3-n_1+j)(n_3-n_2+l)+j(n_3-n_2+l).$$

The generating function of $L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x))$ can be evaluated from Theorem 3, yielding

$$\sum_{n_1,n_2,n_3} L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x))\frac{t_1^{n_1}}{[n_1]!_q}\frac{t_2^{n_2}}{[n_2]!_q}\frac{t_3^{n_3}}{[n_3]!_q} =$$

(4.6)
$$(-t_3;q)_\infty(-at_1t_2(1-q);q)_\infty\, {}_2\phi_1\left(\begin{matrix} at_1(1-q),\ at_2(1-q) \\ -at_1t_2(1-q)\end{matrix};q,-t_3\right).$$

Letting $q \to 1$ in (4.6) gives back (4.3) for $k = 3$. This generating function can also be evaluated directly using the measure ([4, p.196]), the generating function (2.3) for the polynomials and a ${}_3\phi_2$ transformation.

More generally, for $k \geq 4$, the generating function of $L_q(C_{n_1}(x) \ldots C_{n_k}(x))$ can be expressed as a difference of two basic hypergeometric series. This has been done by Ismail and Stanton [14] for the Al-Salam Carlitz polynomials, so an equivalent formula can be deduced for the q-Charlier polynomials using (2.2).

Let us set up the combinatorial context in which Theorem 3 will be proven. We first introduce notations and conventions that will be used throughout the proof. Define

$$\begin{aligned} L_q(n_1, n_2, n_3) = \{((B_i, \sigma_i); \pi) = ((B_1, \sigma_1), (B_2, \sigma_2), (B_3, \sigma_3); \pi) | \\ (B_i, \sigma_i) \text{ is a partial permutation on the set } \{i\} \times [n_i], \\ \text{and } \pi \text{ is a partition on the cycles of } \sigma_1,\ \sigma_2 \text{ and } \sigma_3\ \}. \end{aligned}$$

We will say that an element of the set $\{i\} \times [n_i]$ is *of color* i. When giving examples of elements of $L_q(n_1, n_2, n_3)$, to simplify notation, pairs $(1, i)$, $(2, i)$ and $(3, i)$ will always be denoted $\underline{i}$, i and $\bar{i}$ respectively. Thus a typical element of $L_q(8, 7, 10)$ would be described in the following way: $B_1 = \{\underline{2}, \underline{3}\}$, $B_2 = \varnothing$, $B_3 = \{\bar{5}, \bar{9}, \overline{10}\}$, and $\pi = (\underline{1}, \underline{5}, \underline{7})(\underline{8})(\bar{1})|(\underline{4})|(\underline{6})(3,5)(\bar{3}, \bar{7})|(1,4,2)(6,7)|(\bar{2}, \bar{8}, \bar{4}, \bar{6})$ (the underlying permutations $\sigma_1 = (\underline{1}, \underline{5}, \underline{7})(\underline{4})(\underline{6})(\underline{8})$, $\sigma_2 = (1,4,2)(3,5)(6,7)$ and $\sigma_3 = (\bar{1})(\bar{2}, \bar{8}, \bar{4}, \bar{6})(\bar{3}, \bar{7})$ can be recovered from π).

Note that the lexicographic order on pairs (i, j) induces a total order on the cycles of σ_1, σ_2 and σ_3, according to their minima. Therefore we can talk about RG-functions. We will always use the letter w to denote the RG-function associated to π. In the above example, $w = 1231434153$. The first $cyc(\sigma_1)$ letters of w correspond to the positions of cycles of color 1 in π, the next $cyc(\sigma_2)$ to the positions of cycles of color 2, and the last $cyc(\sigma_3)$ letters to the positions of cycles of color 3. We will denote by w_a, w_b and w_c respectively these portions of w. In the above example, we have $w_a = 1231$, $w_b = 434$, $w_c = 153$, and $w = w_a w_b w_c$, the concatenation of words w_a, w_b and w_c.

Finally, we will use the notation $Supp(w)$ (or $Supp(\sigma)$ or $Supp(\pi_i)$) to denote the underlying set of letters of a word w (or a permutation σ or a block π_i of a partition π respectively).

From Theorems 1 and 2, we deduce that

$$L_q(C_{n_1}(x) C_{n_2}(x) C_{n_3}(x)) = \sum_{((B_i, \sigma_i); \pi) \in L_q(n_1, n_2, n_3)} \omega_q((B_i, \sigma_i); \pi), \tag{4.7}$$

where

$$\omega_q((B_i, \sigma_i); \pi) = \omega_q(B_1, \sigma_1) \omega_q(B_2, \sigma_2) \omega_q(B_3, \sigma_3) q^{rs(\pi)} a^{\#blocks(\pi)}, \tag{4.8}$$

and $\omega_q(B, \sigma)$ was defined in Theorem 1, as a signed monomial in the variables a and q. This gives a combinatorial interpretation of the right-hand side of (4.5).

For $q = 1$, the negative coefficients of a are counterbalanced by the positive coefficients of a, and (4.7) is a polynomial with positive coefficients. Indeed, in that case, it is not hard to find a weight-preserving sign-reversing involution on $L_q(n_1, n_2, n_3)$ (cf [5]) whose fixed points $((B_i, \sigma_i); \pi)$ are characterized by

i) $B_i = \varnothing$ and $\sigma_i =$ Identity, for $i = 1, 2, 3$;

ii) the word w_a (respectively w_b and w_c) contains all distinct letters, and $Supp(w_a) \subseteq Supp(w_b w_c)$ (respectively $Supp(w_b) \subseteq Supp(w_a w_c)$ and $Supp(w_c) \subseteq Supp(w_a w_b)$).

Identity (4.4) easily follows from ω_1-counting these fixed points.

However, the general q-case is much harder, and some negative weights remain. The sign of $\omega_q((B_i, \sigma_i); \pi)$ comes from the cardinalities of the sets B_i and the signs of the permutations σ_i. In our proof, we successively apply five weight-preserving sign-reversing involutions Φ_i to $L_q(n_1, n_2, n_3)$, each one acting on the fixed points of the preceding one. Φ_1 forces $\sigma_3 = Id$, Φ_2 forces $B_3 = \varnothing$, Φ_3 forces $\sigma_1 = Id$, Φ_4 forces $B_1 = \varnothing$, and Φ_5 forces $\sigma_2 = Id$, leaving B_2 arbitrary. Hence the negative part of (4.7) is due only to B_2.

The final set of fixed points, $Fix\Phi_5$, does not contain the fixed point set (above) for $q = 1$. Instead there is a bijection from a subset of $Fix\Phi_5$ to this set, but it does not preserve the powers of q.

The five weight-preserving sign-reversing involutions Φ_i and their respective fixed points sets $Fix\Phi_i$ are given in the next section and the complete characterization of $Fix\Phi_5$ is given by the conditions Fix.1 through Fix.4, stated at the beginning of §6. In §6, we show that the ω_q-weight of $Fix\Phi_5$ is equal to the right-hand side of (4.5), thus establishing Theorem 3.

5. The weight-preserving sign-reversing involutions Φ_i.

Let us recall that a *weight-preserving sign-reversing involution* (or *WPSR-involution*) Φ with weight function ω is an involution such that for any $e \notin Fix\Phi$, $\omega(\Phi(e)) = -\omega(e)$.

Involution Φ_1. This WPSR-involution will kill any $((B_i, \sigma_i); \pi)$ such that σ_3 is not the identity.

Remember that the cycles of σ_3 are ordered by increasing minima. Find the greatest cycle c_{i_0} such that either this cycle is of length ≥ 2 or it lies in the same block π_i of π as some other 1-cycle greater than it. If c_{i_0} satisfies the latter condition, the 1-cycle greater than c_{i_0} in the leftmost block π_h of partition π is glued to the end of c_{i_0}. Then, if $h = h_0 < h_1 < \ldots < h_m = i$ denote the indices of the blocks between π_h and π_i containing 1-cycles greater than c_{i_0}, these 1-cycles are moved from block π_{h_l} to block $\pi_{h_{l-1}}$.

For example, for $((B_i, \sigma_i); \pi) \in L_q(9, 0, 10)$ such that $B_1 = B_2 = \varnothing$, $B_3 = \{\overline{10}\}$ and $\pi = (\underline{1})(\overline{1}, \overline{2})(\overline{6})|(\underline{2}, \underline{8})(\overline{5})|(\underline{3})(\underline{4})|(\underline{5})(\overline{3}, \overline{9})(\overline{8})|(\underline{6}, \underline{9})(\underline{7})(\overline{7})|(\overline{4})$, we have $\sigma_3 = (\overline{1}, \overline{2})(\overline{3}, \overline{9})(\overline{4})(\overline{5})(\overline{6})(\overline{7})(\overline{8})$, $c_{i_0} = (\overline{3}, \overline{9})$, and $\Phi_1((B_i, \sigma_i); \pi)$ is given by the same B_i's, σ_3 becomes $(\overline{1}, \overline{2})(\overline{3}, \overline{9}, \overline{6})(\overline{4})(\overline{5})(\overline{7})(\overline{8})$, and $\pi = (\underline{1})(\overline{1}, \overline{2})(\overline{5})|(\underline{2}, \underline{8})(\overline{8})|(\underline{3})(\underline{4})|(\underline{5})(\overline{3}, \overline{9}, \overline{6})|(\underline{6}, \underline{9})(\underline{7})(\overline{7})|(\overline{4})$.

Note that the number of inversions gained in σ_3 is counterbalanced by the loss in the statistic $rs(\pi)$. Conversely, if c_{i_0} is of length ≥ 2 and does not lie in the same block as any other greater cycles, its image is defined in the obvious way so that Φ_1 is an involution. For more details, see [5].

Fixed points for Φ_1. The cycle c_{i_0} is not defined if and only if σ_3 contains only 1-cycles which all lie in different blocks of π. Therefore,

$$Fix\Phi_1 = \{((B_i, \sigma_i); \pi) \in L_q(n_1, n_2, n_3) \mid \sigma_3 \text{ is the identity}$$
$$\text{and } w_c \text{ contains all distinct letters}\}.$$

Involution Φ_2. This WPSR-involution is designed to discard all $((B_i,\sigma_i);\pi) \in Fix\Phi_1$ such that B_3 is not empty.

Let $((B_i,\sigma_i);\pi) \in Fix\Phi_1$ and let $k = \#blocks(\pi)$.

Denote by j_0, $0 \le j_0 \le (n_3-1)$, the integer such that $\overline{j_0+1} = \min(B_3)$. If $B_3 = \varnothing$, we let $j_0 = \infty$. Likewise, denote by j_1, $1 \le j_1 \le n_3$, the maximum integer such that the 1-cycle $(\bar{j}_1)$ forms a singleton block in π. Remember that $\sigma_3 = Id$ and w_c contains all distinct letters. By maximality, $(\bar{j}_1)$ lies in the k-th block of π. Denote by j'_1 its contribution to the statistic rs, that is the number of (different) letters after the only occurrence of k in w_c (and in w). If there are no such singleton blocks in π, let $j_1 = j'_1 = \infty$.

There are two cases: $j_0 \le j'_1$, or $j_0 > j'_1$. If $j_0 \le j'_1$, $\Phi_2((B_i,\sigma_i);\pi)$ is obtained by inserting the 1-cycle $(\overline{j_0+1})$ in σ_3 and by inserting the letter $(k+1)$ in w_c at the (j_0+1)-th position from the end of w_c, leaving everything else fixed.

For example, for $((B_i,\sigma_i);\pi)$ defined by $B_1 = \varnothing = B_2$, $B_3 = \{\bar{2},\bar{6},\bar{8}\}$ and $\pi = (\underline{1},\underline{6})(5)(\bar{7})|(\underline{2})(1,3,2)(\bar{4})|(\underline{3},\underline{5},\underline{4})|(4)(\bar{9})|(\bar{1})|(\bar{3})|(\bar{5})$, $w_c = 562714$, $j_0 = 1$, $j_1 = 5$ and $j'_1 = 2$. Then the new w_c in $\Phi_2((B_i,\sigma_i);\pi)$ is $w_c = 5627184$, and $\Phi_2((B_i,\sigma_i);\pi)$ is defined by $B_1 = \varnothing = B_2$, $B_3 = \{\bar{6},\bar{8}\}$ and $\pi = (\underline{1},\underline{6})(5)(\bar{5})|(\underline{2})(1,3,2)(\bar{3})|(\underline{3},\underline{5},\underline{4})|(4)(\bar{9})|(\bar{1})|(\bar{2})|(\bar{4})|(\bar{7})$.

Note that $\Phi_2((B_i,\sigma_i);\pi)$ has its j'_1 equal to the j_0 associated to $((B_i,\sigma_i);\pi)$. Conversely, if $j'_1 < j_0$, the image of $((B_i,\sigma_i);\pi)$ is defined in the obvious way so that Φ_2 is an involution. Φ_2 is also weight-preserving and sign-reversing. For more details, see [5].

Fixed points for Φ_2. Fixed points correspond to the case $j_0 = j'_1 = \infty$. This means that $B_3 = \varnothing$ and there are no singleton blocks in π of color 3. Therefore,

$$Fix\Phi_2 = \{((B_i,\sigma_i);\pi) \in Fix\Phi_1 \mid B_3 = \varnothing \text{ and } Supp(w_c) \subseteq Supp(w_a w_b)\}.$$

Note that $Supp(w_c) \subseteq Supp(w_a w_b)$ is equivalent to the condition that the $w_a w_b$ is an RG-function whose maximum equals $\#blocks(\pi)$.

To do Φ_3 and later Φ_5, we need to describe the contribution to the statistic rs of the elements of color 1 and 2 in partition π. Let w be a word on the alphabet $[k]$. Let w_{ij} denote the subword of w obtained by discarding letters not equal to i or j, $1 \le i < j \le k$. For instance, if $w = 123144124$, $w_{12} = 12112$. Then we can write

$$rs(w) = \sum_{1 \le i < j \le k} rs(w_{ij}).$$

Claim. *Let w be an RG-function of maximum k and suppose $w = vv'$. Then v is an RG-function and*

$$\begin{aligned} rs(w) &= \sum_{\substack{1 \le i<j \le k,\\ i \notin Supp(v')}} rs(v_{ij}) + \sum_{\substack{1 \le i<j \le k,\\ i \in Supp(v')}} ls(v_{ij}) + rs(v') \\ &=: rs(w)|_v + rs(v'). \end{aligned}$$

Thus the contribution to the statistic $rs(w)$ of the initial word v, $rs(w)|_v$, is indeed an interpolation between the hard statistic rs and the easy statistic ls, as was studied by White in [22]. He showed in particular that these specific interpolating statistics were q-Stirling distributed, meaning that their generating functions over $RG(n,k)$ are the q-Stirling numbers of the second kind $S_q(n,k)$, up to a power of q. He provides a bijection on $RG(n,k)$ such that the mixed statistic is sent to the easy statistic ls (up to a constant). More precisely,

Lemma 4. *Let $S = \{s_1 < s_2 < \ldots < s_m\} \subseteq [k]$. There is a bijection $\Psi_S : RG(n,k) \to RG(n,k)$ such that for any $w \in RG(n,k)$,*

$$\sum_{\substack{1\le i<j\le k,\\ i\in S}} rs(w_{ij}) + \sum_{\substack{1\le i<j\le k,\\ i\in [k]-S}} ls(w_{ij}) = ls(\Psi_S(w)) - \sum_{j=1}^{m}(k-s_j). \tag{5.1}$$

Proof. Define $\Psi_i : RG(n,k) \to RG(n,k)$, $1 \le i \le k-1$ as follow:

i) if $w \in RG(n,k)$ has a letter i to the right of the first occurrence of $(i+1)$, then the rightmost letter i is switched to $(i+1)$ and any $(i+1)$ to its right is changed to i. For example, $\Psi_1(111212332122) = 111212332211$.

ii) if w does not have a letter i to the right of the first occurrence of $(i+1)$, then all $(i+1)$'s to its right are switched to i's. For example, $\Psi_1(1112232) = 1112131$.

For convenience, we will set $\Psi_k : RG(n,k) \to RG(n,k)$ to be the identity. Now, given $S = \{s_1 < s_2 < \ldots < s_m\} \subseteq [k]$, Ψ_S is defined as follow:

$$\Psi_S = (\Psi_k \circ \Psi_{k-1} \circ \ldots \circ \Psi_{s_1}) \circ (\Psi_k \circ \Psi_{k-1} \circ \ldots \circ \Psi_{s_2}) \circ \ldots \circ (\Psi_k \circ \ldots \circ \Psi_{s_m}).$$

Note that Ψ_S preserves the positions of the first occurrences. For more details, the reader is referred to [22]. □

Involution Φ_3. This next involution is designed to kill any element $((B_i, \sigma_i); \pi)$ such that σ_1 is not the identity. Note that since the interpolating statistics on w_a are q-Stirling distributed, it reduces to proving the orthogonality relation

$$\sum_{k=m}^{n} (-1)^{n-k} c_q(n,k) S_q(k,m) = \delta_{n,m}.$$

But this formula was deduced in Proposition 3.1 of [6] from a weight-preserving sign-reversing involution on appropriate pairs of 0–1 tableaux. The general idea is to map σ_1 and w_a bijectively into a pair of 0–1 tableaux, using Ψ_S defined in the previous lemma and the correspondences described in §1. Then we can apply the WPSR-involution, essentially shifting the rightmost shortest column from one 0–1 tableau to the other. $\Phi_3((B_i, \sigma_i); \pi)$ is then obtained by replacing σ_1 and w_a by the new decoded pair of 0–1 tableaux. Involution Φ_5 will use similar ideas.

We need only specify the bijective coding of (σ_1, w_a) into a pair of 0–1 tableaux. Let $((B_i, \sigma_i); \pi) \in Fix\Phi_2$ and let $n = n_1 - |B_1|$, $k = cyc(\sigma_1)$ and $m = max(Supp(w_a))$.

i) For σ_1, simply use the correspondence described in §3 to get a 0–1 tableau φ_1 with $(n-k)$ columns of distinct length $\le (n-1)$. Note that $inv(\sigma_1) = inv(\varphi_1)$.

ii) For w_a, we first want to reduce the interpolating statistic $rs(w)|_{w_a}$ to the easy statistic $ls(w_a)$. This is done by applying Ψ_S defined in the previous lemma to w_a, for $S = [m] \setminus Supp(w_b w_c)$. We then use the correspondence described in §3 to get a 0–1 tableau φ_2 with $(k-m)$ columns of length $\le m$. There is one last technicality: the statistic ls is sent to the non-inversion statistic on 0–1 tableaux (up to the constant $\binom{m}{2}$), therefore we will apply to φ_2 the symmetry involution exchanging non-inversions and inversions, so that for its image $\tilde{\varphi}_2$, we have

$$rs(w)|_{w_a} = inv(\tilde{\varphi}_2) + \binom{m}{2} - \sum_{i\in S}(m-i).$$

Note that m is not modified by the WPSR-involution applied to pairs of 0–1 tableaux, thus insuring that the overall involution Φ_3 is well-defined (the new w is still an RG-function) and weight-preserving. It is also sign-reversing. Details are left to the reader.

Fixed points for Φ_3. At the 0–1 tableau level, the only fixed pair of 0–1 tableaux is $(\varnothing, \varnothing)$, because in that case, it is impossible to move columns. But this can happen if and only if $(n-k) = (k-m) = 0$, and therefore $n = k = m = n_1 - |B_1|$, σ_1 is the identity on $[n_1] - B_1$, and $w_a = 12\ldots(n_1 - |B_1|)$. Therefore

$$Fix\Phi_3 = \{((B_i, \sigma_i); \pi) \in Fix\Phi_2 \mid \sigma_1 \text{ is the identity and } w_a = 12\ldots(n_1 - |B_1|)\}.$$

Involution Φ_4. This involution is the simplest. Its task is to eliminate elements $((B_i, \sigma_i); \pi)$ such that $B_1 \neq \varnothing$.

Let $((B_i, \sigma_i); \pi) \in Fix\Phi_3$ and let i_0 be the smallest integer, $1 \leq i_0 \leq n_1$, such that either $\underline{i_0} \in B_1$, or the 1-cycle $(\underline{i_0})$ forms a singleton block in π. Then if $\underline{i_0} \in B_1$, insert it as a 1-cycle in σ_1 and as a singleton block in π, and vice-versa.

For example, if $B_1 = \{\underline{2}\}, B_2 = B_3 = \varnothing$, and $\pi = (\underline{1})(\overline{1}, \overline{3})|(\underline{3})|(1,2)(\overline{2})$, then $i_0 = 2$ and the image of $((B_i, \sigma_i); \pi)$ under Φ_4 is $B_1 = \varnothing, B_2 = B_3 = \varnothing$, and $\pi = (\underline{1})(\overline{1}, \overline{3})|(\underline{2})|(\underline{3})|(1,2)(\overline{2})$. Details are left to the reader.

Fixed points for Φ_4.

$$Fix\Phi_4 = \{((B_i, \sigma_i); \pi) \in Fix\Phi_3 \mid B_1 = \varnothing \text{ and } Supp(w_a) = [n_1] \subseteq Supp(w_b w_c)\}.$$

Involution Φ_5. This final WPSR-involution will annihilate the remaining $((B_i, \sigma_i); \pi)$ such that σ_2 is not the identity. It is the only one using the hypothesis $n_3 \geq n_1 \geq n_2$. The principle of the involution is similar to Φ_3: we will reduce the problem to finding an involution for the easy statistic ls.

Let $((B_i, \sigma_i); \pi) \in Fix\Phi_4$, and let $\#blocks(\pi) = n_3 + s$. First, encode σ_2 as a 0–1 tableau φ with $(n_2 - |B_2| - cyc(\sigma_2))$ columns of distinct lengths $\leq (n_2 - |B_2| - 1)$, using the correspondence described in §3. Note that $inv(\sigma_2) = inv(\varphi)$ and that the shortest column of φ is of length at most $cyc(\sigma_2)$.

For w_b, we reduce the interpolating statistic $rs(w)|_{w_a w_b}$ to the easy statistic ls by applying Ψ_S defined in Lemma 4 to $w_a w_b$, with $S = [n_3 + s] - Supp(w_c)$. Note that since $w_a = 12\ldots n_1$ and Ψ_S preserves first occurrences, $\Psi_S(w_a w_b) = w_a \tilde{w}_b$ for some word $\tilde{w}_b = \tilde{b}_1 \tilde{b}_2 \ldots \tilde{b}_k$. Note also that we must have $\{n_1+1, \ldots, n_3+s\} \subseteq Supp(\tilde{w}_b)$ (because $w_a w_b$ has maximum $(n_3 + s)$).

For example, if $((B_i, \sigma_i); \pi) \in Fix\Phi_4$ is defined by $B_1 = B_2 = B_3 = \varnothing$, and $\pi = (\underline{1})(2)|(\underline{2})(3)(\overline{5})|(\underline{3})(\overline{2})|(\underline{4})(\overline{4})|(\underline{5})(5)(\overline{1})|(1)(4)(\overline{3})$, we have $w_a = 12345$, $w_b = 61265$, $w_c = 53642$, and $\sigma_2 = (1)(2)(3)(4)(5)$. Then σ_2 corresponds to the empty 0–1 tableau $\varphi = \varnothing$, and we successively compute $S = [6] - Supp(53642) = \{1\}$, $\Psi_{\{1\}}(w_a w_b) = 1234566154$, and $\tilde{w}_b = 66154$.

Let i_0 denote the length of the shortest column in φ, $1 \leq i_0 \leq cyc(\sigma_2)$. If $\varphi = \varnothing$, let $i_0 = \infty$. Likewise, let h_0 denote the smallest integer, $1 \leq h_0 \leq cyc(\sigma_2)$, such that $\tilde{b}_{h_0} < h_0$. If no such $\tilde{b}_i$ exists, set $h_0 = \infty$.

There are two cases: $i_0 \geq h_0$ or $i_0 < h_0$. If $i_0 \geq h_0$, then delete $\tilde{b}_{h_0}$ from the word $\tilde{w}_b$ and add a column of length $(h_0 - 1)$ to φ, with a 1 in position $\tilde{b}_{h_0}$, from bottom to top, thus obtaining a new pair $(\tilde{w}'_b, \varphi')$. Since the letter removed from $\tilde{w}_b$ is at most equal to $(cyc(\sigma_2) - 1) < (n_2 - |B_2|) < (n_1 + 1)$, $w_a \tilde{w}'_b$ is still an RG-function of maximum $(n_3 + s)$, and the new i_0 associated to φ' is equal to $(h_0 - 1)$. $\Phi_5((B_i, \sigma_i); \pi)$ is then obtained by applying Ψ_S^{-1} to $w_a \tilde{w}'_b$ and by decoding the 0–1 tableau φ'.

In the above example, $i_0 = \infty$ and $h_0 = 3$. Hence (corresponding to the new permutation $\sigma_2 = (1)(2,3)(4)(5)$) and $\tilde{w}'_b = 6654$. From $\Psi_{\{1\}}^{-1}(w_a \tilde{w}_b) = 123456165$, we get $\Phi_5((B_i,\sigma_i);\pi)$ equals $B_1 = B_2 = B_3 = \varnothing$, and $\pi = (\underline{1})(2,3)|(\underline{2})(\bar{5})|(\underline{3})(\bar{2})|(\underline{4})(\bar{4})|(\underline{5})(5)(\bar{1})|(1)(4)(\bar{3})$.

If $i_0 < h_0$, the image of $((B_i,\sigma_i);\pi)$ is defined in the obvious way so that Φ_5 is an involution. The proof that Φ_5 is weight-preserving and sign-reversing is quite straight-forward, and the details will be left to the reader. It remains to show that Φ_5 is well-defined. Remember that if $((B_i,\sigma_i);\pi) \in Fix\Phi_4$, we must have $Supp(w_a) \subseteq Supp(w_b w_c)$. We have to show that Φ_5 preserves this property. What complicates matters is the application of Ψ_S and Ψ_S^{-1} to the RG-functions $w_a w_b$. In Lemma 5, we explicitly find the set of images $w_a\tilde{w}_b$ (which we will denote by $\tilde{W}(S)$) of all possible $w_a w_b$ under Ψ_S. We will then show that the deletion or insertion of a letter whose value is strictly less than its position in $\tilde{w}_b$ yields new RG-functions $w_a\tilde{w}'_b$ which remain in the set $\tilde{W}(S)$.

Fix $n_3 \geq n_1 \geq n_2 \geq 0$, $0 \leq t \leq n_2$, and $0 \leq s \leq n_1 + n_2 - n_3$. Let $S \subseteq [n_3 + s]$ such that $|S| \leq s$, and fix $w_a = 12\ldots n_1$. We denote by

$$\begin{aligned} W(S) &= \{w_b \mid w_a w_b \in RG(n_1+n_2-t, n_3+s), \text{ and} \\ &\qquad [n_1] \subseteq ([n_3+s]-S) \cup Supp(w_b)\}, \\ \tilde{W}(S) &= \{\tilde{w}_b \mid \Psi_S(w_a w_b) = w_a\tilde{w}_b \text{ for } w_b \in W(S)\}, \end{aligned}$$

and

$$\begin{aligned} w_a W(S) &= \{w_a w_b \mid w_b \in W(S)\}, \\ w_a\tilde{W}(S) &= \{w_a\tilde{w}_b \mid \tilde{w}_b \in \tilde{W}(S)\}. \end{aligned}$$

In particular, when $|S| = s$, if w_c is a word containing the letters in $([n_3+s]-S)$ in any order, with no repetition, and (B_2,σ_2) is a partial permutation of $\{2\}\times[n_2]$ with $cyc(\sigma_2) = n_2 - t$, $W(S)$ contains all possible words w_b such that $w = 12\ldots n_1 w_b w_c$ is the RG-function associated to some $((B_i,\sigma_i);\pi) \in Fix\Phi_4$ having these fixed (B_2,σ_2) and w_c.

Lemma 5. *(characterization of $\tilde{W}(S)$) Let $S \subseteq [n_3+s]$ such that $|S| \leq s$. The set $\tilde{W}(S)$ depends only upon the cardinality $j = |S \cap [n_1]|$. More precisely, we have*

(i) $\tilde{W}(S) = \tilde{W}(S \cap [n_1])$,

(ii) If $j = 0$, $\tilde{W}(\varnothing) = W(\varnothing)$, and $\tilde{w}_b \in \tilde{W}(\varnothing)$ has the following form:

(5.2)
$$\tilde{w}_b = \underbrace{*\ldots*}_{\text{entries} \leq n_1}\ (n_1+1)\ \underbrace{*\ldots*}_{\leq (n_1+1)}\ (n_1+2)\ldots(n_3+s-1)\ \underbrace{*\ldots*}_{\leq (n_3+s-1)}\ (n_3+s)\ \underbrace{*\ldots*}_{\leq (n_3+s)}.$$

(iii) If $j = 1$, then $\tilde{W}(\{i\}) = \tilde{W}(\{1\})$ is obtained from $\tilde{W}(\varnothing)$ by keeping only the words $\tilde{w}_b$ of the form (5.2) such that one of the stars $*$ is set to its maximum and the maximum value of all the stars to its right is lowered by 1. So any $\tilde{w}_b$ has the form

$$\tilde{w}_b = \underbrace{*\ldots*}_{\text{entries} \leq n_1}\ (n_1+1)\ \underbrace{*\ldots*}_{\leq (n_1+1)}\ (n_1+2)\ldots(n_1+h)\ \underbrace{*\ldots*}_{\leq (n_1+h)}\ (n_1+h)\ \underbrace{*\ldots*}_{\leq (n_1+h-1)}$$

(5.3)
$$(n_1+h+1)\ldots(n_3+s-1)\ \underbrace{*\ldots*}_{\leq (n_3+s-2)}\ (n_3+s)\ \underbrace{*\ldots*}_{\leq (n_3+s-1)}.$$

(iv) If $j \geq 2$, then $\tilde{W}(S) = \tilde{W}(\{1, 2, \ldots, j\})$ is obtained from $\tilde{W}(\{1, 2, \ldots, j-1\})$ by the same construction as the one described in (iii).

Proof.
(i). First we show that $\tilde{W}(S) = \tilde{W}(S \cap [n_1])$. From the definition of $W(S)$, it is clear that $W(S) = W(S \cap [n_1])$. Moreover, if $S = \{s_1 < \ldots < s_j < s_{j+1} < \ldots < s_n\}$, where $s_j \leq n_1$ and $s_{j+1} > n_1$, since $\Psi_{S \setminus [n_1]}$ is a bijection on $RG(n_1 + n_2 - t, n_3 + s)$, preserving first occurrences and leaving all letters $\leq n_1$ fixed, we must have

$$\Psi_{S \setminus [n_1]}(w_a W(S)) = w_a W(S).$$

Therefore,

$$\begin{aligned} w_a \tilde{W}(S) &= \Psi_S(w_a W(S)) = \Psi_{S \cap [n_1]} \circ \Psi_{S \setminus [n_1]}(w_a W(S)) \\ &= \Psi_{S \cap [n_1]}(w_a W(S \cap [n_1])) = w_a \tilde{W}(S \cap [n_1]). \end{aligned}$$

(ii). If $j = 0$, $\Psi_\varnothing$ is the identity map and

$$\tilde{W}(\varnothing) = W(\varnothing) = \{w_b \mid 12 \ldots n_1 w_b \in RG(n_1 + n_2 - t, n_3 + s)\},$$

in which typical elements (tails of RG-functions) are given by (5.2).
(iii). If $j = 1$, suppose $S = \{i\}$, $1 \leq i \leq n_1$. Then

$$W(S) = \{w_b \mid w_a w_b \in RG(n_1 + n_2 - t, n_3 + s), \text{ and } i \in Supp(w_b)\}.$$

Let $w_b \in W(S)$ and suppose the rightmost occurrence of i lies in position p of w_b, between the first occurrence of $(n_1 + h)$ and the first occurrence of $(n_1 + h + 1)$. Thus $w_a w_b$ has the form

$$w_a w_b = 12 \ldots n_1 \underbrace{* \ldots *}_{\text{entries } \leq n_1} (n_1 + 1) \underbrace{* \ldots *}_{\leq (n_1+1)} (n_2 + 2) \ldots (n_1 + h) \underbrace{* \ldots *}_{\leq (n_1+h)} \underbrace{i}_{\substack{\text{position} \\ (n_1+p)}} \underbrace{* \ldots *}_{\substack{\leq (n_1+h), \\ \text{entries } \neq i}}$$

(5.4)

$$(n_1 + h + 1) \ldots (n_3 + s - 1) \underbrace{* \ldots *}_{\substack{\leq (n_3+s-1), \\ \neq i}} (n_3 + s) \underbrace{* \ldots *}_{\substack{\leq (n_3+s), \\ \neq i}} .$$

Apply $\Psi_{\{i\}} = \Psi_{n_3+s} \circ \Psi_{n_3+s-1} \circ \ldots \circ \Psi_i$ to $w_a w_b$. The last occurrence of i in $w_a w_b$ (in position $(n_1 + p)$) lies to the right of the first occurrence of $(i + 1)$ (case (i) in the definition of Ψ_m), so it is changed to $(i + 1)$ by Ψ_i, and any $(i + 1)$ to its right is changed to i. Thus the last occurrence of $(i + 1)$ in $\Psi_i(w_a w_b)$ now appears in position $(n_1 + p)$, again to the right of the first occurrence of $(i + 2)$. So all $(i + 2)$'s to its right are changed to $(i + 1)'s$ by Ψ_{i+1}, the $(i + 1)$ in position $(n_1 + p)$ is switched to $(i + 2)$, and every other letter remain fixed.

The same argument applies until we reach Ψ_{n_1+h}. At this point in $\Psi_{n_1+h-1} \circ \ldots \circ \Psi_i(w_a w_b)$, there is a $(n_1 + h)$ in position $(n_1 + p)$ and no occurrence of $(n_1 + h)$ to its right. This means that there are no letters $(n_1 + h)$ to the right of the first occurrence of $(n_1 + h + 1)$ (case (ii) in the definition of Ψ_m). Hence Ψ_{n_1+h} changes every occurrence of $(n_1 + h + 1)$, except for the first one, to $(n_1 + h)'s$, and fixes everything else. Once again in the RG-function obtained, there are no occurrences

of (n_1+h+1) to the right of the first occurrence of (n_1+h+2). It is clear that by applying successively $\Psi_{n_1+h+1}, \dots, \Psi_{n_3+s}$ respectively, we will get $\Psi_{\{i\}}(w_a w_b)$ exactly of the form (5.3). This shows that the set defined in (iii) is equal to $\tilde{W}(\{i\})$. Note that the definition of the set $\tilde{W}(\{i\})$ is independent of the actual value of i, so $\tilde{W}(\{i\}) = \tilde{W}(\{1\})$.
(iv). The proof is an easy induction based on the proof of (iii). Note that if $S = \{s_1 < s_2 < \dots < s_j\}, s_j \leq n_1$, the positions of the last occurrences of $s_1, s_2, \dots, s_j$ respectively in $w_a w_b$ correspond exactly to the positions of the stars successively fixed to their maximum in $\Psi_s(w_a w_b)$. □

We can show now that Φ_5 is well-defined.

Let $\tilde{w}_b \in \tilde{W}(\{1,2,\dots,j\})$. The letters of $\tilde{w}_b$ can be divided into two categories: the *fixed letters* (first occurrences of (n_1+1) up to (n_3+s), and j stars that were fixed to their maximum in the construction described in the preceding lemma), and the *free letters* (corresponding to stars in the description of $\tilde{w}_b$ in Lemma 5). So in order to be in $\tilde{W}(\{1,2,\dots,j\})$, a word $\tilde{w}_b$ must have (n_3+s-n_1+j) fixed letters (appearing in some fixed relative order), and possibly some free letters, depending on its length.

On one hand, note that the fixed letters of $\tilde{w}_b$ are always greater or equal to their positions in $\tilde{w}_b$. Indeed, we have already seen that the first occurrences were necessarily greater than their position p ($(n_3+s) \geq (n_1+1) > n_2 \geq p$). As for the j stars fixed to their maximum, the way to minimize their value in the construction of Lemma 5 is to fix them successively by increasing order of their positions. Then, if they all lie before the first occurrences of (n_1+1) up to (n_3+s), the j-th star fixed will have minimum value (n_1-j+1), and the rightmost position where it can be located is, for example, the one in the following word:

$$\tilde{w}_b = \underbrace{* * \dots *}_{\text{entries} \leq n_1} n_1 \, (n_1-1) \dots (n_1-j+1)(n_1+1)(n_1+2) \dots (n_3+s).$$

But from the relations $n_3 \geq n_2$, $t \geq 0$, and $j \leq s$, we deduce that its position p,

$$p = |\tilde{w}_b| - (n_3+s-n_1) = n_1 + (n_2-n_3) - t - s \leq n_1 - j + 1.$$

Therefore in that case, all fixed stars are greater or equal to their positions. More generally, if a fixed star is rather located to the right of a first occurrence, its value is increased by one, so the letter remains greater or equal to its position.

On the other hand, note that the allowed maxima for the free letters are also greater or equal to their positions in $\tilde{w}_b$. The same type of argument (with same inequalities) applies. Details are left to the reader.

Now, the "involutive step" of Φ_5 was to add or to delete a letter from $\tilde{w}_b$, and this letter had the property of being strictly smaller than its position in $\tilde{w}_b$.

If the involutive step deleted a letter from $\tilde{w}_b$ ($w_a\tilde{w}_b \in RG(n_1+n_2-t, n_3+s)$), then it had to be one of its free letters because the fixed ones are greater or equal to their positions. Therefore the new $\tilde{w}_b'$ obtained is in the set $\tilde{W}(\{1,2,\dots,j\})$ (with $w_a\tilde{w}_b' \in RG(n_1+n_2-(t+1), n_3+s)$). Likewise, if the involutive step added a letter to $\tilde{w}_b$, the new letter is in the right range to be considered a free letter, and the fixed letters (and their relative order) are not modified, so the new $\tilde{w}_b'$ is in the set $\tilde{W}(\{1,2,\dots,j\})$ as well (with $w_a\tilde{w}_b' \in RG(n_1+n_2-(t-1), n_3+s)$).

Fixed points for Φ_5. The fixed points of Φ_5 correspond to the case $i_0 = h_0 = \infty$. Clearly,

$$Fix\Phi_5 = \{((B_i,\sigma_i);\pi) \in Fix\Phi_4 \mid \sigma_2 = Id \text{ and for } S = [\#blocks(\pi)] - Supp(w_c), \text{ the word } \tilde{w}_b \text{ in } \Psi_S(w_a w_b) = w_a\tilde{w}_b \text{ has its } i\text{-th letter } \geq i, \forall i\}.$$

6. Combinatorial evaluation of $L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x))$.

An expression of $L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x))$ can now be computed by ω_q-counting of the remaining fixed points $Fix\Phi_5$. More precisely, $((B_i,\sigma_i);\pi) \in Fix\Phi_5$ if and only if

Fix.1 $B_1 = B_3 = \varnothing$,

Fix.2 $\sigma_i = Id$ for $i = 1,2,3$,

Fix.3 w_a (respectively w_c) has all distinct letters and $Supp(w_a) \subseteq Supp(w_b w_c)$ (respectively $Supp(w_c) \subseteq Supp(w_a w_b)$),

Fix.4 for $S = [\#blocks(\pi)] - Supp(w_c)$, the word $\tilde{w}_b = \tilde{b}_1\tilde{b}_2\ldots\tilde{b}_{n_2-|B_2|}$ in $\Psi_S(w_a w_b) = w_a\tilde{w}_b$ has all $\tilde{b}_i$'s $\geq i$, where Ψ_S was defined in Lemma 4.

Clearly, for such elements, the weight (as was defined in (4.8)) reduces to

$$\omega_q((B_i,\sigma_i):\pi) = (-1)^{|B_2|} q^{inv(B_2)+rs(\pi)} a^{|B_2|+\#blocks(\pi)}. \tag{6.1}$$

By ω_q-counting this fixed point set, we will show that

(6.2)

$$L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x)) = \sum_{l=0}^{n_1+n_2-n_3} \sum_{s=0}^{l}\sum_{j=0}^{s} a^{n_3+l}(-1)^{l-s} q^L [n_3]!_q \begin{bmatrix} n_2 \\ l-s \end{bmatrix}_q$$
$$\begin{bmatrix} n_1 \\ j \end{bmatrix}_q \begin{bmatrix} n_3-n_1+s \\ s-j \end{bmatrix}_q \begin{bmatrix} n_2-l+s \\ n_3-n_1+s \end{bmatrix}_q \frac{[j]!_q[n_1-j]!_q}{[n_3-n_2+l]!_q} \begin{bmatrix} n_1+n_2-n_3-l \\ j \end{bmatrix}_q,$$

where

$$L = \binom{n_1}{2} + \binom{l-s}{2} + j(-n_3-s+1) + \binom{j}{2} - \binom{s-j}{2} - (s-j)(n_3-n_1+j)$$
$$+ \binom{n_2-l+s}{2} + (n_3-n_1+s)(n_3-n_2+l) + j(n_3-n_2+l).$$

Evaluating the s-sum by the q-binomial theorem (which has a simple bijective proof) gives the right-hand side of (4.5) and thus Theorem 3.

The main difficulty here is to transpose the condition Fix.4 into the ω_q-counting. Using Lemmas 4 and 5, we will see that this corresponds to the q-counting of some special sets of RG-functions according to the statistic ls, which is the object of Lemma 6.

Let us first group the elements of $Fix\Phi_5$ by powers of a. The power of a ranges from a minimum of n_3 (expressing the fact that w_c has n_3 distinct letters) to a maximum of (n_1+n_2) (being the maximum value of $\{\max(Supp(w_a w_b)) + |B_2|\}$). Now,

(6.3)

$$L_q(C_{n_1}C_{n_2}C_{n_3}) = \sum_{l=0}^{n_1+n_2-n_3} a^{n_3+l} \sum_{s=0}^{l} (-1)^{l-s} q^{\binom{l-s}{2}} \begin{bmatrix} n_2 \\ l-s \end{bmatrix}_q \sum_{\pi} q^{rs(\pi)},$$

where the last sum ranges over all partitions π corresponding to $((B_i, \sigma_i); \pi) \in Fix\Phi_5$ such that $\#blocks(\pi) = (n_3 + s)$ and B_2 is any fixed subset of $\{2\} \times [n_2]$ of cardinality $(l - s)$. The s-sum is the generating function for the subsets B_2, as was established in §2. But

$$\begin{aligned} rs(\pi) = rs(w) &= rs(w_c) + rs(w)|_{w_a w_b} \\ &= rs(w_c) + ls(w_a \tilde{w}_b) - \sum_{u \in ([n_3+s] - Supp(w_c))} (n_3 + s - u). \end{aligned} \tag{6.4}$$

Note that for any fixed set $Supp(w_c)$, there are no constraints on the positions of the letters in w_c, so $rs(w_c)$ is simply the number of inversions of the word w_c, whose distribution is mahonian (i.e. the generating function equals $[n_3]!_q$). From Lemma 5, we also know that the possible choices for $\tilde{w}_b$ only depend on the cardinality j of the set $([n_3 + s] - Supp(w_c)) \cap [n_1]$, not on the actual set $Supp(w_c)$ itself. Hence, if we let

$$Fix\tilde{W}(j) = \{\tilde{w}_b \mid \tilde{w}_b = \tilde{b}_1 \ldots \tilde{b}_{n_2-l+s} \in \tilde{W}(\{1, 2, \ldots, j\}) \text{ and } \tilde{b}_i \geq i, \forall i\},$$

where $\tilde{W}(\{1, 2, \ldots, j\})$ was characterized in Lemma 5, we get that the last sum on the right-hand side of (6.3) equals

$$\begin{aligned} \sum_{\pi} q^{rs(\pi)} = [n_3]!_q \sum_{j=0}^{s} \sum_{\tilde{w}_b \in Fix\tilde{W}(j)} & q^{ls(w_a \tilde{w}_b)} \\ & q^{j(-n_3-s+1)+\binom{j}{2}} \begin{bmatrix} n_1 \\ j \end{bmatrix}_q q^{-\binom{s-j}{2}-(s-j)(n_3-n_1+j)} \begin{bmatrix} n_3 - n_1 + s \\ s - j \end{bmatrix}_q . \end{aligned} \tag{6.5}$$

Finally, we show that

Lemma 6. *If* $w_a = 12 \ldots n_1$, $\tilde{w}_b = \tilde{b}_1 \ldots \tilde{b}_{n_2-l+s}$ *and* $max\,(Supp(w_a \tilde{w}_b)) = n_3 + s$, *then*

(6.6)

$$\sum_{\tilde{w}_b \in Fix\tilde{W}(j)} q^{ls(w_a \tilde{w}_b)} = q^A \begin{bmatrix} n_2 - l + s \\ n_3 - n_1 + s \end{bmatrix}_q \frac{[j]!_q [n_1 - j]!_q}{[n_3 - n_2 + l]!_q} \begin{bmatrix} n_1 + n_2 - n_3 - l \\ j \end{bmatrix}_q ,$$

where

$$A = \binom{n_1}{2} + \binom{n_2 - l + s}{2} + (n_3 - n_1 + s + j)(n_3 - n_2 + l).$$

Proof. Note that the statistic ls of any RG-function is just the sum of the values of the letters minus one, so

$$ls(w_a \tilde{w}_b) = \binom{n_1}{2} + \sum_{i=1}^{n_2-l+s} (\tilde{b}_i - 1).$$

To visualize more easily where the various factors of (6.6) come from, let us encode $\tilde{w}_b$ as a 0–1 tableau φ in the following manner: start with a $(n_3 + s) \times (n_2 - l + s)$ rectangular Ferrers diagram. Fill it with a 1 in position j (from bottom to top) of column i if $\tilde{b}_i = j$, and with 0's elsewhere. For example, if $(n_3 + s) = 8$, $n_1 = 6 = (n_2 - l + s)$ and $\tilde{w}_b = 175787$, φ is the 0–1 tableau on the left of Figure 3.

Obviously, we have $\sum(\tilde{b}_i - 1) = inv(\varphi)$. Note also that the 0's in the shaded staircase shape of φ in Figure 3 always count as inversions, expressing the fact that $\tilde{b}_i \geq i$. They account for the factor $q^{\binom{n_2-l+s}{2}}$ in (6.6). We can therefore drop them from φ without loss of generality, and compute the inversion number of the reduced 0–1 tableau $\tilde{\varphi}$. We now use Lemma 5 to characterize the possible fillings of $\tilde{\varphi}$ according to j.

Case 1: $j = 0$**.** From Lemma 5 (ii), $\tilde{w}_b \in \widetilde{W}(\varnothing)$ simply means that it is the tail of an RG-function. Thus the only restrictions on $\tilde{w}_b$ are that the first occurrences of $(n_1 + 1), (n_1 + 2), \ldots, (n_3 + s)$ appear in the right order.

If we set $x = (n_3 + s)$, $y = n_1$, and $z = (n_2 - l + s)$, in the context of 0–1 tableaux, we want to q-count all 0–1 tableaux $\tilde{\varphi}$ with z columns of lengths $x, (x - 1), \ldots, (x - z + 1)$ respectively, such that when we look at the top $(x - y)$ rows of $\tilde{\varphi}$ from left to right, the leftmost 1 in any row must always occur before the ones in the rows above it. Grouping the tableaux according to these leftmost occurrences of 1's, we get "typical" 0–1 tableaux $\tilde{\varphi}_{typ}$, corresponding exactly to the typical words $\tilde{w}_b$ described in (5.2) of Lemma 5. For instance, the typical 0–1 tableau $\tilde{\varphi}_{typ}$ containing our previous example is illustrated in Figure 3 (stars $*$ correspond to possible positions of 1's). Carrying out the q-counting, observe that

1.1 Each column containing a number m of stars contributes a factor $[m]_q$ to the q-counting of inversions. No matter which $(x - y)$ columns are chosen to be first occurrences of upper 1's, the number of stars in the remaining columns is $y, (y - 1), \ldots$ and $(x - z + 1)$ respectively, contributing to an overall factor of

$$\frac{[y]!_q}{[x - z]!_q} = \frac{[n_1]!_q}{[n_3 - n_2 + l]!_q}.$$

1.2 The 0's below the leftmost occurrences of upper 1's (shaded in Figure 3) form a partition μ with $(x-y)$ parts of length at least $(x-z)$ and at most y, determined by the positions of the first occurrences. Summing over all possible choices, it contributes a factor

$$q^{(x-y)(x-z)} \begin{bmatrix} z \\ x - y \end{bmatrix}_q = q^{(n_3-n_1+s)(n_3-n_2+l)} \begin{bmatrix} n_2 - l + s \\ n_3 - n_1 + s \end{bmatrix}_q .$$

Case 2: $j \geq 1$**.** Recall that Lemma 5 (iii) and (iv) provides a method to construct all the elements of $\widetilde{W}(\{1, 2, \ldots, j\})$ uniquely from $\widetilde{W}(\varnothing)$. In the 0–1 tableau context, if we extract only the cells filled with stars in $\tilde{\varphi}_{typ}$ (hence obtaining a tableau $\tilde{\psi}_{typ}$ with $(n_1 + n_2 - n_3 - l)$ columns of lengths $n_1, (n_1 - 1), \ldots, (n_3 - n_2 + l + 1)$ respectively), the manipulation described in Lemma 5 (iii) corresponds to replacing the top star of a column by a 1, and all the top stars to its right and the stars below it by a 0. Repeating this procedure j times and reinserting the columns of $\tilde{\psi}_{typ}$ in $\tilde{\varphi}_{typ}$ yields to "typical" 0–1 tableaux that correspond to the elements of $Fix\widetilde{W}(j)$. For example, Figure 4 shows the above manipulations on the third and the first columns respectively of the stars extracted from $\tilde{\varphi}_{typ}$ of Figure 3.

Proceeding to q-counting, the part (1.2) of case 1 is left unchanged and the part (1.1) is replaced by the contribution of the different choices of $\tilde{\psi}_{typ}$. But observe that

2.1 All the 0–1 tableaux $\tilde{\psi}_{typ}$ such that a star has been changed to a 1 in columns $c_1, c_2, \ldots$ and c_j contribute to $[j]!_q$ times the q-counting of the 0–1 tableaux $\tilde{\psi}_{typ}$ such that this procedure was done in increasing order of the c_i's. Therefore, we can restrict to this latter case. This explains the factor $[j]!_q$ in (6.6).

2.2 It is not hard to see that in that case, we are q-counting all 0–1 tableaux $\tilde{\psi}$ containing $(n_1+n_2-n_3-l)$ columns of lengths $n_1, (n_1-1), \ldots, (n_3-n_2+l+1)$ respectively, such that when we look at the top j rows from right to left, the rightmost 1 in any row has to occur before the ones in the rows above it. There is a simple weight-preserving bijection between this class of 0–1 tableaux and the one that was q-counted in case 1, for $x = n_1$, $y = (n_1-j)$ and $z = (n_1+n_2-n_3-l)$ (this class is defined by interchanging "left" and "right"). Given $\tilde{\psi}$ in the first class of 0–1 tableaux, just leave all the 1's below the j-th row fixed and "reverse the order" of the 1's in the top j rows, within the columns where they appear. Figure 5 gives an example of this for $j = 2$.

This is clearly an involution that preserves the number of 0's below 1's. Therefore, we can simply use case 1 to compute the q-contribution of the $\tilde{\psi}$'s. We obtain

$$q^{j(n_3-n_2+l)} \frac{[n_1-j]!_q}{[n_3-n_2+l]!_q} \begin{bmatrix} n_1+n_2-n_3-l \\ j \end{bmatrix}_q . \qquad \square$$

Finally, putting together Lemma 6, identities (6.5) and (6.3) yields identity (6.2), thus completing the proof of Theorem 3.

Note that if we take $n_2 = 0$ and apply Φ_1 and Φ_2 to $L_q(n_1, 0, n_3)$ (assuming $n_3 \geq n_1$), the set $Fix\Phi_2$ is easily seen to be empty unless $n_1 = n_3$, in which case it can be proven to be ω_q-counted by the right-hand side of (4.2) ($n = n_1, m = n_3$), thus proving orthogonality and Theorem 2. These results can also be obtained by applying directly Φ_3 and Φ_4 to the set $L_q(n_1, 0, n_3)$, assuming this time $n_1 \geq n_3$. We also have a weight-preserving sign-reversing involution proving orthogonality when the q-statistic for the moments is taken to be lb instead of rs, but we do not know how to generalize it to the linearization problem.

Corollary 7. *Let $n_1 \geq n_2 \geq \ldots \geq n_k$. The coefficient of the lowest power of a, a^{n_1} in $L_q(C_{n_1} C_{n_2} \ldots C_{n_k})$ is a polynomial in q with positive coefficients.*

Proof. The proof of Theorem 3 can be generalized to a product of k q-Charlier polynomials, any additional color being treated as was color 2, the middle color. It is easy to see then that the fixed points contributing to the lowest power of a must have all $B_i = \varnothing$, and therefore have all positive weights. $\square$

Corollary 8. *Let $n_3 \geq n_1 \geq n_2$. The coefficient of $a^{n_1+n_2-i}$ in $L_q(C_{n_1} C_{n_2} C_{n_3})$ is equal to $(q-1)^{n_1+n_2-n_3-2i}$ times the coefficient of a^{n_3+i}, for $0 \leq i \leq \lfloor (n_1+n_2-n_3)/2 \rfloor$.*

Our proof of Corollary 8 is analytical, but we would like to have a combinatorial explanation of this "symmetry" property.

Note that $Fix\Phi_5$ is not an optimal set of fixed points, in the sense that there are still some terms that cancel each other when we proceed to ω_q-counting of $Fix\Phi_5$. For example, for $n_1 = n_2 = n_3 = 2$, the two elements of $Fix\Phi_5$ such that $B_2 = \{2\}$, $w = 12121$ and $B_2 = \varnothing$, $w = 123123$ have weight $-a^3q^3$ and a^3q^3 respectively. However, we do not believe that an attempt to reduce $Fix\Phi_5$ would be worthwhile.

Corollary 9. *Let* $n_1 \geq n_2 \geq \ldots \geq n_k$. *If* $q = 1 + r$, $L_q(C_{n_1} C_{n_2} \ldots C_{n_k})$ *is a polynomial in* r *with positive coefficients.*

7. The classical q-Charlier polynomials.

We contrast the results of the previous sections with those for the classical q-Charlier polynomials [11, p.187]

$$c_n(x; a; q) = {}_2\phi_1(q^{-n}, x; 0; q, -q^{n+1}/a). \tag{7.1}$$

The monic form of these polynomials, $cc_n(x; a; q)$ satisfies

$$cc_{n+1}(x; a; q) = (x - b_n)cc_n(x; a; q) - \lambda_n cc_{n-1}(x; a; q),$$

where

$$\lambda_n = -aq^{1-2n}(1 - q^{-n})(1 + aq^{-n}), \quad b_n = aq^{-1-2n} + q^{-n} + aq^{-2n} - aq^{-n}.$$

A calculation (see [11, p.187]) shows that the moments for these polynomials are

$$\mu_n = \prod_{i=1}^{n}(1 + aq^{-i}).$$

We need to rescale x and a so that b_n and λ_n are q-analogues of $a + n$ and an respectively. If we put $x = 1 + z(1 - q)$, and multiply a by $(1 - q)$, and call the resulting monic polynomials $\hat{C}_n(z; a; q)$, the explicit formula from (7.1) is

$$\hat{C}_n(z; a; q) = q^{-n^2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} q^{\binom{k+1}{2}} \prod_{i=0}^{k-1}(q^i z - [i]_q) \tag{7.2}$$

The three term recurrence relation coefficients are

$$b_n = q^{-n}[n]_q(1+a(1-q)q^{-n})+aq^{-1-2n}, \quad \lambda_n = aq^{1-3n}[n]_q(1+a(1-q)q^{-n}). \tag{7.3}$$

A calculation using the measure in [11, p.187] gives

$$\mu_n = \sum_{j=1}^{n} q^{-\binom{j}{2}-n} S_{1/q}(n, j)a^j. \tag{7.4}$$

Again we find q-Stirling numbers for the moments. Zeng [24] has also derived (7.2) and (7.3) from the continued fraction for the moment generating function.

We see that the individual terms in (7.3) do not have constant sign. This means that the Viennot theory must involve a sign-reversing involution for its combinatorial versions of (7.3) and (7.4). Nonetheless we can give combinatorial interpretations of (7.2) and (7.4), but have no perfect analog of Theorem 3.

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