AN ELEMENTARY APPROACH TO THE MACDONALD IDENTITIES*

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Abstract. Elementary proofs are given for the infinite families of Macdonald identities. The reflections of the Weyl group provide sign-reversing involutions which show that all terms not related to the constant term cancel.

1. Introduction. The purpose of this paper is to give an elementary approach to the Macdonald identities, accessible to all combinatorialists. Sign-reversing involutions have been used recently on a variety of combinatorial problems. The Macdonald identities can be proved with such involutions, and we give the details in this paper. The basic idea of the proof does not differ from Macdonald's original proof for affine root systems [7]. Since that paper can be difficult for a novice in root systems, we offer an elementary approach for the infinite families of affine root systems. This approach was used in [9] to find new Macdonald identities in the low rank cases.

The Macdonald identities [7] are the analogues of the Weyl denominator formula for affine root systems. The Weyl formula [7] expresses a finite sum as a finite product for root systems R:

$$\sum_{w\in W} det(w)e^{w(\rho)-\rho} = \prod_{\substack{a>0\\ a\in R}} (1-e^{-a}). \tag{1.1}$$

For type A_n, (1.1) is just Vandermonde's determinant: a determinant, which is a sum of $n!$ terms, can be written as a product of $n(n-1)/2$ terms. Vandermonde's determinant can be proved by using the antisymmetry of the product to eliminate all terms in the expansion which have equal exponents. This leaves terms with different exponents (the $n!$ terms of the determinant). Their coefficients can be found by finding any one coefficient, and using the antisymmetry.

The same program works to prove the Macdonald identities. We are expanding infinite products instead of finite ones, so first we need to reduce the problem to a finite one. Then we use antisymmetry to show that all of the terms cancel, except for those related to a single term. Finally, we find the coefficient of that term. These three steps, and their relation to the Macdonald identities, are explained at the end of §2.

For the Weyl denominator formula the sign reversing involutions can be interpreted as acting upon combinatorial objects related to the root system [2],[3]. Since

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the same program works for the Macdonald identities, there should be combinatorial models and proofs in this case too (see [12]).

A summary of the recent work in this area can be found in [6] or [11]. Two papers related two special functions are [8] and [5]. The very recent work of Gustafson [4] should also be consulted.

The rest of the paper is organized in the following way. The Macdonald identities for affine root systems are stated in §2. We also explain the three main steps of the proof there. These steps are completed for types C_n and B_n in §3 and §4. We have included type B_n because the evaluation of the constant term is slightly different from type C_n. Nevertheless, all of the infinite families have arguments similar to type B_n or C_n. These identities are explicitly given in §5.

2. Notation. In this section we establish the notation for the affine root systems, Weyl groups, and functions to be considered for the Macdonald identities.

Root systems are certain finite subsets of Euclidean n-space; thus a root is a vector, or dually a linear functional on Euclidean n-space. An affine root can be thought of as an affine functional, that is, a root plus a constant. An affine root system is a set of affine roots that satisfies certain conditions [7]. These conditions will not concern us, since we will use the explicit form of the infinite families of affine root systems.

Macdonald [7, Appendix 1] lists the reduced irreducible affine root systems S. Each such system S is attached to a finite root system R, thus $S = S(R)$, and can be explicitly given. We let $\{e_1, \ldots, e_n\}$ be the standard basis for Euclidean n-space. For example, if $R = C_n (n \geq 2)$, then

$$S(C_n) = \{k \pm 2e_i : k \in \mathbb{Z}, 1 \leq i \leq n\} \cup \{k \pm e_i \pm e_j : k \in \mathbb{Z}, 1 \leq i < j \leq n\}, \tag{2.1}$$

The Macdonald identities expand the product

$$F(S) = \prod_{\substack{a>0 \\ a \in S}} (1 - e^{-a}) \tag{2.2}$$

as a sum. For (2.2) to make sense, some roots in S must be positive and others negative. This is defined by taking a certain basis for S, $B = \{a_0, \ldots, a_n\}$, so that any element of S has coefficients with respect to B that are all non-positive or all non-negative. For $S(C_n)$, $a_0 = 1 - 2e_1$, $a_n = 2e_n$, $a_i = e_i - e_{i+1}$, $1 \leq i \leq n-1$. We also can choose the a_i's so that $\sum_{i=0}^n k_i a_i = c$ for unique positive integers k_i with no common divisor. For C_n, $c = 1$. The expansion of (2.2) takes place in the formal series ring $\mathbb{Z}[[e^{-a_0}, e^{-a_1}, \ldots, e^{-a_n}]]$.

It will be more natural to consider the factors in (2.2) as functions of $x_i = e^{e_i}$ and $q = e^{-c}$. (Macdonald labels e^{-c} with X.) In this paper we put

$$F_S(x_1, \ldots, x_n, q) = \prod_{\substack{a>0 \\ a \in S}} (1 - e^{-a}) \tag{2.3}$$

Let

$$(x; q)_\infty = \prod_{i=0}^{\infty} (1 - xq^i). \tag{2.4}$$

We frequently will write $(x)_\infty$ for $(x;q)_\infty$. Then (2.1), (2.3), and (2.4) imply
(2.5)

$$F_{C_n}(x_1,\ldots,x_n,q)=\prod_{i=1}^{n}(1/x_i^2)_\infty(qx_i^2)_\infty\prod_{1\le i<j\le n}(1/x_ix_j)_\infty(qx_ix_j)_\infty(x_j/x_i)_\infty(qx_i/x_j)_\infty$$

Clearly (2.5) can be interpreted as an element of the formal power series ring $A[[q]]$, $A=\mathbb{Z}[x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}]$. This implies that, for some $f(m_1,\ldots,m_n,q)\in\mathbb{Z}[[q]]$

$$F_{C_n}(x_1,\ldots,x_n,q)=\sum_{m_1,\ldots,m_n=-\infty}^{\infty}f(m_1,\ldots,m_n,q)x_1^{m_1}\ldots x_n^{m_n}. \tag{2.6}$$

In fact we shall see that (2.6) converges as a complex function of $(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n$, $x_i\neq 0$, if $|q|<1$ is fixed. The basic reason is that $f(m_1,\ldots,m_n,q)$ converges quadratically in q.

Our proof will explicitly find $f(m_1,\ldots,m_n,q)$. At first glance (2.6) appears different from Macdonald's [7, (0.4)] as an element of $A[[q]]$:

$$P(q)F_S(x_1,\ldots,x_n,q)=\sum_{\mu\in M}\tilde{\chi}(\mu)q^{(\|\mu+\rho\|^2-\|\rho\|^2)/2g} \tag{2.7}$$

where $P(q)\in\mathbb{Z}[[q]]$, M is a certain lattice in $\mathbb{R}^n$, g is a fixed constant, and ρ is the half sum of positive roots of R. The function $\tilde{\chi}(\mu)$ is given by

$$\tilde{\chi}(\mu)=\sum_{w\in W}det(w)e^{w(\mu+\rho)-\rho} \tag{2.8}$$

where W is the Weyl group of R, and $det(w)$ is the sign of w. The x-dependence in (2.7) is contained in $\tilde{\chi}(\mu)$.

In the course of the proof we identify M, g, and $\tilde{\chi}(\mu)$ for each type R. We shall see that the proof also clearly gives (2.7).

There are three basic steps to the proof.

(I) Use the affine part of the affine root system $S(R)$ to find functional equations for $F_S(x_1,\ldots,x_n,q)$ or $f(m_1,\ldots,m_n,q)$. These functional equations reduce the unknown functions $f(m_1,\ldots,m_n,q)$ to a *finite* number, in effect, just a fundamental domain for the lattice M. The number g is also found in this step.

(II) Use the Weyl group W of the finite root system R, to show that only *one* function, $f(0,0,\ldots,0,q)$, is necessary. Sign-reversing involutions are given by W which show that all other terms are either zero, or a multiple of this constant term. The action $w\circ\mu=w(\mu+\rho)-\rho$ is identified here.

(III) The constant term $f(0,0,\ldots,0,q)$ is found by specializing the identity in (II). We use a specialization which is simpler than Macdonald's.

We carry out this program for types C_n and B_n §3 and §4.

3. Type $C_n (n\ge 2)$. Once we have fixed the affine root system, we shall drop the type S from the function $F_S(x_1,\ldots,x_n,q)$. We also suppress the q dependence. So for type C_n

$$F(x_1,\ldots,x_n)=\prod_{i=1}^{n}(1/x_i^2)_\infty(qx_i^2)_\infty\prod_{1\le i<j\le n}(1/x_ix_j)_\infty(qx_ix_j)_\infty(x_j/x_i)_\infty(qx_i/x_j)_\infty.$$

For step (I), let $S(C_n)^+$ be the positive affine roots of $S(C_n)$, and $S_i(C_n)^+$ be the elements of $S(C_n)^+$ with e_i replaced by $e_i - 1$. It is easy to check from (2.1) that, except for signs, $S(C_n)^+ - S_i(C_n)^+ = S_i(C_n)^+ - S(C_n)^+$. This shows that $F(x_1, \ldots, qx_i, \ldots, x_n)$ is closely related to F; in fact

$$F(x_1, \ldots, qx_i, \ldots, x_n)x_i^{2n+2}q^{2n-i+2} = F(x_1, \ldots, x_n) \tag{3.1}$$

which implies

$$f(m_1, \ldots, m_n, q) = f(m_1, \ldots, m_i - 2n - 2, \ldots, m_n, q)q^{m_i - i}. \tag{3.2}$$

So f is uniquely determined by $f(m_1, \ldots, m_n, q)$, $0 \leq m_i \leq 2n+1$, and $M = \{(2n+2)(m_1, \ldots, m_n) : m_i \in \mathbb{Z}\}$. Let the fundamental domain $FD = \{(m_1, \ldots, m_n) : 0 \leq m_i \leq 2n+1, 1 \leq i \leq n\}$. This finishes (I).

For (II), note that the Weyl group of type C_n is the hyperoctahedral group $W = \{(\sigma, \pi) : \sigma \in S_n, \pi \in \mathbb{Z}_2^n\}$. The generators of W are the transpositions $\sigma_i = ((i, i+1), id)$, $1 \leq i \leq n-1$, and $\sigma_n = (id, (1, \ldots, 1, -1))$, which changes the sign of the last coordinate. So we see that

$$-x_{i+1}F(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) = x_iF(x_1, \ldots, x_n) \tag{3.3a}$$

and

$$-x_n^{-2}F(x_1, \ldots, x_{n-1}, 1/x_n) = F(x_1, \ldots, x_n). \tag{3.3b}$$

Clearly (3.3a) and (3.3b) are equivalent to

$$f(m_1, \ldots, m_n, q) = -f(m_1, \ldots, m_{i+1} - 1, m_i + 1, \ldots, m_n, q) \tag{3.4a}$$

and

$$f(m_1, \ldots, m_n, q) = -f(m_1, \ldots, -m_n - 2, q). \tag{3.4b}$$

Thus, for $w = (\sigma, \pi) \in W$,

$$f(m_1, \ldots, m_n, q) = det(w)f(w(m_1, \ldots, m_n), q)$$

where the ith component of $w(m_1, \ldots, m_n)$ is

$$w(m_1, \ldots, m_n)(i) = \pi(i)(n + 1 - \sigma(i) + m_{\sigma(i)}) - (n + 1 - i). \tag{3.5}$$

Since $\rho = (n, n-1, \ldots, 1)$, we see that $w(m_1, \ldots, m_n) = w(\mu + \rho) - \rho$ if $\mu = (m_1, \ldots, m_n)$. This is precisely the action given by Macdonald in (2.8).

Our goal is to show that all terms $f(m_1, \ldots, m_n, q)$ in FD are zero, except for those corresponding to a W orbit of $(0, \ldots, 0)$. Thus we need to find the representative of $w\mu$ in FD. We define $m = (m_1, \ldots, m_n) \sim (m'_1, \ldots, m'_n) = m'$ if $m - m' \in M$.

Proposition 3.6.

(i) *For* $\mu, \mu' \in M$, *and* $w, w' \in W$, *if* $w\mu = w'\mu'$, *then* $\mu = \mu'$ *and* $w = w'$.

(ii) *For* $\mu \in M$, *and* $w, w' \in W$, *if* $w\mu \sim w'\mu$ *then* $w = w'$.

Proof. Both assertions are clear from (3.5).

A consequence of Proposition 3.6(ii) is that the W-orbit of $(0, 0, \ldots, 0)$ contains $n!2^n$ inequivalent elements, which can be mapped to FD. We now show that all other $(m_1, \ldots, m_n) \in FD$ satisfy $f(m_1, \ldots, m_n, q) = 0$. To do this we use the reflections of W: the transpositions $((ij), (1, 1, \ldots, 1))$ $1 \le i < j \le n$, the inversions $e_i \to -e_i$, and $\theta_{ij} : e_i \to -e_j$, $e_j \to -e_i$. If $w(m_1, \ldots, m_n) \sim (m_1, \ldots, m_n)$ for any of these elements w, which all have $det(w) = -1$, then $f(m_1, \ldots, m_n, q) = 0$.

Proposition 3.7. *The number of* $(m_1, \ldots, m_n) \in FD$ *which satisfy*

(i) $m_i \not\equiv m_j + i - j \mod 2n+2$,

(ii) $m_i \not\equiv 2i - m_i \mod 2n+2$, *and*

(iii) $m_i \not\equiv i + j - m_j \mod 2n+2$,

for all $1 \le i \ne j \le n$ *is* $n!2^n$.

Proof. Let $\tilde{m}_i = m_i - i$, so that (i) and (iii) becomes $\tilde{m}_i \not\equiv \pm\, \tilde{m}_j \mod 2n+2$, $i \ne j$, and (ii) becomes $\tilde{m}_i \not\equiv 0$ or $n+1 \mod 2n+2$. The remaining residue classes mod $2n+2$ for $\tilde{m}_i$ are $\{(1, 2n+1), \ldots, (n, n+2)\}$ so there are $n!2^n$ solutions $(\tilde{m}_1, \ldots, \tilde{m}_n)$. $\square$

Clearly, Propositions 3.7 and 3.6(i), and (3.2) imply that

(3.8)
$$F(x_1, \ldots, x_n) = c(q) \sum_{\mu \in M} \tilde{\chi}(\mu) x_1^{m_1(2n+2)} \ldots\ x_n^{m_n(2n+2)} q^{\sum_{i=1}^n [\binom{m_i+1}{2}(2n+2) - im_i]}$$

where $\mu = (2n+2)(m_1, \ldots, m_n)$. This agrees with (2.7). For (2.6), we note that the W-orbit of $\mu = (0, 0, \ldots, 0)$ is $w(\rho) - \rho$. The Weyl denominator formula evaluates this sum

$$\sum_{w \in W} det(w) e^{w\rho - \rho} = \prod_{\substack{a > 0 \\ a \in R}} (1 - e^{-a}) = \Delta(x_1, \ldots, x_n). \tag{3.9}$$

Combining (3.9) with (3.2), we have

$$F(x_1, \ldots, x_n) = c(q) \sum_{(m_1, \ldots, m_n) = -\infty}^{\infty} x_1^{(2n+2)m_1} \ldots x_n^{(2n+2)m_n} q^{\sum_{i=1}^n \binom{m_i+1}{2}(2n+2) - im_i}$$
$$\prod_{i=1}^n (1 - \frac{1}{x_i^2 q^{2m_i}}) \prod_{1 \le i < j \le n} (1 - \frac{1}{x_i x_j q^{m_i+m_j}})(1 - \frac{x_j q^{m_j - m_i}}{x_i}). \tag{3.10}$$

Note that the inner product is $\Delta(x_1 q^{m_1}, \ldots, x_n q^{m_n})$. This completes step (II).

It remains to evaluate the constant term $c(q)$. Clearly $(0, \ldots, 0)$ is the only element of FD all of whose entries are $0 \mod 2n+2$. So we just $(2n+2)$-sect both sides of (3.10), i.e. replace each x_i by ω^j, $0 \le j \le 2n+1$, $\omega = \exp(2\pi i/(2n+2))$ and add.

Under this operation the right side is evaluable by the Jacobi triple product identity [1] to

$$RS = (2n+2)^n c(q)(q^{2n+2};q^{2n+2})_\infty^n(-q)_\infty/(-q^{n+1};q^{n+1})_\infty. \tag{3.11}$$

For the left side, note that $F(x_1,\dots,x_n)=0$ unless $x_i \neq x_j$, $x_i \neq x_j^{-1}$, and $x_i^2 \neq 1$. So each $x_i = \omega^j$, where $j \in \{1, 2n+1, 2, 2n, \dots, n, n+2\}$. As in Proposition 3.7, there are $n!2^n$ such n-tuples $(x_1,\dots,x_n)$, which are precisely the orbit of W on $(\omega^1,\omega^2,\dots,\omega^n)$. Since $F(x_1,\dots,x_n)/\Delta(x_1,\dots,x_n)$ is invariant under W, the left side is

$$LS = F(x_1,\dots,x_n)\sum_{w\in W} w\cdot\Delta(x_1,\dots,x_n) \text{ at } (\omega^1,\omega^2,\dots,\omega^n) \tag{3.12}$$

The Weyl denominator formula (3.9) implies

$$\sum_{w\in W} w\Delta(x_1,\dots,x_n) = \Delta(x_1,\dots,x_n)\Delta(1/x_1,\dots,1/x_n) \tag{3.13}$$

so that the left side becomes

$$LS = \prod_{i=1}^{n}(\omega^{-2i})_\infty(\omega^{2i})_\infty \prod_{1\le i<j\le n}(\omega^{-i-j})_\infty(\omega^{i+j})_\infty(\omega^{j-i})_\infty(\omega^{i-j})_\infty. \tag{3.14}$$

It is easy to see that the factor $(\omega^k)_\infty$, $0\le k\le 2n+1$, occurs $n-1$ times if k is odd, and n times if k is even. There is also an additional $(\omega^{n+1})_\infty$ factor. Since $\prod_{i=1}^{2n+1}(1-\omega^i) = 2n+2$, the left side becomes

$$LS = 2(2n+2)^{n-1}(n+1)(-q)_\infty(q^{2n+2};q^{2n+2})_\infty^{n-1}(q^{n+1};q^{n+1})_\infty/(q)_\infty^n. \tag{3.15}$$

Combining (3.15) and (3.11), we have

$$c(q) = q/(q)_\infty^n. \tag{3.16}$$

This completes step (III).

4. Type $B_n (n\ge 3)$. For type B_n we proceed in the same way. However the lattice M must be slightly modified if the analogues of Propositions 3.6 and 3.7 are to be true.

In this case

$$F(x_1,\dots,x_n) = \prod_{i=1}^{\infty}(1/x_i)_\infty(qx_i)_\infty \prod_{1\le i<j\le n}(1/x_ix_j)_\infty(qx_ix_j)_\infty(x_j/x_i)_\infty(qx_i/x_j)_\infty$$

Instead of (3.1)-(3.4) we have

$$-F(x_1,\dots,qx_i,\dots,x_n)x_i^{2n-1}q^{2n-i} = F(x_1,\dots,x_n), \tag{4.1}$$

$$f(m_1,\dots,m_n,q) = -f(m_1,\dots,m_i-2n+1,\dots,m_n,q)q^{m_i-i+1}, \tag{4.2}$$

$$-x_{i+1}F(x_1,\dots,x_{i+1},x_i,\dots,x_n)=x_iF(x_1,\dots,x_n), \tag{4.3a}$$

$$-x_n^{-1}F(x_1,\dots,x_{n-1},1/x_n)=F(x_1,\dots,x_n), \tag{4.3b}$$

$$f(m_1,\dots,m_n,q)=-f(m_1,\dots,m_{i+1}-1,m_i+1,\dots,m_n,q),\text{and} \tag{4.4a}$$

$$f(m_1,\dots,m_n,q)=-f(m_1,\dots,m_{n-1},-m_n-1,q). \tag{4.4b}$$

The lattice M is replaced by $\tilde{M}=\{(2n-1)(m_1,\dots,m_n):m_i\in\mathbb{Z}\}$. The Weyl group action is

$$w(m_1,\dots,m_n)(i)=\pi(i)(n+1/2-\sigma(i)+m_{\sigma(i)})-(n+1/2-i) \tag{4.5}$$

where $w=(\sigma,\pi)$ as in type C_n. Again this is the action $w(\mu+\rho)-\rho$. This completes step (I).

Proposition 3.6(ii) is not true for $\tilde{M}$: there are non-zero elements of W which map $\tilde{M}$ to itself, e.g. $w=(id,(-1,1,\dots,1)$. However, if we restrict $\tilde{M}$ to

$$M=\{(2n-1)(m_1,\dots,m_n):m_i\in\mathbb{Z},m_1+\cdots+m_n\equiv 0 \mod 2\}$$

then Propositions 3.6(i) and (ii) are true. As a fundamental domain take

$$FD=\{(m_1,\dots,m_n):1-2n\le m_1\le 2n-2,0\le m_i\le 2n-2,2\le i\le n\}.$$

Again we show that there are $n!2^n$ elements of the orbit $(0,0,\dots,0)$ of D not fixed by a reflection. The conditions that replace those in Proposition 3.7 are $m_i-i\ne m_j-j \pmod{2n-1}$, $m_i\ne -2n-1+2i-m_i \pmod{4n-2}$, and $m_i-i\ne j-m_j-2n-1 \pmod{2n-1}$. If we put $\tilde{m}_i=m_i-i+1$ these are $\tilde{m}_i\ne\pm\tilde{m}_j$, $i\ne j$. So for $0\le\tilde{m}_i\le 2n-2$, the classes $\{(0),(1,2n-2),\dots,(n-1,n)\}$ give $n!2^{n-1}$ solutions. Allowing $1-2n\le m_1\le -1$ gives $n!2^{n-1}+n!2^{n-1}=n!2^n$ solutions $(m_1,\dots,m_n)$. This establishes (2.7), and (2.6) becomes

$$F(x_1,\dots,x_n)=c(q)\sum_{\mu\in M}x_1^{(2n-1)m_1}\dots x_n^{(2n-1)m_n}q^{\sum_{i=1}^n\binom{m_i+1}{2}(2n-1)-(i-1)m_i}$$

$$\prod_{i=1}^n(1-\frac{q^{-m_i}}{x_i})\prod_{1\le i<j\le n}(1-\frac{q^{-m_i-m_j}}{x_ix_j})(1-\frac{x_jq^{m_j-m_i}}{x_i}). \tag{4.6}$$

This completes step (II).

For step (III), if we were to $(2n-1)$-sect the right side of (4.6), the product

$$\Delta(x_1q^{m_1},\dots,x_nq^{m_n})$$

would contain two terms: $1-(x_1q^{m_1})^{1-2n}$. If we put each $x_i=-\omega^j$, $0\le j\le 2n-2$, $\omega=\exp(2\pi i/(2n-1))$, and sum, we obtain

$$RS=c(q)(2n-1)^n\sum_{\mu\in\tilde{M}}q^{\sum_{i=1}^n\binom{m_i+1}{2}(2n-1)-(i-1)m_i}, \tag{4.7}$$

which is evaluable by the Jacobi triple product formula to

$$RS = (2n-1)^n c(q)(q^{2n-1};q^{2n-1})_\infty^n(-1;q^{2n-1})_\infty(-q)_\infty. \tag{4.8}$$

On the left side, again $x_i \neq x_j$ or x_j^{-1}, so the residue classes are $\{(0),(1,2n-2),\ldots,(n-1,n)\}$. Thus, $n!2^{n-1}$ values are allowed for $(x_1,\ldots,x_n)$. Since $\omega^0=\omega^{-0}$, we again can average over all of W if we divide by 2. Equation (3.13) implies

$$LS = \frac{1}{2}\prod_{i=1}^n(-\omega^{i-1})_\infty(-\omega^{1-i})_\infty \prod_{1\le i<j\le n}(\omega^{2-i-j})_\infty(\omega^{i+j-2})_\infty(\omega^{j-i})_\infty(\omega^{i-j})_\infty. \tag{4.9}$$

In (4.9), $(\omega^k)_\infty$, $1\le k\le 2n-2$, occurs n times, so $\prod_{i=1}^{2n-2}(1+\omega^i)=1$ implies

$$LS = \frac{1}{2}(2n-1)^n(q^{2n-1};q^{2n-1})_\infty^n 4(-q)_\infty(-q^{2n-1};q^{2n-1})_\infty/(q)_\infty^n \tag{4.10}$$

Clearly (4.8) and (4.10) imply

$$c(q)=1/(q)_\infty^n. \tag{4.11}$$

5. The other infinite families. We give the Macdonald identities for types A_{n-1}, D_n, $B_n^\vee$, $C_n^\vee$, and BC_n in the form of (2.6).

Type A_{n-1} $(n\ge 2)$

$$F(x_1,\ldots,x_n) = \prod_{1\le i<j\le n}(x_j/x_i)_\infty(qx_i/x_j)_\infty. \tag{5.1}$$

$$\begin{aligned} F(x_1,\ldots,x_n) = c(q)\sum_{\mu\in M} x_1^{nm_1}\ldots x_n^{nm_n} q^{\frac{1}{2}\sum_{i=1}^n nm_i^2+m_i(n+1-2i)} \\ \prod_{1\le i<j\le n}(1-x_jq^{m_j-m_i}/x_i) \end{aligned} \tag{5.2}$$

where

$$M=\{n(m_1,\ldots,m_n): m_i\in\mathbb{Z}, \sum_{i=1}^n m_i=0\}, \text{ and} \tag{5.3}$$

$$c(q)=1/(q)_\infty^{n-1}.$$

Type D_n $(n\ge 4)$

$$F(x_1,\ldots,x_n) = \prod_{1\le i<j\le n}(x_j/x_i)_\infty(qx_i/x_j)_\infty(1/x_ix_j)_\infty(qx_ix_j)_\infty \tag{5.4}$$

$$\begin{aligned} F(x_1,\ldots,x_n) = c(q)\sum_{\mu\in M} x_1^{(2n-2)m_1}\ldots x_n^{(2n-2)m_n} q^{\sum_{i=1}^n \binom{m_i+1}{2}(2n-2)-(i-1)m_i} \\ \prod_{1\le i<j\le n}\left(1-\frac{x_jq^{m_j-m_i}}{x_i}\right)\left(1-\frac{q^{-m_i-m_j}}{x_ix_j}\right) \end{aligned} \tag{5.5}$$

where

$$M = \{(2n-2)(m_1,\ldots,m_n) : m_i \in \mathbb{Z}, \sum_{i=1}^n m_i \equiv 0 \pmod 2\} \tag{5.6}$$

and

$$c(q) = 1/(q)_\infty^n. \tag{5.7}$$

Type $B_n^\vee$ $(n \geq 3)$.

(5.8)

$$F(x_1,\ldots,x_n) = \prod_{i=1}^n (1/x_i^2;q^2)_\infty (q^2x_i^2;q^2)_\infty \prod_{1\leq i<j\leq n} (1/x_ix_j)_\infty (qx_ix_j)_\infty (x_j/x_i)_\infty (qx_i/x_j).$$

$$\begin{aligned} F(x_1,\ldots,x_n) = c(q) \sum_{\mu\in M} x_1^{2nm_1}\ldots x_n^{2nm_n} q^{\sum_{i=1}^n \binom{m_i+1}{2}2n-(i-1)m_i} \\ \prod_{i=1}^n (1-\frac{1}{x_i^2q^{2m_i}}) \prod_{1\leq i<j\leq n} (1-\frac{x_jq^{m_j-m_i}}{x_i})(1-\frac{q^{-m_i-m_j}}{x_ix_j}) \end{aligned} \tag{5.9}$$

where

$$M = \{2n(m_1,\ldots,m_n) : m_i \in \mathbb{Z}, \sum_{i=1}^n m_i \equiv 0 \mod 2\} \tag{5.10}$$

and

$$c(q) = 1/(q)_\infty^{n-1}(q^2;q^2)_\infty \tag{5.11}$$

Type $C_n^\vee$ $(n \geq 2)$

$$\begin{aligned} F(x_1,\ldots,x_n) = \prod_{i=1}^n (1/x_i;q^{1/2})_\infty (q^{1/2}x_i;q^{1/2})_\infty \\ \prod_{1\leq i<j\leq n} (1/x_ix_j)_\infty (qx_ix_j)_\infty (x_j/x_i)_\infty (qx_i/x_j)_\infty \end{aligned} \tag{5.12}$$

$$\begin{aligned} F(x_1,\ldots,x_n) = c(q) \sum_{\mu\in M} x_1^{2nm_1}\ldots x_n^{2nm_n} q^{\sum_{i=1}^n \binom{m_i+1}{2}2n-(i-1/2)m_i} \\ \prod_{i=1}^n (1-\frac{1}{x_iq^{m_i}}) \prod_{1\leq i<j\leq n} (1-\frac{q^{-m_i-m_j}}{x_ix_j})(1-\frac{x_jq^{m_j-m_i}}{x_i}) \end{aligned} \tag{5.13}$$

where

$$M = \{2n(m_1,\ldots,m_n) : m_i \in \mathbb{Z}\} \tag{5.14}$$

and

$$c(q) = 1/(q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty. \tag{5.15}$$

Type BC_n $(n \geq 1)$

$$F(x_1,\dots,x_n) = \prod_{i=1}^{n}(1/x_i)_\infty(qx_i)_\infty(qx_i^2;q^2)_\infty(q/x_i^2;q^2)_\infty$$
$$\prod_{1\le i<j\le n}(1/x_ix_j)_\infty(qx_ix_j)_\infty(x_j/x_i)_\infty(qx_i/x_j)_\infty. \tag{5.16}$$

$$F(x_1,\dots,x_n) = c(q)\sum_{\mu\in M} x_1^{(2n+1)m_1}\dots x_n^{(2n+1)m_n}q^{\sum_{i=1}^n\binom{m_i+1}{2}(2n+1)-im_i}$$
$$\cdot\prod_{i=1}^{n}(1-\frac{1}{x_iq^{m_i}})\prod_{1\le i<j\le n}(1-\frac{q^{-m_i-m_j}}{x_ix_j})(1-\frac{x_j}{x_i}q^{m_j-m_i}) \tag{5.17}$$

where

$$M = \{(2n+1)(m_1,\dots,m_n): m_i\in\mathbb{Z}\} \tag{5.18}$$

and

$$c(q) = 1/(q)_\infty^n. \tag{5.19}$$

A few remarks are in order. For each type, the sum over $\mu \in M$ refers to the allowed $(m_1,\dots,m_n)$ in M. For type D_n, the constant term is evaluated as in type B_n. The non-trivial element of W which maps $\tilde{M}$ to $\tilde{M}$ is $(id,(-1,1,\dots,1,-1)$. A $(2n-2)$-section inserts $1+q^{2m_1(n-1)}$ for the inner product on the sum side. This changes the sum to one over $\tilde{M}$ which is evaluable by the Jacobi triple product identity.

In type $B_n^\vee$ a $2n$-section fails to evaluate $c(q)$. For $F(x_1,\dots,x_n)\neq 0$ we need $x_1^2\neq 1$, $x_i\neq x_j$ or x_j^{-1}, and if $x_i=\omega^j$, $\omega=\exp(2\pi i/2n)$, this is impossible. Instead, we $2n$-sect over $x_2,\dots,x_n$ and fix x_1. The product is replaced by $1-(x_1q^{m_1})^{2n}$, which changes the sum to $\tilde{M}$ and inserts a factor of $(-1)^{m_1+\cdots+m_n}$. So, here we have

$$RS = c(q)(2n)^{n-1}(q^{2n};q^{2n})_\infty^{n-1}(q)_\infty(-q^n;q^n)_\infty(1/x_1^{2n};q^{2n})_\infty(q^{2n}x_1^{2n};q^{2n})_\infty. \tag{5.20}$$

The left side has $(n-1)!2^{n-1}$ terms, which is the Weyl group action on $(x_2,\dots,x_n)$ evaluated at $(\omega^1,\omega^2,\dots,\omega^{n-1})$. For any such choice of $(x_2,\dots,x_n)$ it is clear that

$$(1-\frac{1}{x_1^2})\prod_{j=2}^{n}(1-\frac{x_j}{x_1})(1-\frac{1}{x_1x_j}) = 1-\frac{1}{x_1^{2n}}. \tag{5.21}$$

So we can apply (3.16) for type C_{n-1} to conclude that

$$LS = (1-\frac{1}{x_1^{2n}})(q^2/x_1^2;q^2)_\infty(q^2x_1^2;q^2)_\infty\prod_{j=2}^{n}(\omega^{-2(j-1)};q^2)_\infty(\omega^{2(j-1)};q^2)_\infty$$
$$\cdot(qx_1\omega^{1-j})_\infty(qx_1\omega^{j-1})_\infty\prod_{2\le i<j\le n}(q\omega^{2-i-j})_\infty(q\omega^{i+j-2})_\infty(q\omega^{j-i})_\infty(q\omega^{i-j})_\infty. \tag{5.22}$$

It is not hard to see that (5.22) is

(5.23)
$$LS = (1/x_1^{2n}; q^{2n})_\infty (q^{2n}x_1^{2n}; q^{2n})_\infty (q^{2n}; q^{2n})_\infty^n (2n)^{n-1} (-q)_\infty / (q)_\infty^{n-3} (q^n; q^n)_\infty (q^2; q^2)_\infty^2.$$

Finally, (5.20) and (5.23) give (5.11).

Macdonald's form (2.7) also follows in these cases.

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