q-Integral and Moment Representations for q-Orthogonal Polynomials *

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Abstract

We develop a method for deriving integral representations of certain orthogonal polynomials as moments. These moment representations are applied to find linear and multilinear generating functions for q-orthogonal polynomials. As a byproduct we establish new transformation formulas for combinations of basic hypergeometric functions, including a new representation of the q-exponential function $\mathcal{E}_q$.

Running Title: Moment Representations

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1 Introduction

The concept of the q-integral has proved to be very useful in analyzing q-special functions. For $|q| < 1$, the q-integral is, [3], [10],

$$\int_a^b f(x)d_qx := b(1-q)\sum_{n=0}^{\infty} q^n f(bq^n) - a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n), \tag{1.1}$$

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with

$$\int_0^\infty f(x)d_qx := (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n). \tag{1.2}$$

We will follow the notation and terminology in [3] and [10]. Some of the technical manipulations are greatly simplified by the q-integration by parts formula

$$\begin{aligned}\int_a^b f(x)g(qx)\,d_qx \\ = q^{-1}\int_a^b g(x)f(x/q)\,d_qx + q^{-1}(1-q)[ag(a)f(a/q) - bg(b)f(b/q)].\end{aligned} \tag{1.3}$$

In our earlier papers [12], [13], [14] we utilized integral representations of orthogonal polynomials as moments to derive linear and multilinear generating functions. The idea is to start with a sequence of polynomials in which we are interested, say $\{p_n(x)\}$, then derive an integral representation of the form

$$p_n(x) = \int_a^b y^n d\mu(y), \tag{1.4}$$

where μ is some measure to be determined. For example we obtain an integral representation for any generating function of the orthogonal polynomials $\{p_n(x)\}$

$$F(x,t) = \sum_{n=0}^\infty \lambda_n p_n(x)t^n = \int_a^b \left[\sum_{n=0}^\infty \lambda_n (ty)^n\right] d\mu(y), \tag{1.5}$$

and any bilinear generating function

$$\sum_{n=0}^\infty \lambda_n p_n(x)p_n(z)t^n = \int_a^b F(z,yt)d\mu(y).$$

Mixed bilinear generating functions of the type

$$\sum_{n=0}^\infty \lambda_n p_n(x)r_n(z)t^n$$

may also be found in this manner.

By changing the normalization of $\{p_n(x)\}$ to $\{c_n p_n(x)\}$, new moment representations may also be found for $\{c_n p_n(x)\}$. A key feature of this paper is giving such alternative moment representations for q-orthogonal polynomials (see for example Theorem 2.1, Corollary 3.1, Theorem 4.1).

In this work we propose a more systematic method to establish representations such as (1.4). Our representations are all q-integrals, that is, μ is a discrete measure whose masses are located at points of the form aq^n or bq^n. The derivations use the fact that every orthogonal polynomial sequence $\{p_n(x)\}$ satisfies a three term recurrence relation of the form

$$\alpha_n p_{n+1}(x) + [\beta_n x + \gamma_n] p_n(x) + \delta_n p_{n-1}(x) = 0. \tag{1.6}$$

If the coefficients in (1.6) are polynomials in q^n, then we let $d\mu(y) = f(y)d_q y$. Now q integration by parts leads to a q-difference equation for f, with the boundary conditions $f(a/q) = f(b/q) = 0$. This method will be illustrated in the subsequent sections.

The method employed here is not completely new. When the coefficients in (1.6) are polynomials in n, integration by parts leads to a differential equation satisfied by $f(y)$ under the boundary conditions $f(a) = f(b) = 0$. This is similar to the Laplace transform method which appears in classical treatises on the subject, for example see Milne-Thomson [17, Chapter 15].

It is important to emphasize that the solution derived this way will be a solution to (1.6) but may or may not be a polynomial. One then needs an independent verification that (1.4) gives the desired polynomial solution. We show by examples that this method is effective for the Al-Salam-Chihara polynomials (§2), the q-Pollaczek polynomials (§3), the continuous q-Hermite polynomials (§4), the associated continuous q-ultraspherical polynomials (§5), and the associated Al-Salam-Chihara polynomials (§6). On the other hand when we try solutions of the form $\int_0^b y^n f(y) d_q y$, $n \geq 0$ we only need to match the boundary condition at b, that is require $f(b/q) = 0$. By varying the boundary conditions we construct two linearly independent solutions to (1.6), which is of independent interest.

Rahman and Tariq [19] used their deep knowledge of basic hypergeometric functions and their transformation theory to derive a representation of the associated q-ultraspherical polynomials as moments of a discrete measure and applied their moment representation to derive a bilinear generating function for the associated q-ultraspherical polynomials introduced in [7]. In §5 we give an elementary proof of the Rahman-Tariq result and state a companion representation of the same polynomials also as moments. Both results are used to establish linear and bilinear generating functions for the associated continuous q-ultraspherical polynomials. The same program is carried out in §6 to treat the associated Al-Salam-Chihara polynomials.

Many of the bilinear generating functions are of the form

$$K(x,y)=\sum_{n=0}^{\infty}a_n r_n(x)s_n(y), \tag{1.7}$$

where $\{r_n(x)\}$ and $\{s_n(x)\}$ are orthonormal polynomials with respect to positive measures ρ and σ, respectively. If $\{r_n(x)\}$ and $\{s_n(x)\}$ are complete in $L^2(\rho)$ and $L^2(\sigma)$, respectively, then

$$\int_{\mathcal{R}} K(x,y)r_n(x)d\rho(x)=a_n s_n(y), \quad \int_{\mathcal{R}} K(x,y)s_n(y)d\sigma(y)=a_n r_n(x).$$

The above are projection formulas involving the integral operators

$$\int_{\mathcal{R}} K(x,y)f(x)d\rho(x), \quad \int_{\mathcal{R}} K(x,y)f(y)d\sigma(x).$$

In the special case $\rho=\sigma$ the kernel K becomes a symmetric kernel, the above two integral operators coincide, and have eigenvalues $\{a_n\}$ and the corresponding eigenfunctions are $\{r_n(x)\}$, see [22]. The completeness of $\{r_n(x)\}$ shows that these are all the eigenfunctions and eigenfunctions of the corresponding integral operator. Thus many of our bilinear generating functions construct kernels of integral operators and in certain cases are Mercer kernels [22].

We now review the Casorati determinant for solutions of difference equations. If u_n and v_n are solutions of

$$a_n y_n = b_n y_{n+1} + c_n y_{n-1}, \tag{1.8}$$

then the Casorati determinant of $\{u_n, v_n\}$ is

$$\Delta_n := u_{n+1}v_n - v_{n+1}u_n. \tag{1.9}$$

By substituting u_n (respectively v_n) for y_n in (1.8), and multiplying by v_n (respectively u_n) then subtracting the results we see that $b_n\Delta_n = c_n\Delta_{n-1}$, hence

$$\Delta_n = \Delta_{m-1}\prod_{k=m}^{n}\left[\frac{c_k}{b_k}\right]. \tag{1.10}$$

Formula (1.10) will be used repeatedly in this paper.

One of the corollaries in §4 gives a new representation of the q-exponential function

$$\mathcal{E}_q(\cos\theta;\alpha) := \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-i\alpha)^n}{(q;q)_n}\, q^{n^2/4} \times(-ie^{i\theta}q^{(1-n)/2}, -ie^{-i\theta}q^{(1-n)/2};q)_n, \tag{1.11}$$

introduced in [15]. The new representation is given in Corollary 4.3. The function $\mathcal{E}_q$ satisfies $\lim_{q\to 1}\mathcal{E}_q(x;(1-q)\alpha/2) = \exp(\alpha x)$, and $\mathcal{E}_q(0;\alpha) = 1$. Ismail and Zhang [15] established a q-plane wave expansion, a special case of which is

$$(q\alpha^2;q^2)_\infty\mathcal{E}_q(x;\alpha) = \sum_{n=0}^{\infty}\frac{q^{n^2/4}\alpha^n}{(q;q)_n}H_n(x|q). \tag{1.12}$$

2 The Al-Salam-Chihara Polynomials

The Al-Salam-Chihara polynomials were introduced in [5] and [2]. We shall follow the notation in our work [13] for the Al-Salam-Chihara polynomials $\{p_n(x;t_1,t_2)\}$,

$$\begin{aligned} p_n(\cos\theta;t_1,t_2) &= {}_3\phi_2\left(\begin{matrix} q^{-n}, t_1e^{i\theta}, t_1e^{-i\theta} \\ t_1t_2, 0\end{matrix}\,\middle|\, q,q\right) \\ &= \frac{(t_2e^{-i\theta};q)_n t_1^n e^{in\theta}}{(t_1t_2;q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n}, t_1e^{i\theta} \\ q^{1-n}e^{i\theta}/t_2\end{matrix}\,\middle|\, q, qe^{-i\theta}/t_2\right). \end{aligned} \tag{2.1}$$

In [13] and [14] two representations for the Al-Salam-Chihara polynomials as moments were given.

Theorem 2.1 *The Al-Salam-Chihara polynomials have the q-integral representations*

$$\begin{aligned} \frac{p_n(\cos\theta;t_1,t_2)}{t_1^n} &= \frac{(t_1e^{i\theta}, t_1e^{-i\theta}, t_2e^{i\theta}, t_2e^{-i\theta};q)_\infty}{(1-q)e^{i\theta}\,(q, t_1t_2, qe^{2i\theta}, e^{-2i\theta};q)_\infty} \\ &\quad\times\int_{e^{-i\theta}}^{e^{i\theta}} y^n\,\frac{(qye^{i\theta}, qye^{-i\theta};q)_\infty}{(t_1y, t_2y;q)_\infty}\,d_qy, \end{aligned} \tag{A}$$

$$\begin{aligned} \frac{(t_1t_2;q)_n}{(q;q)_n}\frac{p_n(\cos\theta;t_1,t_2)}{t_1^n} &= \frac{(t_1e^{i\theta}, t_1e^{-i\theta}, qe^{i\theta}/t_1, qe^{-i\theta}/t_1;q)_\infty}{2(1-q)i\sin\theta\,(q,q,qe^{2i\theta}, qe^{-2i\theta};q)_\infty} \\ &\quad\times\int_{e^{-i\theta}}^{e^{i\theta}} y^n\,\frac{(qye^{i\theta}, qye^{-i\theta}, t_2/y;q)_\infty}{(qy/t_1, t_1y, q/(yt_1);q)_\infty}\,d_qy. \end{aligned} \tag{B}$$

The derivation of these results was by an ad-hoc method. In this section we show that the representing measures can be easily found from the three-term recurrence relation. In particular, in this section we derive the second measure and give some generating functions as corollaries (Corollaries 2.2, 2.3, and 2.4).

We use the fact that the Al-Salam-Chihara polynomials may be renormalized in two ways so that the three-term recurrence relation is linear in q^n. Specifically if,

$$\hat{p}_n(x; t_1, t_2) := p_n(x; t_1, t_2)/t_1^n$$
$$c_n(x; t_1, t_2) := \frac{(t_1 t_2; q)_n}{(q; q)_n t_1^n}\, p_n(x; t_1, t_2)$$

then [16],

$$\begin{aligned} &2x\hat{p}_n(x; t_1, t_2) \\ &\quad = (1 - t_1 t_2 q^n)\hat{p}_{n+1}(x; t_1, t_2) + (1 - q^n)\hat{p}_{n-1}(x; t_1, t_2) \\ &\qquad + (t_1 + t_2)q^n \hat{p}_n(x; t_1, t_2), \quad n > 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &2xc_n(x; t_1, t_2) \\ &\quad = (1 - q^{n+1})c_{n+1}(x; t_1, t_2) + (1 - t_1 t_2 q^{n-1})c_{n-1}(x; t_1, t_2) \\ &\qquad + (t_1 + t_2)q^n c_n(x; t_1, t_2), \quad n > 0, \end{aligned} \tag{2.3}$$

with the initial conditions

$$\begin{aligned} &\hat{p}_0(x; t_1, t_2) = 1 = c_0(x; t_1, t_2), \\ &(1 - t_1 t_2)\hat{p}_1(x; t_1, t_2)/(1 - q) = (2x - t_1 - t_2)/(1 - q) = c_1(x; t_1, t_2). \end{aligned}$$

We now show how (2.3) leads to Theorem 2.1B. We seek an integral representation

$$c_n(x; t_1, t_2) = \int_a^b y^n f(y) d_q y, \tag{2.4}$$

with f satisfying the boundary conditions

$$f(a/q) = f(b/q) = 0. \tag{2.5}$$

Assume that a and b are finite, hence the moment problem is determinate, that is the moments determine f in (2.4) uniquely, [20]. Substitute the representation (2.4) for the c's in (2.3), then equate the coefficients of y^n. The result, after applying (1.3), is that f must satisfy the functional equation

$$f(y) = \frac{q}{t_1 t_2} \frac{(1 - 2xyq + q^2y^2)}{(1 - qy/t_1)(1 - qy/t_2)} f(qy). \tag{2.6}$$

Recall that

$$u(y) = \frac{(\lambda y, q/(\lambda y); q)_\infty}{(\mu y, q/(\mu y); q)_\infty} \text{ implies } \frac{u(u)}{u(qy)} = \frac{\lambda}{\mu}. \tag{2.7}$$

Thus a solution to (2.6) which satisfies the boundary conditions (2.5) is given by

$$f(y) = \frac{(qye^{i\theta}, qye^{-i\theta}, \lambda y, q/(\lambda y); q)_\infty}{(qy/t_1, qy/t_2, y\mu, q/(y\mu); q)_\infty}, \quad \text{with} \quad q\mu = t_1 t_2 \lambda \tag{2.8}$$

where $x = \cos\theta$, $a = e^{-i\theta}$ and $b = e^{i\theta}$. Observe that here a and b are finite, hence if f exists it will be unique. We then choose $\mu = t_1$ and $\lambda = q/t_2$ so that

$$g(\cos\theta) c_n(\cos\theta; t_1, t_2) = \frac{1}{1-q} \int_{e^{-i\theta}}^{e^{i\theta}} y^n \frac{(qye^{i\theta}, qye^{-i\theta}, t_2/y; q)_\infty}{(qy/t_1, t_1 y, q/(yt_1); q)_\infty} d_q y, \tag{2.9}$$

for some function $g(\cos\theta)$, independent of n.

We now give a rigorous proof of (2.9) and determine g. The proof is based on the three term transformation formula [10, (III.31)]

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix} A, B \\ C \end{matrix} \,\middle|\, q, Z \right) &= \frac{(ABZ/C, q/C; q)_\infty}{(AZ/C, q/A; q)_\infty} {}_2\phi_1\left(\begin{matrix} C/A, Cq/ABZ \\ qC/AZ \end{matrix} \,\middle|\, q, qB/C \right) \\ &\quad - \frac{(B, q/C, C/A, AZ/q, q^2/AZ; q)_\infty}{(C/q, qB/C, q/A, AZ/C, qC/AZ; q)_\infty} \\ &\quad \times {}_2\phi_1\left(\begin{matrix} qA/C, qB/C \\ q^2/C \end{matrix} \,\middle|\, q, Z \right). \end{aligned} \tag{2.10}$$

Proof of (2.9). By the definition of the q-integral, the right-hand side R of (2.9) is

$$\begin{aligned} R &= e^{i\theta} \sum_{m=0}^{\infty} \frac{(q^{m+1}e^{2i\theta}, q^{m+1}, q^{-m}t_2 e^{-i\theta}; q)_\infty}{(q^{m+1}e^{i\theta}/t_1, q^m t_1 e^{i\theta}, q^{1-m}e^{-i\theta}/t_1; q)_\infty} e^{in\theta} q^{m(n+1)} \\ &\quad - \text{a similar term with } \theta \text{ replaced by } -\theta. \end{aligned}$$

The above expression simplifies to

$$R = \frac{e^{i(n+1)\theta}(qe^{2i\theta}, q, e^{-i\theta}t_2; q)_\infty}{(qe^{i\theta}/t_1, t_1e^{i\theta}, qe^{-i\theta}/t_1; q)_\infty} {}_2\phi_1\left(\begin{matrix} qe^{i\theta}/t_1, qe^{i\theta}/t_2 \\ qe^{2i\theta} \end{matrix} \,\middle|\, q, t_1t_2q^n\right)$$
$$- \text{ a similar term with } \theta \text{ replaced by } -\theta,$$

which is

(2.11)
$$R = e^{i(n+1)\theta}\frac{(qe^{2i\theta}, q, e^{-i\theta}t_2; q)_\infty}{(qe^{i\theta}/t_1, t_1e^{i\theta}, qe^{-i\theta}/t_1; q)_\infty}$$
$$\times \left[{}_2\phi_1\left(\begin{matrix} qe^{i\theta}/t_1, qe^{i\theta}/t_2 \\ qe^{2i\theta} \end{matrix} \,\middle|\, q, t_1t_2q^n\right) + e^{-2in\theta}\frac{(t_2e^{i\theta}, e^{-2i\theta}, t_1e^{i\theta}; q)_\infty}{(t_2e^{-i\theta}, e^{2i\theta}, t_1e^{-i\theta}; q)_\infty}\right.$$
$$\left.\times\ {}_2\phi_1\left(\begin{matrix} qe^{-i\theta}/t_1, qe^{-i\theta}/t_2 \\ qe^{-2i\theta} \end{matrix} \,\middle|\, q, t_1t_2q^n\right)\right]$$

In (2.10) we make the parameter identification

(2.12) $\quad A = qe^{i\theta}/t_1, \quad B = qe^{i\theta}/t_2, \quad C = qe^{2i\theta}, \quad Z = t_1t_2q^n.$

The expression between square brackets in (2.11), with the parameter identification (2.12) is

$${}_2\phi_1\left(\begin{matrix} A, B \\ C \end{matrix} \,\middle|\, q, Z\right)$$
$$+\frac{(B, q/C, C/A, AZ/q, q^2/AZ; q)_\infty}{(C/q, qB/C, q/A, AZ/C, qC/AZ; q)_\infty} {}_2\phi_1\left(\begin{matrix} qA/C, qB/C \\ q^2/C \end{matrix} \,\middle|\, q, Z\right).$$

Thus

$$R = e^{i(n+1)\theta}\frac{(qe^{2i\theta}, q, t_2e^{-i\theta}; q)_\infty}{(qe^{i\theta}/t_1, t_1e^{i\theta}, qe^{-i\theta}/t_1; q)_\infty}\frac{(q^{n+1}, e^{-2i\theta}; q)_\infty}{(q^nt_2e^{-i\theta}, t_1e^{-i\theta}; q)_\infty}$$
$$\times {}_2\phi_1\left(\begin{matrix} t_1e^{i\theta}, q^{-n} \\ q^{1-n}e^{i\theta}/t_2 \end{matrix} \,\middle|\, q, qe^{-i\theta}/t_2\right),$$

which simplifies to

$$R = e^{in\theta}\frac{(e^{2i\theta}, e^{-2i\theta}, q, q; q)_\infty}{(qe^{i\theta}/t_1, qe^{-i\theta}/t_1, t_1e^{i\theta}, t_1e^{-i\theta}; q)_\infty}\frac{i(t_2e^{-i\theta}; q)_n}{2\sin\theta(q; q)_n}$$
$$\times {}_2\phi_1\left(\begin{matrix} t_1e^{i\theta}, q^{-n} \\ q^{1-n}e^{i\theta}/t_2 \end{matrix} \,\middle|\, q, qe^{-i\theta}/t_2\right).$$

In view of (2.1) we have

$$R = \frac{i(e^{2i\theta}, e^{-2i\theta}, q, q; q)_\infty}{2\sin\theta(qe^{i\theta}/t_1, qe^{-i\theta}/t_1, t_1e^{i\theta}, t_1e^{-i\theta}; q)_\infty} \frac{(t_1t_2; q)_n}{t_1^n(q; q)_n} p_n(\cos\theta; t_1, t_2),$$

and Theorem 2.1B follows.

We next give some generating functions which follow from Theorem 2.1B. The analogous corollaries for Theorem 2.1A appear in [13].

Corollary 2.2 *We have the linear generating function*

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(t_1t_2, \lambda/\mu; q)_n}{(q, q; q)_n} p_n(\cos\theta; t_1, t_2)\mu^n \\ &\quad = \frac{e^{i\theta}(t_2e^{-i\theta}, t_1e^{-i\theta}, t_1\lambda e^{i\theta}; q)_\infty}{2i\sin\theta(q, qe^{-2i\theta}, t_1\mu e^{i\theta}; q)_\infty} \\ &\quad\quad \times {}_3\phi_2\left(\begin{matrix} qe^{i\theta}/t_1, qe^{i\theta}/t_2, t_1\mu e^{i\theta} \\ qe^{2i\theta}, t_1\lambda e^{i\theta}\end{matrix} \,\middle|\, q, t_1t_2\right) \\ &\quad\quad - \textit{a similar term with } \theta \textit{ replaced by } -\theta.\end{aligned}$$

Proof. Theorem 2.1B and the q-binomial theorem show that the left-hand side of Corollary 2.2 is

$$\begin{aligned}&\frac{(t_1e^{i\theta}, t_1e^{-i\theta}, qe^{i\theta}/t_1, qe^{-i\theta}/t_1; q)_\infty}{2(1-q)i\sin\theta\,(q, q, qe^{2i\theta}, qe^{-2i\theta}; q)_\infty} \\ &\quad \times \int_{e^{-i\theta}}^{e^{i\theta}} \frac{(qye^{i\theta}, qye^{-i\theta}, t_2/y, \lambda t_1 y; q)_\infty}{(qy/t_1, t_1y, q/(yt_1), \mu y t_1; q)_\infty}\, d_qy.\end{aligned} \tag{2.13}$$

It is easy to see that

$$\begin{aligned}&\int_{e^{-i\theta}}^{e^{i\theta}} \frac{(qye^{i\theta}, qye^{-i\theta}, t_2/y, \lambda t_1 y; q)_\infty}{(qy/t_1, t_1y, q/(yt_1), \mu y t_1; q)_\infty} \frac{d_qy}{1-q} \\ &\quad = \sum_{m=0}^{\infty} \frac{(q^{m+1}e^{2i\theta}, q^{m+1}, q^{-m}t_2e^{-i\theta}, t_1\lambda e^{i\theta}q^m; q)_\infty}{(q^{m+1}e^{i\theta}/t_1, q^m t_1 e^{i\theta}, q^{1-m}e^{-i\theta}/t_1, t_1\mu e^{i\theta}q^m; q)_\infty} e^{i\theta}q^m \\ &\quad\quad - \text{a similar term with } \theta \to -\theta. \\ &\quad = \frac{e^{i\theta}(qe^{2i\theta}, q, t_2e^{-i\theta}, t_1\lambda e^{i\theta}; q)_\infty}{(qe^{i\theta}/t_1, t_1e^{i\theta}, qe^{-i\theta}/t_1, t_1\mu e^{i\theta}; q)_\infty} \\ &\quad\quad \times {}_3\phi_2\left(\begin{matrix} qe^{i\theta}/t_1, qe^{i\theta}/t_2, t_1\mu e^{i\theta} \\ qe^{2i\theta}, t_1\lambda e^{i\theta}\end{matrix} \,\middle|\, q, t_1t_2\right) \\ &\quad\quad - \text{a similar term with } \theta \text{ replaced by } -\theta.\end{aligned}$$

Therefore (2.13) and the above calculation indicate that the left-hand side of Corollary 2.2 is

$$\frac{e^{i\theta}(t_2e^{-i\theta}, t_1e^{-i\theta}, t_1\lambda e^{i\theta}, ; q)_\infty}{2i\sin\theta(q, qe^{-2i\theta}, t_1\mu e^{i\theta}; q)_\infty} {}_3\phi_2\left(\begin{matrix} qe^{i\theta}/t_1, qe^{i\theta}/t_2, t_1\mu e^{i\theta} \\ qe^{2i\theta}, t_1\lambda e^{i\theta}\end{matrix}\,\middle|\, q, t_1t_2\right)$$
$$- \text{ a similar term with } \theta \text{ replaced by } -\theta,$$

and Corollary 2.2 follows.

Recall that the Al-Salam-Chihara polynomials have the generating function [13, (3.18)]

$$\sum_{n=0}^{\infty} \frac{(t_1t_2; q)_n t^n}{(q; q)_n t_1^n} p_n(\cos\theta; t_1, t_2) = \frac{(tt_1, tt_2; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty}. \tag{2.14}$$

Corollary 2.3 *The Al-Salam-Chihara polynomials have the following bilinear generating function*

$$\sum_{n=0}^{\infty} \frac{(t_1t_2, s_1s_2; q)_n}{(q, q; q)_n} p_n(\cos\theta; t_1, t_2) p_n(\cos\phi; s_1, s_2) \left(\frac{t}{t_1s_1}\right)^n$$
$$= \frac{(t_1e^{-i\theta}, t_2e^{-i\theta}, ts_1e^{i\theta}, ts_2e^{i\theta}; q)_\infty}{(q, e^{-2i\theta}, te^{i(\theta+\phi)}, te^{i(\theta-\phi)}; q)_\infty}$$
$$\times {}_4\phi_3\left(\begin{matrix} te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, qe^{i\theta}/t_1, qe^{i\theta}/t_2 \\ ts_1e^{i\theta}, ts_2e^{i\theta}, qe^{2i\theta}\end{matrix}\,\middle|\, q, t_1t_2\right)$$
$$+ \text{ a similar term with } \theta \text{ replaced by } -\theta.$$

Proof. Replace $p_n(\cos\theta; t_1, t_2)$ by its integral representation in Theorem 2.1B then use (2.14) to see that the left-hand side of Corollary 2.3 is

$$\frac{(t_1e^{i\theta}, t_1e^{-i\theta}, qe^{i\theta}/t_1, qe^{-i\theta}/t_1; q)_\infty}{2(1-q)i\sin\theta\,(q, q, qe^{2i\theta}, qe^{-2i\theta}; q)_\infty}$$
$$\times \int_{e^{-i\theta}}^{e^{i\theta}} \frac{(qye^{i\theta}, qye^{-i\theta}, t_2/y, ts_1y, ts_2y; q)_\infty}{(qy/t_1, t_1y, q/(yt_1), tye^{i\phi}, tye^{-i\phi}; q)_\infty} d_qy.$$

This expression simplifies to the right-hand side of Corollary 2.3.

An unexpected transformation formula results from the above corollary, namely the fact that its right-hand side is invariant under the interchanges

$$(\theta, \phi, t_1, t_2, s_1, s_2) \to (\phi, \theta, s_1, s_2, t_1, t_2).$$

This establishes the next corollary.

Corollary 2.4 *The combination*

$$\begin{aligned}&\frac{(t_1e^{-i\theta},t_2e^{-i\theta},ts_1e^{i\theta},ts_2e^{i\theta};q)_\infty}{(q,e^{-2i\theta},te^{i(\theta+\phi)},te^{i(\theta-\phi)};q)_\infty}\\&\qquad\times{}_4\phi_3\left(\begin{matrix}te^{i(\theta+\phi)},te^{i(\theta-\phi)},qe^{i\theta}/t_1,qe^{i\theta}/t_2\\ ts_1e^{i\theta},ts_2e^{i\theta},qe^{2i\theta}\end{matrix}\,\middle|\,q,t_1t_2\right)\\&\quad+\text{ a similar term with }\theta\text{ replaced by }-\theta,\end{aligned}$$

is invariant under the permutation $(\theta,\phi,t_1,t_2,s_1,s_2)\to(\phi,\theta,s_1,s_2,t_1,t_2)$.

It is important to emphasize that the ${}_4\phi_3$'s appearing in the transformation Corollary 2.4 are not balanced and most of the known transformations of this type involve balanced series.

The moment representations not only give an integral representation for the Al-Salam-Chihara polynomials but also they give q-integral representations for other solutions to the same three term recurrence relation. For example the argument preceding (2.8) shows that

(2.15)
$$\phi_n^{\pm}(x):=\frac{1}{1-q}\int_0^{e^{\pm i\theta}}y^n\,\frac{(qye^{i\theta},qye^{-i\theta},\lambda y,q/(\lambda y);q)_\infty}{(qy/t_1,qy/t_2,\mu y,q/(\mu y);q)_\infty}\,d_qy,\quad n>1,$$

are solutions to (2.3), where $q\mu=t_1t_2\lambda$,

$$e^{\pm i\theta}=x\pm\sqrt{x^2-1}, \tag{2.16}$$

and the branch of the square root is chosen in such a way that $|e^{-i\theta}|\le|e^{i\theta}|$. Thus

$$\begin{aligned}\psi_n^{\pm}(x):=&\frac{(q;q)_nt_1^n}{(t_1t_2)_n(1-q)}\\&\times\int_0^{e^{\pm i\theta}}y^n\,\frac{(qye^{i\theta},qye^{-i\theta},\lambda y,q/(\lambda y);q)_\infty}{(qy/t_1,qy/t_2,\mu y,q/(\mu y);q)_\infty}\,d_qy,\quad n>0,\end{aligned} \tag{2.17}$$

are solutions to the recurrence relation satisfied by the Al-Salam-Chihara polynomials. Therefore

$$e^{\pm i(n+1)\theta}\frac{(q;q)_nt_1^n}{(t_1t_2;q)_n}{}_2\phi_1\left(\begin{matrix}qe^{\pm i\theta}/t_1,qe^{\pm i\theta}/t_2\\ qe^{\pm 2i\theta}\end{matrix}\,\middle|\,q,q^nt_1t_2\right), \tag{2.18}$$

are linearly independent solutions of the Al-Salam-Chihara three term recurrence relation (2.19), which are multiples of $\psi_n^{\pm}(x)$. The polynomial solution in (2.1) together with any one of the solutions in (2.18) form a basis of solutions to the three term recurrence relation

(2.19)
$$\begin{aligned} 2xz_n(x;t_1,t_2) &= (t_1^{-1} - t_2q^n)z_{n+1}(x;t_1,t_2) + t_1(1-q^n)z_{n-1}(x;t_1,t_2) \\ &\quad + (t_1+t_2)q^n z_n(x;t_1,t_2), \quad n > 0. \end{aligned}$$

We next state a bibasic version of Corollary 2.3. Let $p_n(x;t_1,t_2|q)$ denote the Al-Salam-Chihara polynomials with base q. Then the bibasic version is:

$$\begin{aligned} &\sum_{n=0}^{\infty} p_n(\cos\theta;t_1,t_2|q)p_n(\cos\phi;s_1,s_2|p)\frac{(t_1t_2;q)_n(s_1s_2;p)_n}{(q;q)_n(p;p)_n}\left(\frac{t}{t_1s_1}\right)^n \\ &= \frac{(t_1e^{-i\theta},t_2e^{-i\theta};q)_\infty}{(q,e^{-2i\theta};q)_\infty}\sum_{k=0}^{\infty}\frac{(qe^{i\theta}/t_1,t_1e^{i\theta},qe^{i\theta}/t_2;q)_k}{(q,qe^{2i\theta},t_1e^{i\theta};q)_k}(t_1t_2)^k \\ &\quad \times \frac{(ts_1q^ke^{i\theta},ts_2q^ke^{i\theta};p)_\infty}{(tq^ke^{i(\theta+\phi)},tq^ke^{i(\theta-\phi)};p)_\infty} \\ &\quad + \text{a similar term with } \theta \text{ replaced by } -\theta. \end{aligned} \tag{2.20}$$

This establishes the following bibasic version of Corollary 2.4.

Corollary 2.5 *The right-hand side of* (2.20) *is symmetric under interchanging*

$$(t_1,t_2,s_1,s_2,\theta,\phi,p,q) \text{ with } (s_1,s_2,t_1,t_2,\phi,\theta,q,p).$$

3 The q-Pollaczek Polynomials

The q-Pollaczek polynomials $\{F_n(x;U,\Delta,V)\}$, or $\{F_n(x)\}$ for short, were introduced in [9], whose notation we shall follow. They are generated by

$$F_0(x) = 1, \quad F_{-1}(x) = 0, \tag{3.1}$$

and

$$\begin{aligned} 2[(1-U\Delta q^n)x + Vq^n]F_n(x) &= (1-q^{n+1})F_{n+1}(x) \\ &\quad + (1-\Delta^2q^{n-1})F_{n-1}(x), \quad n > 0. \end{aligned} \tag{3.2}$$

Charris and Ismail [9] gave the generating function

$$\sum_{n=0}^{\infty} F_n(\cos\theta)t^n = \frac{(t/\xi,t/\eta;q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}, \tag{3.3}$$

where

$$(3.4) \quad 1 + 2q(V - x\Delta U)\Delta^{-2}y + q^2\Delta^{-2}y^2 = (1 - q\xi y)(1 - q\eta y),$$

and ξ and η depend on x, and satisfy

$$(3.5) \qquad \xi\eta = \Delta^{-2}.$$

The generating function (3.3) implies the explicit representation

$$(3.6) \quad F_n(\cos\theta) = e^{in\theta}\frac{(e^{-i\theta}/\xi;q)_n}{(q;q)_n}{}_2\phi_1\left(\begin{matrix} q^{-n}, e^{i\theta}/\eta \\ q^{1-n}e^{i\theta}\xi \end{matrix}\,\middle|\, q, qe^{-i\theta}\xi\right).$$

From (3.6) and (2.1) it follows that

$$(3.7) \qquad F_n(x;U,\Delta,V) = \frac{(1/(\xi\eta);q)_n}{(q;q)_n}\eta^n p_n(x;1/\eta,1/\xi),$$

and we can apply the results of §2 to state similar results for the q-Pollaczek polynomials.

Corollary 3.1 *The q-Pollaczek polynomials have the q-integral representations*

$$\text{(A)} \quad \begin{aligned} \frac{(q;q)_n}{(\Delta^2;q)_n}F_n(x;U,\Delta,V) &= \frac{(e^{i\theta}/\eta, e^{-i\theta}/\eta, e^{i\theta}/\xi, e^{-i\theta}/\xi;q)_\infty}{(1-q)e^{i\theta}\,(q, qe^{2i\theta}, qe^{-2i\theta};q)_\infty} \\ &\times \int_{e^{-i\theta}}^{e^{i\theta}} y^n\,\frac{(qye^{i\theta}, qye^{-i\theta};q)_\infty}{(y/\eta, y/\xi;q)_\infty}\,d_qy, \end{aligned}$$

$$\text{(B)} \quad \begin{aligned} F_n(x;U,\Delta,V) &= \frac{(q\eta e^{i\theta}, q\eta e^{-i\theta}, e^{i\theta}/\eta, e^{-i\theta}/\eta;q)_\infty}{2(1-q)i\sin\theta\,(q,q,qe^{2i\theta}, qe^{-2i\theta};q)_\infty} \\ &\times \int_{e^{-i\theta}}^{e^{i\theta}} y^n\,\frac{(qye^{i\theta}, qye^{-i\theta}, 1/(\xi y);q)_\infty}{(qy\eta, y/\eta, q\eta/y;q)_\infty}\,d_qy, \end{aligned}$$

It was shown in [9] that the orthogonality relation of the F_n's is

$$(3.8) \quad \begin{aligned} &\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(e^{i\theta}/\xi, e^{-i\theta}/\xi, e^{i\theta}/\eta, e^{-i\theta}/\eta;q)_\infty} \\ &\quad\times F_m(\cos\theta;U,\Delta,V)F_n(\cos\theta;U,\Delta,V)d\theta \\ &= \frac{2\pi}{(q,\Delta^2;q)_\infty}\,\frac{(\Delta^2;q)_n}{(1-U\Delta q^n)(q;q)_n}\,\delta_{m,n}. \end{aligned}$$

We next record two reproducing kernels for the q-Pollaczek polynomials. Corollary 3.1B shows that the q-Pollaczek polynomials have the bilinear generating functions

$$\sum_{n=0}^{\infty} F_n(\cos\theta; U_1, \Delta_1, V_1) F_n(\cos\phi; U_2, \Delta_2, V_2) t^n \tag{3.9}$$
$$= \frac{(q\eta_1 e^{-i\theta}, e^{-i\theta}/\eta_1, te^{i\theta}/\xi_2, te^{i\theta}/\eta_2, ; q)_\infty}{(q, e^{2i\theta}, te^{i(\theta+\phi)}, te^{i(\theta-\phi)}; q)_\infty}$$
$$\times {}_4\phi_3\left(\begin{matrix} q\eta_1 e^{i\theta}, q\xi_1 e^{i\theta}, te^{i(\theta+\phi)}, te^{i(\theta-\phi)} \\ qe^{2i\theta}, te^{i\theta}/\xi_2, te^{i\theta}/\eta_2, \end{matrix} \middle| q, \frac{1}{\xi_1\eta_1}\right)$$
$$+ \text{ a similar term with } \theta \text{ replaced by } -\theta,$$

where

$$1 + 2q(V_1 - \cos\theta\Delta_1 U_1)\Delta_1^{-2} y + q^2\Delta_1^{-2} y^2 = (1 - q\xi_1 y)(1 - q\eta_1 y), \tag{3.10}$$
$$1 + 2q(V_2 - \cos\phi\Delta_2 U_2)\Delta_2^{-2} y + q^2\Delta_2^{-2} y^2 = (1 - q\xi_2 y)(1 - q\eta_2 y). \tag{3.11}$$

Another reproducing kernel follows from Corollary 3.1A and the generating function (3.3). The result is

$$\sum_{n=0}^{\infty} F_n(\cos\theta; U_1, \Delta_1, V_1) F_n(\cos\phi; U_2, \Delta_2, V_2) \frac{(q;q)_n t^n}{(\Delta^2; q)_n} \tag{3.12}$$
$$= \frac{(e^{-i\theta}/\eta_1, e^{-i\theta}/\xi_1, te^{i\theta}/\xi_2, te^{i\theta}/\eta_2, ; q)_\infty}{(q, e^{2i\theta}, te^{i(\theta+\phi)}, te^{i(\theta-\phi)}; q)_\infty}$$
$$\times {}_4\phi_3\left(\begin{matrix} e^{i\theta}/\xi_1, e^{i\theta}/\eta_1, te^{i(\theta+\phi)}, te^{i(\theta-\phi)} \\ qe^{2i\theta}, te^{i\theta}/\xi_2, te^{i\theta}/\eta_2, \end{matrix} \middle| q, q\right)$$
$$+ \text{ a similar term with } \theta \text{ replaced by } -\theta.$$

The Poisson kernel is similar to (3.12) except that the summand on left-hand side will have the additional factor $(1 - U_1\Delta_1 q^n)$. The Poisson kernel can be evaluated by taking appropriate combinations of the right-hand side of (3.12). The same phenomenon occurs for continuous q-ultraspherical polynomials which corresponds to $U = 1$, $V = 0$, and $\Delta = \beta$. Thus $\xi_1 = e^{i\theta}/\beta, \eta_1 = e^{-i\theta}/\beta$, and similarly for the ξ_2 and η_2. For details see [10, §8.6].

It is worth noting the integral evaluation equivalent to the orthogonality relation (3.8). Multiply (3.8) by $s^m(1 - U\Delta q^n)t^n$ and sum over m, n, $m, n \geq 0$. The right-hand side can be summed by the q-binomial theorem to

$$\frac{2\pi(st\Delta^2; q)_\infty}{(q, \Delta^2, st; q)_\infty}.$$

Applying the generating function (3.3) the integrand on the left-hand side involves the factor

$$(1 - t/\xi)(1 - t/\eta) - U\Delta(1 - te^{i\theta})(1 - te^{-i\theta})$$

which in view of (3.4) and (3.5) is $1 - U\Delta + 2tV + \Delta t^2(\Delta - U)$. This establishes the following theorem.

Theorem 3.2 *We have the integral evaluation*

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}, s/\xi, s/\eta, qt/\xi, qt/\eta; q)_\infty}{(e^{i\theta}/\xi, e^{-i\theta}/\xi, e^{i\theta}/\eta, e^{-i\theta}/\eta, se^{i\theta}, se^{-i\theta}, te^{i\theta}, te^{-i\theta}; q)_\infty}\, d\theta$$
$$= \frac{1}{[1 - U\Delta + 2tV + \Delta t^2(\Delta - U)]} \frac{2\pi(st\Delta^2; q)_\infty}{(q, \Delta^2, st; q)_\infty}.$$

We do not know of a direct way of evaluating this integral. The evaluation of the integral via a moment problem was given in [9]. This also occurred in Chapters 6 and 7 of [5], where the identities obtained through solving a moment problem do not seem to be amenable to direct proofs.

4 The Continuous q-Hermite Polynomials

The continuous q-Hermite polynomials satisfy

(4.1) $\quad H_{-1}(x|q) = 0, \quad H_0(x|q) = 1,$

(4.2) $\quad 2xH_n(x|q) = H_{n+1}(x|q) + (1 - q^n)H_{n-1}(x|q), \quad n > 0.$

We clearly have $H_n(x|q) = \hat{p}_n(x; 0, 0)$, so that Theorem 2.1 gives integral representations for the q-Hermite polynomials. For Theorem 2.1A this is immediate, while it is not clear how to let $t_1 = t_2 = 0$ in Theorem 2.1B. In this section we carry out this limit, and we also give two additional q-integral representations. One surprising result is Corollary 4.3 which gives ${}_2\phi_1$ representations of the function $\mathcal{E}_q$.

Theorem 4.1 *The q-Hermite polynomials have the q-integral representations*

(A)
$$H_n(\cos\theta \mid q) = \frac{1}{(1 - q)e^{i\theta}\,(q, qe^{2i\theta}, e^{-2i\theta}; q)_\infty} \times \int_{e^{-i\theta}}^{e^{i\theta}} y^n\,(qye^{i\theta}, qye^{-i\theta}; q)_\infty d_q y,$$

(B)
$$\frac{H_n(\cos\theta \mid q)}{(q;q)_n} = \frac{(\lambda e^{i\theta}, qe^{i\theta}/\lambda, \lambda e^{-i\theta}, qe^{-i\theta}/\lambda; q)_\infty}{2(1-q)i\sin\theta(q, qe^{2i\theta}, qe^{-2i\theta}; q)_\infty} \times \int_{e^{-i\theta}}^{e^{i\theta}} y^n \frac{(qye^{i\theta}, qye^{-i\theta}; q)_\infty}{(\lambda y, qy/\lambda, \lambda/y, q/(\lambda y); q)_\infty} d_q y,$$

(C)
$$\frac{H_n(\cos\theta \mid q^2)}{(q;q)_n} = \frac{(\sqrt{q}e^{i\theta}, \sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta}, \sqrt{q}e^{-i\theta}; q)_\infty}{2(1-q)i\sin\theta(q, q, qe^{2i\theta}, qe^{-2i\theta}; q)_\infty} \times \int_{e^{-i\theta}}^{e^{i\theta}} y^n \frac{(qye^{i\theta}, qye^{-i\theta}, -\sqrt{q}/y; q)_\infty}{(\sqrt{q}y, \sqrt{q}y, \sqrt{q}/y; q)_\infty} d_q y.$$

(D)
$$\frac{H_n(\cos\theta \mid q^2)}{(-q;q)_n} = \frac{(qe^{2i\theta}, qe^{-2i\theta}; q^2)_\infty}{2(1-q)i\sin\theta(q, -q, qe^{2i\theta}, qe^{-2i\theta}; q)_\infty} \times \int_{e^{-i\theta}}^{e^{i\theta}} y^n \frac{(qye^{i\theta}, qye^{-i\theta}; q)_\infty}{(qy^2; q^2)_\infty} d_q y.$$

Note that the right-hand side of Theorem 4.1B is independent of λ.
Proof of Theorem 4.1B. First we motivate the integral for Theorem 4.1B. If $\hat{H}_n(x|q) = H_n(x|q)/(q;q)_n$, then (4.2) becomes

$$2x\hat{H}_n(x|q) = (1-q^{n+1})\hat{H}_{n+1}(x|q) + \hat{H}_{n-1}(x|q). \tag{4.3}$$

Here again we see that writing $\hat{H}_n(x|q) = \int_a^b y^n f(y) d_q y$ requires f to satisfy

$$f(y) = (1-qye^{i\theta})(1-qye^{-i\theta})(qy^2)^{-1} f(qy).$$

Solving the above functional equation gives rise to the two solutions

$$\int_0^{e^{\pm i\theta}} y^n \frac{(qye^{i\theta}, qye^{-i\theta}; q)_\infty}{(\lambda y, \lambda/y, q/(\lambda y), qy/\lambda; q)_\infty} d_q y$$

and the integral in Theorem 4.1B is linear combination of these two solutions.

We next show that Theorem 4.1A implies Theorem 4.1B. From Theorem 4.1A we have

$$H_n(\cos\theta|q) = \frac{e^{-i\theta}(q, qe^{2i\theta}; q)_\infty}{(q, qe^{2i\theta}, qe^{-2i\theta}; q)_\infty} e^{i(n+1)\theta} \times {}_2\phi_1(0,0; qe^{2i\theta}; q, q^{n+1}) + \text{ a similar term with } \theta \text{ replaced by } -\theta. \tag{4.4}$$

However a limiting case of Heine's transformation [10, (III.3)] implies

$$(q;q)_\infty \, {}_2\phi_1(0,0,qe^{2i\theta};q,q^{n+1}) = (q;q)_n \, {}_0\phi_1(-;qe^{2i\theta};q,q^{n+2}e^{2i\theta}), \tag{4.5}$$

so that (4.4) becomes

$$\begin{aligned} \frac{H_n(\cos\theta|q)}{(q;q)_n} &= \frac{e^{in\theta}}{(e^{-2i\theta};q)_\infty} {}_0\phi_1\left(-;qe^{2i\theta};q,q^{n+2}e^{2i\theta}\right) \\ &\quad + \text{a similar term with } \theta \text{ replaced by } -\theta, \end{aligned} \tag{4.6}$$

which is the equivalent form of Theorem 4.1B.

Proof of Theorem 4.1C. This time if $p_n(x|q) = H_n(x|q^2)/(q;q)_n$, then (4.2) becomes

$$2xp_n(x|q) = (1-q^{n+1})p_{n+1}(x|q) + (1+q^n)p_{n-1}(x|q). \tag{4.7}$$

In the notation of (2.3) we find that $p_n(x|q) = c_n(x;\sqrt{q},-\sqrt{q})$, so that Theorem 4.1C is a special case of Theorem 2.1B.

Proof of Theorem 4.1D. This follows from Theorem 2.1A and the proof of Theorem 4.1C.

The limit $(t_1,t_2) \to (0,0)$ **in Theorem 2.1B**. The limit $t_2 \to 0$ is Theorem 2.1B in straightforward. To let $t_1 \to 0$ we set $t_1 = \lambda q^m$ then let $m \to \infty$. Theorem 4.1B follows from letting $m \to \infty$ in

$$\begin{aligned} &\frac{(q^{1-m}e^{i\theta}/\lambda, q^{1-m}e^{-i\theta}/\lambda;q)_\infty}{(q^{1-m}y/\lambda, q^{1-m}/(\lambda y);q)_\infty} \\ &\quad = \frac{(q^{1-m}e^{i\theta}/\lambda, q^{1-m}e^{-i\theta}/\lambda;q)_m (qe^{i\theta}/\lambda, qe^{-i\theta}/\lambda;q)_\infty}{(q^{1-m}y/\lambda, q^{1-m}/(\lambda y);q)_m (qy/\lambda, q/(\lambda y);q)_\infty} \\ &\quad = \frac{(\lambda e^{i\theta}, \lambda e^{-i\theta};q)_m (qe^{i\theta}/\lambda, qe^{-i\theta}/\lambda;q)_\infty}{(\lambda y, \lambda/y;q)_m (qy/\lambda, q/(\lambda y);q)_\infty}. \end{aligned}$$

We now give two generating functions which follow from Theorem 4.1B and one which follows from Theorem 4.1D.

(4.8)

$$\begin{aligned} \sum_{n=0}^\infty \frac{H_{n+k}(\cos\theta|q)}{(q;q)_{n+k}} t^n &= \frac{e^{ik\theta}}{(1-te^{i\theta})(e^{-2i\theta};q)_\infty} {}_1\phi_2\left(\begin{matrix} te^{i\theta} \\ qe^{2i\theta}, qte^{i\theta} \end{matrix} \,\middle|\, q, q^{k+2}e^{2i\theta} \right) \\ &\quad + \text{ a similar term with } \theta \text{ replaced by } -\theta. \end{aligned}$$

In fact one can get the more general result

$$(4.9) \qquad \sum_{n=0}^{\infty} \frac{H_{n+k}(\cos\theta|q)}{(q;q)_{n+k}} \frac{(\lambda;q)_n\, t^n}{(q;q)_n}$$
$$= \frac{(\lambda t e^{i\theta};q)_\infty\, e^{ik\theta}}{(te^{i\theta}, e^{-2i\theta};q)_\infty} {}_1\phi_2\left(\begin{matrix} te^{i\theta} \\ qe^{2i\theta}, \lambda t e^{i\theta} \end{matrix} \middle| q, q^{k+2}e^{2i\theta} \right)$$
$$+ \text{ a similar term with } \theta \text{ replaced by } -\theta.$$

Corollary 4.2 *A generating function for the q-Hermite polynomials is*

$$\sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q^2;q^2)_n} H_n(x|q^2) t^n$$
$$= \frac{(\lambda t e^{i\theta};q)_\infty}{(te^{i\theta};q)_\infty} {}_3\phi_2\left(\begin{matrix} \lambda, \sqrt{q}\, e^{i\theta}, -\sqrt{q}\, e^{i\theta} \\ \lambda t e^{i\theta}, -q \end{matrix} \middle| q, te^{-i\theta} \right).$$

Sketch of proof of Corollary 4.2. Multiply both sides of the equation in Theorem 4.1D by $(\lambda;q)_n t^n/(q;q)_n$, sum on n and use the q-binomial theorem. The right-hand side becomes a combination of two ${}_3\phi_2$'s with argument q. This can be transformed to a multiple of a ${}_3\phi_2$ using [10, (III.34)].

Observe that the ${}_3\phi_2$ in Corollary 4.2 is essentially bibasic on base q and q^2. If $\lambda = 0$ or $\lambda = -q$ the ${}_3\phi_2$ may be summed to infinite products, these are known results. Furthermore [10, (III.9)] shows that the right-hand side of Corollary 4.2 is a function of $\cos\theta$.

Corollary 4.3 *The q-exponential function $\mathcal{E}_q$ is essentially a ${}_2\phi_1$ function, that is*

$$\mathcal{E}_q(\cos\theta;t)$$
$$= \frac{(-t;q^{1/2})_\infty}{(qt^2;q^2)_\infty} {}_2\phi_1\left(\begin{matrix} q^{1/4}\, e^{i\theta}, q^{1/4}\, e^{-i\theta} \\ -q^{1/2} \end{matrix} \middle| q^{1/2}, -t \right)$$
$$= \frac{(t;q^{1/2})_\infty}{(qt^2;q^2)_\infty} {}_2\phi_1\left(\begin{matrix} -q^{1/4}\, e^{i\theta}, -q^{1/4}\, e^{-i\theta} \\ -q^{1/2} \end{matrix} \middle| q^{1/2}, t \right).$$

Consequently if either $0 \le t < 1$, $x \ge -(q^{1/4} + q^{-1/4})/2$, *or* $-1 < t \le 0$, $x \le (q^{1/4} + q^{-1/4})/2$, *then* $\mathcal{E}_q(x;t) > 0$.

Proof. In Corollary 4.2 replace t by $-t/\lambda$ then let $\lambda \to \infty$. The result is the generating function

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q^2;q^2)_n} H_n(\cos\theta|q^2)t^n = (-te^{i\theta};q)_\infty {}_2\phi_2\left(\begin{matrix} \sqrt{q}\, e^{i\theta}, -\sqrt{q}\, e^{i\theta} \\ -q, -te^{i\theta} \end{matrix} \,\middle|\, q, -te^{-i\theta} \right). \tag{4.10}$$

The transformation [10, (III.4)] reduces the above equation to

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q^2;q^2)_n} H_n(\cos\theta|q^2)t^n = (-tq^{-1/2};q)_\infty \, {}_2\phi_1\left(\begin{matrix} \sqrt{q}\, e^{i\theta}, \sqrt{q}\, e^{-i\theta} \\ -q \end{matrix} \,\middle|\, q, -tq^{-1/2} \right). \tag{4.11}$$

Now (1.12) and (4.10) imply the first of the representation. The second equality follows from the first and [10, (III.3)]. The statement about the zeros follows from the second equation in Corollary 4.3.

The two equations of Corollary 4.3 are q-analogue of the identities $e^{xt} = e^{\pm t}e^{\mp t(1+\mp x)}$.

At the end of this section we will come back to Corollaries 4.2 and 4.3 and give a direct proof of Corollary 4.2, which also proves Corollary 4.3. It is worth pointing out that Corollary 4.3 is an important result and yields some quadratic transformations, which will be the subject of a future work. In the same work we establish a Taylor series type expansion in the basis $\{(q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}; q^{1/2})_n\}, n = 0, 1 \ldots,$ and use the Taylor type expansion to study transformation formulas, expansions and identities for q-series.

Recall the Poisson kernel [4]

$$\sum_{n=0}^{\infty} H_n(\cos\theta|q)H_n(\cos\phi|q)\frac{t^n}{(q;q)_n} = \frac{(t^2;q)_\infty}{(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{-i(\theta-\phi)}, te^{-i(\theta+\phi)};q)_\infty}. \tag{4.12}$$

Using Theorem 4.1A we can derive a trilinear generating function for the continuous q-Hermite polynomials. If we replace t by ty in (4.12), then multiply by y^k, and then use Theorem 4.1A we find

(4.13)

$$
\begin{aligned}
&\sum_{n=0}^{\infty} H_{n+k}(\cos\psi|q)H_n(\cos\theta|q)H_n(\cos\phi|q)\frac{t^n}{(q;q)_n} \\
&= \frac{e^{ik\psi}\,(t^2e^{2i\psi};q)_\infty}{(e^{-2i\psi}, te^{i(\psi+\theta+\phi)}, te^{i(\psi+\theta-\phi)}, te^{i(\psi+\phi-\theta)}, te^{i(\psi-\theta-\phi)};q)_\infty} \\
&\times {}_6\phi_5\left(\left.\begin{matrix} te^{i(\psi+\theta+\phi)}, te^{i(\psi+\theta-\phi)}, te^{i(\psi+\phi-\theta)}, te^{i(\psi-\theta-\phi)}, 0, 0 \\ qe^{2i\psi}, te^{i\psi}, -te^{i\psi}, \sqrt{q}te^{i\psi}, -\sqrt{q}te^{i\psi}\end{matrix}\right| q, q^{k+1}e^{i\psi}\right) \\
&\qquad + \text{a similar term with } \psi \text{ replaced by } -\psi.
\end{aligned}
$$

It is clear that both sides of (4.13) are symmetric in θ and ϕ. When $k = 0$ the left-hand side is clearly symmetric in θ and ψ, but the form of the right-hand side does not make its symmetry obvious. This leads to the following theorem.

Theorem 4.4 *The expression*

$$
\begin{aligned}
&\frac{(t^2e^{2i\psi};q)_\infty}{(e^{-2i\psi}, te^{i(\theta+\phi+\psi)}, te^{i(\theta-\phi+\psi)}, te^{-i(\phi+\psi)}, te^{-i(\theta+\psi+\phi)};q)_\infty} \\
&\times {}_6\phi_5\left(\left.\begin{matrix} te^{i(\theta+\phi+\psi)}, te^{i(\theta+\phi-\psi)}, te^{i(\theta+\psi-\phi)}, te^{i(\theta-\psi-\phi)}, 0, 0 \\ qe^{2i\psi}, te^{i\psi}, -te^{i\psi}, \sqrt{q}te^{i\psi}, -\sqrt{q}te^{i\psi}\end{matrix}\right| q, qe^{i\psi}\right) \\
&\qquad + \textit{a similar term with } \psi \textit{ replaced by } -\psi.
\end{aligned}
$$

is symmetric under any permutation of θ, ϕ, *and* ψ.

Similarly using Theorem 4.1B and (4.12) we establish the following theorem.

Theorem 4.5 *We have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{H_{n+k}(\cos\psi|q)}{(q;q)_{n+k}(q;q)_n}\, H_n(\cos\theta|q)H_n(\cos\phi|q)\, t^n \\
&= \frac{e^{ik\psi}\,(t^2e^{2i\psi};q)_\infty}{(e^{-2i\psi}, te^{i(\psi+\theta+\phi)}, te^{i(\psi+\theta-\phi)}, te^{i(\psi+\phi-\theta)}, te^{i(\psi-\theta-\phi)};q)_\infty} \\
&\times {}_4\phi_5\left(\left.\begin{matrix} te^{i(\psi+\theta+\phi)}, te^{i(\psi+\theta-\phi)}, te^{i(\psi+\phi-\theta)}, te^{i(\psi-\theta-\phi)} \\ qe^{2i\psi}, te^{i\psi}, -te^{i\psi}, \sqrt{q}te^{i\psi}, -\sqrt{q}te^{i\psi}\end{matrix}\right| q, q^{k+2}e^{3i\psi}\right) \\
&\qquad + \textit{a similar term with } \psi \textit{ replaced by } -\psi.
\end{aligned}
$$

Furthermore when $k = 0$ *the right-hand side of the above equality is symmetric in* θ, ϕ, ψ.

The trilinear generating function (4.13) contains two important product formulas for the continuous q-Hermite polynomials which will be stated in the next theorem.

Theorem 4.6 *With $K(\cos\theta, \cos\phi, \cos\psi)$ denoting the right-hand side of* (4.13)*, we have the product formulas*

$$
\begin{aligned}
H_n(\cos\theta|q)H_n(\cos\phi|q) = \frac{(q;q)_\infty (q;q)_n}{2\pi t^n (q;q)_{n+k}} \int_0^\pi & K(\cos\theta, \cos\phi, \cos\psi) \\
& \times H_{n+k}(\cos\psi|q)(e^{2i\psi}, e^{-2i\psi};q)_\infty \, d\psi,
\end{aligned} \tag{4.14}
$$

and

$$
\begin{aligned}
H_n(\cos\theta|q)H_{n+k}(\cos\psi|q) = \frac{(q;q)_\infty}{2\pi t^n} \int_0^\pi & K(\cos\theta, \cos\phi, \cos\psi) \\
& \times H_n(\cos\phi|q)(e^{2i\phi}, e^{-2i\phi};q)_\infty \, d\phi.
\end{aligned} \tag{4.15}
$$

We now return to Corollary 4.2 and give a direct proof of it.

Proof of Corollary 4.2. Expand the ${}_3\phi_2$ on the right-hand side of Corollary 4.2 as a sum over k, say, then use the q-binomial theorem to expand $(\lambda q^k te^{i\theta};q)_\infty/(te^{i\theta};q)_\infty$ as a power series in t. Thus the coefficient of t^n on the right-hand side of Corollary 4.2 is

$$
(\lambda;q)_n \sum_{k=0}^n \frac{(qe^{2i\theta};q^2)_k}{(q^2;q^2)_k (q;q)_{n-k}} \, e^{i(n-2k)\theta} = (\lambda;q)_n S_n,
$$

say. Now the q-binomial theorem gives

$$
\sum_{n=0}^\infty S_n t^n = \frac{1}{(te^{i\theta};q)_\infty} \frac{(qte^{i\theta};q^2)_\infty}{(te^{-i\theta};q^2)_\infty}
$$

which is the generating function for $H_n(\cos\theta|q^2)/(q^2;q^2)_n$ and the result follows.

Observe that in the above proof we have established the representation

$$
\frac{H_n(\cos\theta|q^2)}{(q^2;q^2)_n} = \sum_{k=0}^n \frac{(qe^{2i\theta};q^2)_k}{(q^2;q^2)_k (q;q)_{n-k}} \, e^{i(n-2k)\theta}. \tag{4.16}
$$

Note that (4.1), (4.2), (2.3), and the initial conditions of $c_n(x;t_1,t_2)$ imply

$$
\begin{aligned}
c_n(\cos 2\theta; -1, -q|q^2) &= \frac{H_{2n}(\cos\theta|q)}{(q^2;q^2)_n}, \\
2 cos\theta \, c_n(\cos 2\theta; -q^2, -q|q^2) &= \frac{H_{2n+1}(\cos\theta|q)}{(q^2;q^2)_n}
\end{aligned}
$$

Thus Theorem 2.1 gives q-integral moment representations for the following functions:

$$\frac{H_{2n}(x|q)}{(q^2;q^2)_n}, \quad \frac{H_{2n+1}(x|q)}{(-q;q^2)_n}, \quad \frac{H_{2n+1}(x|q)}{(q^2;q^2)_n}, \quad \frac{H_{2n+1}(x|q)}{(-q^3;q^2)_n}.$$

One can also derive several generating functions involving $H_{2n}(x|q)$ and $H_{2n+1}(x|q)$ from the corresponding results in §2.

5 The Associated Continuous q-ultraspherical Polynomials

The associated continuous q-ultraspherical polynomials $\{C_n^{(\alpha)}(x;\beta|q)\}$ [7] satisfy the three term recurrence relation

$$\begin{aligned} 2x(1-\alpha\beta q^n)C_n^{(\alpha)}(x;\beta|q) &= (1-\alpha q^{n+1})C_{n+1}^{(\alpha)}(x;\beta|q) \\ &\quad + (1-\alpha\beta^2 q^{n-1})C_{n-1}^{(\alpha)}(x;\beta|q),\ n>0, \end{aligned} \tag{5.1}$$

and the initial conditions

$$C_0^{(\alpha)}(x;\beta|q) = 1, \quad C_1^{(\alpha)}(x;\beta|q) = \frac{2(1-\alpha\beta)}{(1-\alpha q)}x. \tag{5.2}$$

In this section we give the moment representation (5.10) for the associated continuous q-ultraspherical polynomials which leads to three new generating functions in Theorems 5.3, 5.4 and 5.5. In §5 we shall always write $x = \cos\theta$.

Here again we set

$$C_n^{(\alpha)}(x;\beta|q) = \int_a^b y^n f(y) d_q y$$

then find out that f satisfies

$$f(y) = \frac{q}{\alpha\beta^2} \frac{(1-qye^{i\theta})(1-qye^{-i\theta})}{(1-qye^{i\theta}/\beta)(1-qye^{-i\theta}/\beta)} f(qy).$$

This suggests that we consider the functions

$$\int_{e^{-i\theta}}^{e^{i\theta}} \frac{y^n}{1-q} \frac{(qye^{i\theta}, qye^{-i\theta}, \lambda y, q/(\lambda y);q)_\infty}{(\mu y, q/(\mu y), qye^{i\theta}/\beta, qye^{-i\theta}/\beta;q)_\infty} d_q y,$$

with

$$q\mu = \lambda\alpha\beta^2. \tag{5.3}$$

We choose $\lambda = qe^{i\theta}/\beta, \mu = \alpha\beta e^{i\theta}$ and consider the functions

$$(5.4) \quad \Phi_n(\theta;\beta,\alpha) = \int_{e^{-i\theta}}^{e^{i\theta}} \frac{y^n}{1-q} \frac{(qye^{i\theta}, qye^{-i\theta}, \beta e^{-i\theta}/y;q)_\infty}{(\alpha\beta e^{i\theta}y, qe^{-i\theta}/(\alpha\beta y), qye^{-i\theta}/\beta;q)_\infty} d_qy.$$

Theorem 5.1 *The functions $\Phi_n(\theta,\beta,\alpha)$ have the hypergeometric representation*

$$\begin{aligned}\Phi_n(\theta,\beta,\alpha) &= e^{i(n+1)\theta}\frac{(q,\alpha q^{n+1}, qe^{2i\theta}, e^{-2i\theta};q)_\infty}{(q/\beta, \alpha\beta q^n, \alpha\beta e^{2i\theta}, qe^{2i\theta}/(\alpha\beta);q)_\infty} \\ &\quad \times {}_2\phi_1\left(\begin{matrix} q^{-n}/\alpha, \beta \\ q^{1-n}/(\alpha\beta) \end{matrix} \,\middle|\, q, \frac{q}{\beta}e^{-2i\theta}\right), \quad n \geq 0.\end{aligned}$$

Proof. From the definition of q-integration we see that the right-hand side of (5.4) is

$$\begin{aligned} & e^{i(n+1)\theta}\frac{(q, qe^{2i\theta}, \beta e^{-2i\theta};q)_\infty}{(q/\beta, \alpha\beta e^{2i\theta}, qe^{-2i\theta}/(\alpha\beta);q)_\infty} {}_2\phi_1\left(\begin{matrix} q/\beta, qe^{2i\theta}/\beta \\ qe^{2i\theta} \end{matrix} \,\middle|\, q, \alpha\beta^2 q^n\right) \\ & -\frac{e^{-i(n+1)\theta}(q, qe^{-2i\theta}, \beta;q)_\infty}{(\alpha\beta, q/(\alpha\beta), qe^{-2i\theta}/\beta;q)_\infty} {}_2\phi_1\left(\begin{matrix} q/\beta, qe^{-2i\theta}/\beta \\ qe^{-2i\theta} \end{matrix} \,\middle|\, q, \alpha\beta^2 q^n\right) \\ &= \frac{e^{i(n+1)\theta}(q, qe^{2i\theta}, \beta e^{-2i\theta};q)_\infty}{(q/\beta, \alpha\beta e^{2i\theta}, qe^{-2i\theta}/(\alpha\beta);q)_\infty}\left[{}_2\phi_1\left(\begin{matrix} q/\beta, qe^{2i\theta}/\beta \\ qe^{2i\theta} \end{matrix} \,\middle|\, q, \alpha\beta^2 q^n\right)\right. \\ & -e^{-2i(n+1)\theta}\frac{(qe^{-2i\theta}, \beta, q/\beta, \alpha\beta e^{2i\theta}, qe^{-2i\theta}/(\alpha\beta);q)_\infty}{(qe^{2i\theta}, \beta e^{-2i\theta}, \alpha\beta, q/(\alpha\beta), qe^{-2i\theta}/\beta;q)_\infty} \\ & \left.\times {}_2\phi_1\left(\begin{matrix} q/\beta, qe^{-2i\theta}/\beta \\ qe^{-2i\theta} \end{matrix} \,\middle|\, q, \alpha\beta^2 q^n\right)\right].\end{aligned}$$

Apply (2.10) with $A = qe^{2i\theta}/\beta, B = q/\beta, C = qe^{2i\theta}, Z = \alpha\beta^2 q^n$ to complete the proof of Theorem 5.1.

Corollary 5.2 *The function $v_n(\theta;\beta,\alpha)$ defined by*

$$v_n(\theta;\beta,\alpha) = \frac{\Phi_n(\theta;\beta,\alpha)}{\Phi_0(\theta;\beta,\alpha)} = e^{in\theta}\frac{(\alpha\beta;q)_n}{(q\alpha;q)_n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}/\alpha, \beta \\ q^{1-n}/(\alpha\beta) \end{matrix} \,\middle|\, q, \frac{q}{\beta}e^{-2i\theta}\right)$$

satisfies the three term recurrence relation (5.1).

When $\alpha = 1$ the extreme right-hand side of Corollary 5.2 reduces to the q-ultraspherical polynomial $C_n(\cos\theta;\beta|q)$. For $\alpha \neq 1$ it may not be a polynomial but nevertheless is a solution to (5.1).

The solution of (5.1) given in Corollary 5.2 has a restricted β domain. We give two other solutions of (5.1) which hold for a wider domain of β. Unlike Φ_n constructing these two solutions will not require the application of transformations of basic hypergeometric series. However we will need to verify the three term recurrence relation for $n = 0$.

Let $y^n f(y, \theta)$ be the integrand in (5.4). Observe that the analysis preceding Theorem 5.1 indicates that both $\int_0^{e^{\pm i\theta}} y^n f(y, \theta) d_q y$, for $n > 0$ satisfy the recurrence (5.1). Define $v_n^{\pm}(\theta; \alpha, \beta)$ by

$$(5.5) \quad v_n^{\pm}(\theta; \alpha, \beta) := e^{\pm(n+1)i\theta} {}_2\phi_1\left(\begin{matrix} q/\beta, qe^{\pm 2i\theta}/\beta \\ qe^{\pm 2i\theta} \end{matrix} \middle| q, \alpha\beta^2 q^n \right).$$

This comes from the integral (5.4) on $[0, e^{\pm i\theta}]$. Both v_n^+ and v_n^- satisfy (5.1) for $n > 0$ and we will see later that are linearly independent functions of n for $\theta \neq k\pi, k = 0, \pm 1, \dots$.

We now verify that v_n^+ and v_n^- satisfy (5.1) if $n = 0$. To do so assume

$$(5.6) \qquad -1 < \alpha\beta^2/q < 1,$$

so that $v_{-1}^{\pm}$ is well-defined. We now go back and reexamine the analysis preceding Theorem 5.1. From (1.3) we see that when $a = 0$, the boundary term in (1.3) will vanish if $ug(u)f(u/q) \to 0$ as $u \to 0$ for u of the form ζq^m for fixed ζ and $m \to \infty$. In our case it suffices to prove that

$$\lim_{m\to\infty} (\beta e^{-i\theta} q^{-m}/\zeta; q)_\infty / (qe^{-i\theta} q^{-m}/(\zeta\alpha\beta); q)_\infty = 0.$$

The above limit is a bounded function times

$$\lim_{m\to\infty} \frac{(\beta e^{-i\theta} q^{-m}/\zeta; q)_m}{(qe^{-i\theta} q^{-m}/(\zeta\alpha\beta); q)_m} = \lim_{m\to\infty} (\alpha\beta^2/q)^m \frac{(q\zeta e^{i\theta}/\beta; q)_m}{(\alpha\beta\zeta e^{i\theta}/q; q)_m} = 0.$$

Note that (5.1), (5.2) and (5.6) imply $C_{-1}^{(\alpha)}(x; \beta|q) = 0$.

It is important to note that one can directly verify that $v_n^{\pm}$ satisfy (5.1) by substituting the right-hand side of (5.5) in (5.1) and equating coefficients of various powers of α. In fact this shows that $v_n^{\pm}$ satisfies (5.1) for all n for which $|\alpha\beta q^{n-1}| < 1$. To go beyond this restriction we need to analytically continue the ${}_2\phi_1$ in (5.5) using transformations of basic hypergeometric series, see Appendix III in [10], for example.

We now show that v_n^+ and v_n^- are linearly independent functions of n by computing the Casorati determinant

$$\Delta_n = v_{n+1}^+(\theta; \beta, \alpha) v_n^-(\theta; \beta, \alpha) - v_n^+(\theta; \beta, \alpha) v_{n+1}^-(\theta; \beta, \alpha).$$

Equation (1.10) implies

$$\Delta_n = \frac{(q\alpha\beta^2;q)_{n-1}}{(q^3\alpha;q)_{n-1}}\,\Delta_1,$$

and since $e^{\mp i(n+1)\theta}v_n^{\pm} \to 1$ as $n \to \infty$, then we have $\Delta_n \to 2i\sin\theta$ as $n \to \infty$. Hence

$$\Delta_n = \frac{(\alpha q^{n+2};q)_\infty}{(\alpha\beta^2 q^n;q)_\infty}\,2i\sin\theta. \tag{5.7}$$

This confirms the linear independence of $v_n^{\pm}$ when $\theta \neq k\pi$. Note also that

$$\Delta_{-1} = \frac{(\alpha q;q)_\infty}{(\alpha\beta^2/q;q)_\infty}\,2i\sin\theta. \tag{5.8}$$

Since both $v_n^{\pm}$ satisfy (5.1) then there exists $A(\theta)$ and $B(\theta)$ such that

$$C_n^{(\alpha)}(\cos\theta;\beta|q) = A(\theta)v_n^{+}(\theta;\beta,\alpha) + B(\theta)v_n^{-}(\theta;\beta,\alpha). \tag{5.9}$$

To determine A and B use the initial conditions

$$C_{-1}^{(\alpha)}(x;\beta|q) = 0, \quad C_0^{(\alpha)}(x;\beta|q) = 1$$

and (5.8). The result is

(5.10)

$$\begin{aligned} C_n^{(\alpha)}(\cos\theta;\beta|q) = & \frac{(\alpha\beta^2/q;q)_\infty}{2i\sin\theta(\alpha q;q)_\infty} \\ & \times \left[v_{-1}^{-}(\theta;\beta,\alpha)v_n^{+}(\theta;\beta,\alpha) - v_{-1}^{+}(\theta;\beta,\alpha)v_n^{-}(\theta;\alpha,\beta)\right]. \end{aligned}$$

Formula (5.10) is Rahman and Tariq's result [19, (3.4)]. They used (5.10) to derive linear and bilinear generating functions. In the reminder of this section, we shall apply (5.10) to derive only results not in Rahman and Tariq's paper [19].

Our first result is the following theorem.

Theorem 5.3 *We have*

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n} C_{n+k}^{(\alpha)}(\cos\theta;\beta|q)t^n \\ &= e^{ik\theta}\frac{(\lambda te^{i\theta},\alpha\beta^2/q;q)_\infty}{(1-e^{2i\theta})(te^{i\theta},\alpha q;q)_\infty}\,{}_2\phi_1\left(\left.\begin{matrix} q/\beta, qe^{-2i\theta}/\beta \\ qe^{-2i\theta} \end{matrix}\right| q, \frac{\alpha\beta^2}{q}\right) \\ &\quad \times {}_3\phi_2\left(\left.\begin{matrix} q/\beta, qe^{2i\theta}/\beta, te^{i\theta} \\ qe^{2i\theta}, \lambda te^{i\theta} \end{matrix}\right| q, \alpha\beta^2 q^k\right) \\ &\qquad + \textit{ a similar term with } \theta \textit{ replaced by } -\theta. \end{aligned}$$

The cases $\lambda = q$ or $k = 0$ of Theorem 5.3 are in [19].

Theorem 5.4 *We have the bilinear generating function*

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_n(\cos\phi;\beta_1|q) C_n^{(\alpha)}(\cos\theta;\beta|q) t^n \\
&\quad = \frac{(\alpha\beta^2/q, \beta_1 t e^{i(\theta+\phi)}, \beta_1 t e^{i(\theta-\phi)};q)_\infty}{(1-e^{-2i\theta})(\alpha q, te^{i(\theta+\phi)}, te^{i(\theta-\phi)};q)_\infty} \\
&\quad \times {}_2\phi_1\left(\begin{matrix} q/\beta, qe^{-2i\theta}/\beta \\ qe^{-2i\theta}\end{matrix} \,\middle|\, q, \frac{\alpha\beta^2}{q}\right) \\
&\quad \times {}_4\phi_3\left(\begin{matrix} q/\beta, qe^{2i\theta}/\beta, te^{i(\theta+\phi)}, te^{i(\theta-\phi)} \\ qe^{2i\theta}, \beta_1 te^{i(\theta+\phi)}, \beta_1 te^{i(\theta-\phi)}\end{matrix} \,\middle|\, q, \alpha\beta^2\right) \\
&\qquad + \text{ a similar term with } \theta \text{ replaced by } -\theta.
\end{aligned}$$

Proof. Multiply (5.10) by $C_n(\cos\phi;\beta_1|q)t^n$ and add then use the generating function (4.16).

The associated continuous q-ultraspherical polynomials have the generating function [7]

$$\sum_{n=0}^{\infty} C_n^{(\alpha)}(\cos\theta;\beta|q)t^n = \frac{1-\alpha}{(1-te^{i\theta})(1-te^{-i\theta})} {}_3\phi_2\left(\begin{matrix} \beta te^{i\theta}, \beta te^{-i\theta}, q \\ qte^{i\theta}, qte^{-i\theta}\end{matrix} \,\middle|\, q, \alpha\right). \tag{5.11}$$

We now give a Poisson-type kernel for the polynomials under consideration.

Theorem 5.5 *A bilinear generating function for the associated continuous q-ultraspherical polynomials is given by*

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_n^{(\alpha_1)}(\cos\phi;\beta_1|q) C_n^{(\alpha)}(\cos\theta;\beta|q) t^n \\
&\quad = \frac{(1-\alpha_1)(\alpha\beta^2/q;q)_\infty}{(1-e^{-2i\theta})(\alpha q;q)_\infty} \, {}_2\phi_1\left(\begin{matrix} q/\beta, qe^{-2i\theta}/\beta \\ qe^{-2i\theta}\end{matrix} \,\middle|\, q, \frac{\alpha\beta^2}{q}\right) \\
&\quad \times \sum_{k=0}^{\infty} \frac{(q/\beta, qe^{2i\theta}/\beta;q)_k \alpha^k \beta^{2k}}{[1-2\cos\phi\, te^{i\theta}q^k + t^2q^{2k}e^{2i\theta}]\,(q, qe^{2i\theta};q)_k} \\
&\quad \times {}_3\phi_2\left(\begin{matrix} q, \beta_1 tq^k e^{i(\theta+\phi)}, \beta_1 tq^k e^{i(\theta-\phi)} \\ q^{k+1}te^{i(\theta+\phi)}, q^{k+1}te^{i(\theta-\phi)}\end{matrix} \,\middle|\, q, \alpha_1\right) \\
&\qquad + \text{ a similar term with } \theta \text{ replaced by } -\theta.
\end{aligned}$$

The case $\alpha = \alpha_1$ of (5.16) is in [19].

Theorem 2.1 gave two moment representations for the Al-Salam-Chihara polynomials. We can also do the same for the associated continuous q-ultraspherical polynomials. Namely if

$$\hat{p}_n(x) = \frac{(\alpha q; q)_n}{(\alpha\beta^2; q)_n} C_n^{(\alpha)}(x; \beta|q)$$

then $\hat{p}_n(x)$ also satisfies a three term recurrence relation whose coefficients are also polynomials in q^n. Thus the technique of this paper applies. However we have

$$C_n^{(\alpha\beta^2/q)}(x; q/\beta|q) = \frac{(\alpha q; q)_n}{(\alpha\beta^2; q)_n} C_n^{(\alpha)}(x; \beta|q). \tag{5.12}$$

So the renormalized moment representations amount to changing the α and β in the associated continuous q-ultraspherical polynomials.

6 The Associated Al-Salam-Chihara Polynomials

These polynomials were first considered in [5] where their generating functions, asymptotics, and their weight function were found. In this section we carry out our program on these polynomials.

The associated Al-Salam-Chihara polynomials $p_n^{(\alpha)}(x; t_1, t_2)$ are generated by

$$p_0^{(\alpha)}(x; t_1, t_2) = 1, \quad p_1^{(\alpha)}(x; t_1, t_2) = \frac{t_1[2x - (t_1 + t_2)\alpha]}{1 - \alpha t_1 t_2}, \tag{6.1}$$

and

(6.2)

$$t_1[2x - (t_1 + t_2)\alpha q^n] p_n^{(\alpha)}(x; t_1, t_2) = [1 - t_1 t_2 \alpha q^n] p_{n+1}^{(\alpha)}(x; t_1, t_2) + t_1^2(1 - \alpha q^n) p_{n-1}^{(\alpha)}(x; t_1, t_2), \quad n > 0,$$

as can be seen from (2.2) and (2.3). Now assume $\int_0^b y^n f(y) d_q y$ is a solution to (6.2). Then (1.3) yields

$$f(y) = \frac{t_1^2 - 2qxt_1 y + q^2 y^2}{\alpha[t_1^2 - t_1(t_1 + t_2)y + t_1 t_2 y^2]} f(qy),$$

which gives

$$f(y) = \frac{(qye^{i\theta}/t_1, qye^{-i\theta}/t_1, \lambda y, q/(\lambda y); q)_\infty}{(y, yt_2/t_1, \alpha\lambda y, q/(\alpha\lambda y); q)_\infty}.$$

This leads us to take $b = t_1 e^{\pm i\theta}$ and to introduce the functions

(6.3)
$$A_n^{\pm}(\theta, t_1, t_2) = t_1^{n+1} e^{\pm i(n+1)\theta} {}_2\phi_1\left(\begin{matrix} t_1 e^{\pm i\theta}, t_2 e^{\pm i\theta} \\ q e^{\pm 2i\theta} \end{matrix} \,\middle|\, q, \alpha q^{n+1} \right), \quad n \geq -1,$$

for $|\alpha| < 1$. We need the assumption $|\alpha| < 1$ in order for $A_{-1}^{\pm}$ to be defined by (6.3). We are also assuming $\alpha \neq 0$. We proceed as before, first verify that $A_n^{\pm}$ satisfies (6.2) for all $n \geq 0$ then compute the Casorati determinant. The only difference here is that $t_1^{-2n-2}\Delta_n \to 2t_1 i \sin\theta$ as $n \to \infty$. We find

$$\Delta_n = 2it_1^{2n+3} \sin\theta \, \frac{(\alpha t_1 t_2 q^{n+1}; q)_\infty}{(\alpha q^{n+1}; q)_\infty}. \tag{6.4}$$

The condition $|\alpha| < 1$, enables us to conclude that $p_{-1}^{(\alpha)} = 0$ if (6.2) is extended to hold for $n = 0$. Thus $p_n^{(\alpha)} = [A_{-1}^{-} A_n^{+} - A_{-1}^{+} A_n^{-}]/\Delta_{-1}$, that is

$$\begin{aligned} p_n^{(\alpha)}(\cos\theta; t_1, t_2) = & \frac{(\alpha; q)_\infty \, t_1^n \, e^{in\theta}}{(1 - e^{-2i\theta})(\alpha t_1 t_2; q)_\infty} \, {}_2\phi_1\left(\begin{matrix} t_1 e^{-i\theta}, t_2 e^{-i\theta} \\ q e^{-2i\theta} \end{matrix} \,\middle|\, q, \alpha \right) \\ & \times {}_2\phi_1\left(\begin{matrix} t_1 e^{i\theta}, t_2 e^{i\theta} \\ q e^{2i\theta} \end{matrix} \,\middle|\, q, \alpha q^{n+1} \right) \\ & + \text{ a similar term with } \theta \text{ replaced by } -\theta. \end{aligned}$$

An immediate consequence of (6.5) is

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(q; q)_n} p_n^{(\alpha)}(\cos\theta; t_1, t_2) t^n \\ & = \frac{(\alpha, \lambda t_1 t e^{i\theta}; q)_\infty}{(1 - e^{-2i\theta})(\alpha t_1 t_2, t_1 t e^{i\theta}; q)_\infty} \, {}_2\phi_1\left(\begin{matrix} t_1 e^{-i\theta}, t_2 e^{-i\theta} \\ q e^{-2i\theta} \end{matrix} \,\middle|\, q, \alpha \right) \\ & \times {}_3\phi_2\left(\begin{matrix} t_1 e^{i\theta}, t_2 e^{i\theta}, t_1 t e^{i\theta} \\ q e^{2i\theta}, \lambda t_1 t e^{i\theta} \end{matrix} \,\middle|\, q, \alpha q \right) \\ & + \text{ a similar term with } \theta \text{ replaced by } -\theta. \end{aligned} \tag{6.5}$$

Another application of (6.5) is to combine it with (2.17) and find

$$(6.6)\quad \sum_{n=0}^{\infty} \frac{(t_3t_4;q)_n t^n}{t_1^n t_3^n (q;q)_n}\, p_n^{(\alpha)}(\cos\theta;t_1,t_2)\, p_n(\cos\phi;t_3,t_4)$$

$$= \frac{(\alpha, t_3te^{i\theta}, t_4te^{i\theta};q)_\infty}{(1-e^{-2i\theta})(\alpha t_1t_2, te^{i(\theta+\phi)}, te^{i(\theta-\phi)};q)_\infty}\, {}_2\phi_1\left(\begin{matrix} t_1e^{-i\theta}, t_2e^{-i\theta} \\ qe^{-2i\theta}\end{matrix}\,\middle|\, q,\alpha\right)$$

$$\times\, {}_4\phi_3\left(\begin{matrix} t_1e^{i\theta}, t_2e^{i\theta}, te^{i(\theta+\phi)}, te^{i(\theta-\phi)} \\ qe^{2i\theta}, t_3te^{i\theta}, t_4te^{i\theta}\end{matrix}\,\middle|\, q,\alpha q\right)$$

$$+\ \text{a similar term with } \theta \text{ replaced by } -\theta.$$

The limiting case $\alpha \to 1^-$ of (6.7) is the result stated as Theorem 4.1 in our paper [13].

A companion representation for the associated Al-Salam-Chihara polynomials may also be found. Similar to §5 it follows from

$$(6.7)\quad p_n^{(\alpha t_1t_2/q)}(x;q/t_1,q/t_2) = \left(\frac{q}{t_1^2}\right)^n \frac{(\alpha t_1t_2;q)_n}{(\alpha q;q)_n}\, p_n^{(\alpha)}(x;t_1,t_2).$$

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