LATTICE PATHS AND POSITIVE TRIGONOMETRIC SUMS

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Abstract. A trigonometric polynomial generalization to the positivity of an alternating sum of binomial coefficients is given. The proof uses lattice paths, and identifies the trigonometric sum as a polynomial with positive integer coefficients. Some special cases of the q-analog conjectured by Bressoud are established, and new conjectures are given.

1. Introduction.

Andrews et al [3] proved the signed sum of binomial coefficients

$$g(M,N,K,i)=\sum_{\mu}\binom{M+N}{M-K\mu}-\binom{M+N}{M-K\mu+i} \tag{1.1}$$

is non-negative if M, N, K and i are positive integers satisfying

$$-i\le M-N\le K-i, \quad 0\le i\le K/2.$$

They proved (1.1) is the number of partitions inside an $M\times N$ rectangle, satisfying certain inequalities involving K and i.

A special case of (1.1) is $i=K/2=k$,

$$\sum_{l}\binom{M+N}{M-kl}(-1)^l\ge 0 \text{ if } |M-N|\le k. \tag{1.2}$$

In this paper we generalize (1.2) in several directions. The first generalization is the following.

Theorem 1. *If $|M-N|\le k$, then*

$$\sum_{l}\binom{M+N}{M-kl}cos(lx)\ge 0$$

for any real x.

In fact we shall prove a stronger statement, that the left side of the inequality in Theorem 1 is a polynomial in $1+cos(x)$, with non-negative coefficients. A combinatorial interpretation for the coefficients, which are integers, is given in Theorem 2.

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The case $M = N$, $k = 1$ of Theorem 1 is the de la Vallée Poussin sum [4],

$$(1.3) \qquad \sum_l \binom{2N}{N-l} cos(lx) = 2^N (1 + cos(x))^N.$$

In [3] a q-analog of (1.1) is given. For $i = K/2 = k$ the polynomial in q becomes,

$$(1.4) \qquad B(M,N,k,a,b) = \sum_l \begin{bmatrix} M+N \\ M-lk \end{bmatrix}_q (-1)^l q^{(l^2k(a+b)+lk(b-a))/2}$$

If M, N, k, a, b are non-negative integers with $a+b < 2k$, $b-k \le N-M \le k-a$, then $B(M,N,k,a,b)$ is a polynomial in q with non-negative coefficients [3]. Bressoud [5] conjectured that a and b may be rational.

Conjecture (Bressoud [5]**).** *If M, N, k, ak, and bk are positive integers such that $1 < a+b < 2k-1$, $b-k \le N-M \le k-a$, then $B(M,N,k,a,b)$ is a polynomial in q with non-negative coefficients.*

In §5-6 we verify (see Theorems 4 and 5) special cases of the Bressoud conjecture. Our proof is rather unusual: the non-negativity of (1.1) (the $q = 1$ case) implies the non-negativity in the q case. For a few cases we show that the monotonicity in k follows from Stenger's theorem on quadrature.

Bressoud's conjecture was motivated by the Borwein conjecture [2]: let

$$\prod_{j=0}^{n-1} (1-q^{1+3j})(1-q^{2+3j}) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3)$$

for some polynomials $A_n(q)$, $B_n(q)$, and $C_n(q)$, in q. Then all coefficients of $A_n(q)$, $B_n(q)$, and $C_n(q)$ are non-negative. We conjecture a cosine version of Bressoud's conjecture in Conjecture 1. It implies that if

$$\prod_{j=0}^{n-1} (1+q^{1+3j}e^{ix})(1+q^{2+3j}e^{-ix}) = A_n(q^3,x) - qe^{ix}B_n(q^3,x) - q^2e^{-ix}C_n(q^3,x),$$

the real part of the polynomials $A_n(q,x)$, $B_n(q,x)$, and $C_n(q,x)$ is non-negative as a polynomial in q and $1 + cos(x)$. If $x = \pi$, Conjecture 1 implies Borwein's conjecture.

We shall follow the notation and terminology in [7], [8].

2. Proof of Theorems 1 and 2.

In this section we prove Theorem 1 combinatorially, the precise results are given in Theorem 2. First we review a combinatorial proof for (1.2), which is well-known [10, p. 6], [11, p. 12]. After finishing the proof of Theorem 2 we give an equivalent restatement in Corollary 1.

We will use lattice paths P in the plane. All lattice paths pass through integer points in the plane, and consist of unit steps of two types: in the north and east directions. Two lattice paths are called *disjoint* if they have no steps in common. (They are allowed to intersect at a point.) We shall at times cut a given lattice path into smaller disjoint paths $P = (P_1, P_2, \cdots, P_s)$, by cutting P at a set of integer points of P.

Proposition 1. *The number of lattice paths from* $(0,0)$ *to* (M,N) *which do not intersect the lines* $y = x \pm k$, $|M-N| \le k$ *is given by (1.2).*

Proof. The total number of lattice paths P from $(0,0)$ to (M,N) is $\binom{M+N}{M}$, the $l=0$ term in (1.2). More generally, let $Path_l$ be the set of lattice paths from $(lk,-lk)$ to (M,N). If we weight each path in $Path_l$ by $(-1)^l$, then the sum of the weights of all paths in $Path = \cup_{l=-\infty}^{\infty} Path_l$ is given by (1.2).

Next we define a sign-reversing involution on $Path$, whose fixed points are the paths in $Path_0$ which do not intersect the lines $y = x \pm k$. Since all terms in $Path_0$ have weight $+1$, the number of these paths is given by (1.2).

For the involution, note that all paths in $Path - Path_0$ must intersect a line of the form $y = x + (2j-1)k$ for some j, since $|M-N| \le k$, as do some paths in $Path_0$. Given such a path, we find the first such intersection, and reflect the initial segment of the path in this line. Under this map, a path beginning at $(lk,-lk)$ would intersect $y = x+(2l-1)k$ or $y = x+(2l+1)k$. The reflected path begins at $((l-1)k, -(l-1)k)$, $((l+1)k, (l+1)k)$ respectively, and has the same intersection point. Thus the sign is reversed. □

For Theorems 1 and 2, we consider the same model. Any path P from $(0,0)$ to (M,N) is a union of disjoint "Catalan" paths $P = (P_1, P_2, \cdots, P_{s+1})$, obtained by cutting P at its intersections with the line $y = x$. The paths P_i, $1 \le i \le s$ intersect the line $y = x$ at only the initial and final points P_i, while the path P_{s+1} begins on the line $y = x$ and terminates at (M,N). If p of the paths P_i intersect the lines $y = x \pm k$, we say P has *class* p. Note that P can have class 0 if P also lies inside the lines $y = x \pm k$.

Theorem 2. *If* $|M-N| \le k$, *then*

$$\sum_l \binom{M+N}{M-kl} cos(lx) = \sum_p a_p (1+cos(x))^p$$

where a_p *is the number of lattice paths from* $(0,0)$ *to* (M,N) *of class* p *which do not intersect the lines* $y = x \pm 2k$.

For example, if $M = N$, $k = 1$ in Theorem 2, any path P must satisfy $P = (P_1, \cdots, P_N)$, where each P_i has 2 steps. So the class of P is always N, and there are 2^N such P, giving (1.3).

Proof. Clearly $a_0 \ge 0$ is given by Proposition 1, so we assume that $p > 0$.

By expanding the Chebyshev polynomial $T_l(cos(x)) = cos(lx)$ in terms of $(1 + cos(x))$, [7], we find that the coefficient of $(1+cos(x))^p$ in Theorem 2 is

$$a_p = \sum_l \binom{M+N}{M-kl} \frac{(-l)_p (l)_p}{p!(1/2)_p} \frac{(-1)^l}{2^p} =$$

$$\sum_l \binom{M+N}{M-kl} (\binom{|l|+p}{|l|-p} + \binom{|l|+p-1}{|l|-p-1}) 2^{p-1} (-1)^{l-p}. \tag{2.1}$$

Fix $p > 0$. We consider the same set of paths $Path$, but weight each path in $Path_l$ by

$$w(l,p) = (\binom{|l|+p}{|l|-p} + \binom{|l|+p-1}{|l|-p-1}) 2^{p-1} (-1)^{|l|-p}.$$

Since the weight is 0 for $|l| < p$, we may assume $|l| \geq p > 0$. We will collect many of these paths (with weights) together to give 0. The remaining paths (with weights) all will have $|l| = p$, and are thus positive. We then give a bijection from this multiset of paths to the paths stated in Theorem 2.

Clearly by symmetry we can take $N \geq M$. The paths $Q \in \cup_{l \geq p} Path_{-l}$ all begin in the second quadrant and end at (M, N). Any $Q \in Path_{-l}$ uniquely defines a sequence of integers $(a_1, a_2, \cdots, a_s)$, where $a_1 = 2l$, $|a_i - a_{i+1}| = 1$, $1 \leq i \leq s-1$, $a_s = 0$ or 1, defined by finding the lines $y = x + a_i k$ that Q successively intersects. We also decompose $Q = (Q_1, \cdots, Q_s, Q_{s+1})$ by cutting Q at the location of the successive initial intersection points with the lines $y = x + a_i k$. Moreover there can be a "tail" path Q_{s+1}, which always exists if $M \neq N$.

If the sequence $(a_1, a_2, \cdots, a_s)$ is strictly decreasing, we say Q has *no violations*. The sequence $(a_1, a_2, \cdots, a_s)$ for paths $Q \in Path_l$ are defined analogously using the lines $y = x - a_i k$.

We first consider the case of $N = M$. Suppose that $Q \in Path_{-l}$ has no violations so that $a_i = 2l + 1 - i$, $s = 2l$, and Q_i is the path starting on the line $y = x + (2l + 1 - i)k$ and ending on the line $y = x + (2l - i)k$. If we choose any v of the Q_i, and interchange all of their edges, we obtain a path $\tilde{Q}$ from $(-(l-v)k, (l-v)k)$ to (M, M) with v violations. We consider all such paths $\tilde{Q}$ obtained from a fixed Q for any $0 \leq v \leq l - p$. All such $\tilde{Q}$ begin in the second quadrant since $l - v \geq p > 0$. The total weight obtained for these paths is

$$\sum_{v=0}^{l-p} \binom{2l}{v} w(l-v, p). \tag{2.2(a)}$$

It is easy to show from the ${}_2F_1(1)$ evaluation [7] that this sum is zero if $l > p$.

Since any path with violations can be obtained from a path with no violations, we may consider only paths $Q \in Path_{-p}$, with no violations.

For paths starting in the fourth quadrant, $\cup_{l \geq p} Path_l$, an analogous argument applies. We can reflect the previous argument in the line $y = x$. The remaining paths whose weights do not sum to 0 are those paths $Q \in Path_p$ with no violations.

Since $w(\pm p, p) = 2^{p-1}$, each non-violating path in $Path_p \cup Path_{-p}$ must be counted 2^{p-1} times. We now give a bijection between the paths of Theorem 2, and this multiset of paths. Suppose that $P = (P_1, \cdots, P_s)$ has class p. If the last path which intersects the lines $y = x \pm k$ intersects $y = x + k$, we map P to $Path_{-p}$, otherwise $Path_p$. By flipping the other $p - 1$ paths P_i which intersect the lines $y = x \pm k$ across the diagonal $y = x$, we obtain 2^{p-1} paths of class p, all of which are mapped to the same path in $Path_p \cup Path_{-p}$. To obtain a path P from a non-violating path $Q = (Q_1, \cdots, Q_{2p}) \in Path_{-p}$, switch all edges of the odd paths $Q_{2i+1} \to \hat{Q}_{2i+1}$, $P = (\hat{Q}_1, Q_2, \cdots, \hat{Q}_{2p-1}, Q_{2p})$. For P, the $p - 1$ paths P_i intersect the line $y = x + k$. An analogous argument works for $Q = (Q_1, \cdots, Q_{2p}) \in Path_p$, which are mapped to paths P whose last intersection with the lines $y = x \pm k$ is with the line $y = x - k$. This completes the proof of the $M = N$ case.

For $M + k \geq N > M$, non-violating paths $Q \in Path_{-l}$ must have either $s = 2l$ or $s = 2l - 1$, and non-violating paths $Q \in Path_l$ have $s = 2l$ or $s = 2l + 1$. Again we choose any v of these subpaths and switch all edges to find violating paths. The cases $s = 2l$ sum to zero as before, leaving only the cases $s = 2l - 1, Q \in Path_{-l}$ and $s = 2l + 1, Q \in Path_l$. The appropriate sum of weights is given by (2.2) with $2l - 1$ and $2l + 1$ replacing $2l$ in the binomial coefficient,

$$\sum_{v=0}^{l-p} \binom{2l-1}{v} w(l-v, p), \tag{2.2(b)}$$

$$\sum_{v=0}^{l-p} \binom{2l+1}{v} w(l-v, p). \tag{2.2(c)}$$

This time the ${}_2F_1(1)$ evaluation implies that (2.2)(b),(c) are respectively

$$-2^{p-1} \frac{(l-p)_{l-p-1}}{(l-p)!}$$

and

$$2^{p-1} \frac{(l-p+1)_{l-p}}{(l-p+1)!},$$

so that the sum for $Path_{-l-1}$ cancels that for $Path_l$. Thus the only remaining paths are the

(1) non-violators in $Path_{-p}$, and
(2) the non-violators in $Path_p$ which have $s = 2l$, so do not intersect $y = x + k$.

Note that if l is maximized, ($l = [M/k]$), paths in $Path_l$ have $s = 2l + 1$ only when $Path_{-l-1}$ has paths with $s = 2l + 1$, so this boundary term also is cancelled.

Finally we use the same bijection between the paths of Theorem 2 and the remaining multiset of non-violating paths. Again each path $Q \in Path_p \cup Path_{-p}$ has weight 2^{p-1}. For $Q = (Q_1, \cdots, Q_{2p}) \in Path_{-p}$, switch edges in all odd paths to obtain $\tilde{Q}$ from $(0, 0)$ to (M, N), whose last intersection with the lines $y = x \pm k$ is with the line $y = x + k$. The remaining $p - 1$ Catalan parts of $\tilde{Q}$ intersecting $y = x + k$ lie above the line $y = x$, flipping them about $y = x$ gives the multiplicity 2^{p-1}. The same idea on the multiset of paths $Q \in Path_p$ which do not intersect $y = x + k$ gives all paths of class p from $(0, 0)$ to (M, N) whose last intersection with the lines $y = x \pm k$ is with the line $y = x - k$. $\square$

If we let

$$f(M, N, k, x) = \sum_l \binom{M+N}{M-lk} cos(lx), \tag{2.3}$$

the Pascal triangle relation for the binomial coefficients implies that

$$f(M, N, k, x) = f(M-1, N, k, x) + f(M, N-1, k, x). \tag{2.4}$$

Thus another approach to Theorem 1 is to verify non-negativity for $f(0, N, k, x), 0 \leq N \leq k$, $f(M, 0, k, x), 0 \leq M \leq k$, and $f(M, M \pm k, k, x)$. The first two cases are trivial, while the last case is not. For $k \in \{1,2,3\}$ there are explicit formulas for the last case which verify Theorem 1,

$$f(M+1,M,1,x) = 2^M(1+cos(x))^{M+1}, \tag{2.5}$$

$$f(M+2,M,2,x) = \sum_{l\geq 0} \binom{M+1}{2l+1} 2^{M-l}((1+cos(x))^{l+1}, \tag{2.6}$$

$$f(M+3,M,3,x) = \sum_{l\geq 0} \binom{M-l}{2l} \frac{2M+3}{2l+1} 2^l 3^{M-1-3l}(1+cos(x))^{l+1}. \tag{2.7}$$

Equations (2.6) and (2.7) can be proven directly from Theorem 2. From (2.3) they represent a quadratic ${}_2F_1$ and a cubic ${}_3F_2$ transformation. The $k=2,3$, $M=N$ versions are

$$f(M,M,2,x) = \sum_{l\geq 0} \binom{M}{2l} 2^{M-l}(1+cos(x))^l.$$

$$f(M,M,3,x) = \sum_{l\geq 0} \binom{M-l}{2l} \frac{M}{M-l} 2^{l+1} 3^{M-1-3l}(1+cos(x))^l.$$

In [3], it is shown that $f(M,N,k,\pi)$ is the number of partitions which lie inside an $M \times N$ rectangle, whose hook differences are $\geq 2-k$ and $\leq k-2$. For example,

$$f(N,N,2,\pi) = 2^N,$$

because there are 2^N self-conjugate partitions inside an $N \times N$ rectangle. From Theorem 2 we have

$$f(M,N,2k,\pi) = f(M,N,k,\pi/2).$$

We can therefore reinterpret the class p of a path P as some statistic on the partitions whose hook differences lie between $2-2k$ and $2k-2$. Such a statistic is given in [9].

3. Extensions of Theorem 2.

In this section we give Theorem 3, which generalizes Theorem 2 to arbitrary polynomials. It is applied to Jacobi polynomials to obtain a sine version of Theorem 1 in Corollary 3.

In the proof of Theorem 2, the non-negativity of the coefficient of $(1+cos(x))^p = z^p$ for $p>0$ follows from the non-negativity of (2.2)(a)(b)(c). Thus the proof of Theorem 2 applies to weights $w(l,p)$ besides the T-Chebyshev weight.

Theorem 3. *Suppose $p_l(z) = \sum_{p=0}^{l} w(l,p)z^p$ is a polynomial in z of degree at most l. If $|M-N| \leq k$, and (2.2)(a)(b)(c) are non-negative, then*

$$\sum_l \binom{M+N}{M-kl} p_{|l|}(z) = \sum_{p\geq 0} a_p z^p,$$

where $a_p \geq 0$ *for* $p > 0$.

For example, if

$$p_l(z) = \frac{(\alpha+\beta+1)_l}{(\beta+1)_l} P_l^{(\alpha,\beta)}(z-1)$$

is a Jacobi polynomial [7], then the ${}_2F_1(1)$ evaluation implies

$$\begin{aligned} (2.2)(a) &= C(l,p)(l-p-\alpha-\beta)_{l-p}, \\ (2.2)(b) &= C(l,p)(l-p-\alpha-\beta-1)_{l-p}, \\ (2.2)(c) &= C(l,p)(l-p-\alpha-\beta+1)_{l-p}, \end{aligned} \tag{3.1}$$

where

$$C(l,p) = \frac{(l+\alpha+\beta+1)_p(\alpha+\beta+1)_l}{(l-p)!p!(\beta+1)_p 2^p (1+\alpha+\beta+2p)_{l-p}}.$$

Clearly if $-1 \leq \alpha+\beta \leq 1$, and $-1 < \beta$, we have non-negativity in (3.1).

Corollary 2. *If* $|M-N| \leq k$, *and* $-1 \leq \alpha+\beta \leq 1$, $-1 < \beta$, *then*

$$\sum_l \binom{M+N}{M-kl} \frac{(\alpha+\beta+1)_{|l|}}{(\beta+1)_{|l|}} P_{|l|}^{(\alpha,\beta)}(z-1) = \sum_{p\geq 0} a_p z^p,$$

where $a_p \geq 0$ *for* $p > 0$.

Note that if $\alpha = \beta$ in Corollary 2, the Jacobi polynomials are normalized to be the Gegenbauer polynomials. The constant terms a_0 are not always positive. For example if $\alpha = \beta$, $M = N = 3$, $k = 2$ then

$$a_0 = 8(1-3\alpha), \quad a_1 = 12(1+2\alpha).$$

One may ask if a sine version of Theorem 1 holds. Clearly Theorem 2 implies that

$$\sum_l \binom{M+N}{M-kl} \frac{l\, sin(lx)}{sin(x)} \geq 0$$

for any real x. Another version is given in Corollary 3.

Theorem 3 may be applied to

$$p_l(z) = \begin{cases} 0 & \text{if } l = 0, \\ \frac{(\alpha+\beta+1)_{l-1}}{(\beta+1)_{l-1}} P_{l-1}^{(\alpha,\beta)}(z-1) & \text{if } l > 0. \end{cases} \tag{3.2}$$

The argument of Theorem 2 also proves the constant term is non-negative in this case. What prevents the argument from always showing $a_0 \geq 0$ is that if $v = l$, a single path $\tilde{Q}$ starting at the origin may be obtained by flipping two different paths, from $Q \in Path_l$ and $Q \in Path_{-l}$. Thus $\tilde{Q}$ is not obtained exactly once. However, if (3.2) applies, then all paths $\tilde{Q}$ starting at the origin have weight 0, and they can be safely counted twice.

To verify that Theorem 3 may be applied, we again use the ${}_2F_1(1)$ evaluation to find

$$\begin{aligned} (2.2)(a) &= C(l-1,p)(l-p-\alpha-\beta+1)_{l-p-1}, \\ (2.2)(b) &= C(l-1,p)(l-p-\alpha-\beta)_{l-p-1}, \\ (2.2)(c) &= C(l-1,p)(l-p-\alpha-\beta+2)_{l-p-1} \end{aligned}$$

so that non-negativity holds if $-1 \leq \alpha+\beta \leq 2$, and $-1 < \beta$. The special case $\alpha = \beta = 1/2$ gives the U-Chebyshev polynomials, thus the next corollary.

Corollary 3. *If $|M-N| \le k$, then*

$$\sum_l \binom{M+N}{M-kl} \frac{sin(|l|x)}{sin(x)} \ge 0$$

for any real x.

4. Discrete Chebyshev polynomials.

For the proof of the special cases of Bressoud's conjecture, we need some facts about roots of unity. These facts are properties of the Chebyshev polynomials

$$T_n(cos((2j+1)\pi/2k)) = cos(n(2j+1)\pi/2k) = cos(n\theta_{j,k}),$$

where

$$\theta_{j,k} = (2j+1)\pi/2k.$$

In this section we prove technical Lemma 1, which is necessary for Theorem 5. We also explain how these results are related to Stenger's theorem on quadrature.

Let $d\alpha(x) = (1-x^2)^{-1/2}dx$ on $[-1,1]$, the T-Chebyshev measure. For a polynomial $p(x)$, let

$$I(p) = \int_{-1}^{1} p(x) d\alpha(x).$$

The Gaussian quadrature approximation [6] to $I(p)$ on k points is

$$I_k(p) = \frac{1}{k}\sum_{j=0}^{k-1} p(cos(\theta_{j,k})).$$

It is clear from the binomial theorem that $f(M,N,k,x)$ may be written as a sum over the kth roots of unity. The result is

$$f(M,N,k,x) = \frac{1}{k}\sum_{j=0}^{k-1} cos((M-N)(x+2\pi j)/2k)(2cos((x+2\pi j)/2k))^{M+N} \tag{4.1}$$

Certainly (4.1) implies that for real x, $f(M,M,k,x) \ge 0$. We also see that

$$f(M,N,k,\pi) = 2^{M+N} I_k(x^{M+N} T_{M-N}(x)). \tag{4.2}$$

Theorem 2 (or [3]) implies that $f(M,N,k,\pi)$ is an increasing function of k for $|M-N| \le k$. This result for $M=N$ follows immediately from (4.2) and Stenger's theorem.

Theorem (Stenger [13]**).** *Suppose that $d\alpha(x) = d\alpha(-x)$ is a probability measure on a finite interval $[-a,a]$ having finite moments of all orders. If $p(x) = \sum_j a_{2j}x^{2j}$ is a polynomial with $a_{2j} \ge 0$, then the Gaussian quadrature approximation on k points*

$$I_k(p) = \sum_{j=0}^{k-1} w_{k,j} p(x_{k,j})$$

to $I(p)$ is a monotonically increasing function of k.

The discrete orthogonality relations for T-Chebyshev polynomials are

$$I_k(T_nT_l) = \begin{cases} 1/2 & \text{if } n \equiv \pm l \pmod{4k}, \\ -1/2 & \text{if } n \equiv \pm(2k-l) \pmod{4k}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Also recall [7] that the T-Chebyshev polynomials satisfy

$$2^l x^l = \sum_{s=0}^{l} \binom{l}{s} T_{l-2s}(x). \tag{4.4}$$

Since $T_{2N}(x) = T_N(T_2(x)) = T_N(2x^2-1)$, (4.4) implies

$$2^l (2x^2-1)^l = \sum_{s=0}^{l} \binom{l}{s} T_{2l-4s}(x). \tag{4.5}$$

We use (4.3) and (4.5) for a positivity lemma.

Lemma 1. *If N, l, k are positive integers such that $N \le k$, and N is even, then*

$$L(N,k,l) = I_k(xT_{N-1}(x)2^l(2x^2-1)^l) \ge 0.$$

Moreover $L(N,k,l)$ is a monotonically increasing function of k.

Proof. First we consider

$$I_k(T_N(x)2^l(2x^2-1)^l) = \sum_{s=0}^{l} \binom{l}{s} I_k(T_N(x)T_{2l-4s}(x)).$$

From (4.3) we have

$$I_k(T_N(x)2^l(2x^2-1)^l) = \sum_t \binom{l}{(2l-N)/4+tk} - \binom{l}{(2l-N-2k)/4+tk}. \tag{4.6}$$

We interpret any non-integer binomial coefficient in (4.6) as 0.

For Lemma 1, we have two terms of the type (4.6), since $2xT_{N-1} = T_N(x) + T_{N-2}(x)$. There are four cases, depending upon the mod 4 values of $2l-N$ and $2k$. The results are (see (1.1))

$$L(N,k,l) = \begin{cases} g(\lceil (2l-N)/4 \rceil, \lfloor (2l+N)/4 \rfloor, k, k/2) \text{ if } k \text{ is even,} \\ g(\lceil (2l-N)/4 \rceil, \lfloor (2l+N)/4 \rfloor, k, (k-1)/2) \text{ if } k \text{ is odd.} \end{cases}$$

Since $N \le k$, each of the four cases is non-negative. The monotonicity follows from the monotonicity in k of (1.1), which is given in [3]. □

5. Analytic proofs for $M = N$.

In this section we give elementary analytic proofs of the Bressoud conjecture for special values of a and b if $M = N$.

Two possible q-analogs of (4.1) (for $M = N$) obtained from kth roots of unity are

$$\sum_l \begin{bmatrix} 2M \\ M-kl \end{bmatrix}_q q^{k^2l^2/2} cos(lx) =$$

$$\frac{1}{k}\sum_{j=0}^{k-1}\prod_{p=0}^{M-1}((1-q^{1/2+p})^2+4q^{1/2+p}cos^2((x+2\pi j)/2k)), \tag{5.1}$$

and

$$\sum_l \begin{bmatrix} 2M \\ M-kl \end{bmatrix}_q q^{\binom{kl}{2}} cos(lx) =$$

$$\frac{1+q^M}{2k}\sum_{j=0}^{k-1}\prod_{p=0}^{M-1}((1-q^{p})^2+4q^{p}cos^2((x+2\pi j)/2k)), \tag{5.2}$$

Clearly the right sides of both (5.1) and (5.2) are non-negative, as real numbers for $q > 0$ and any real x. But we can also use (5.1) and (5.2) to verify an extension of a special case of the Bressoud conjecture.

Theorem 4. *The Bressoud conjecture holds if $k = 2K$ is even,*

(1) $M = N$, $a = (k+1)/2 = b+1$,
(2) $M = N$, $a = b = k/2$.

Moreover in these cases, if K increases, the coefficients weakly increase.

Proof. Put $x = \pi$ in (5.2) so that $M = N$, $a = (k+1)/2 = b+1$, and consider

$$S = \frac{1+q^M}{2k}\sum_{j=0}^{k-1}\prod_{p=0}^{M-1}((1+q^{2p}+2q^p(2cos^2(\frac{(2j+1)\pi}{2k})-1)), \tag{5.3}$$

Then

$$S = \sum_{l=0}^{M} c_l(q) I_k((2x^2-1)^l),$$

for some polynomials $c_l(q)$ with non-negative coefficients. We have

$$I_k((2x^2-1)^l) = \begin{cases} I_K(x^l) > 0 \text{ if } l \text{ is even,} \\ 0 \qquad \text{if } l \text{ is odd,} \end{cases}$$

so S is a polynomial in q with non-negative coefficients. The monotonicity in K follows from Stenger's theorem.

The second case is done in the same way, using (5.1). □

We can use (5.3) to find explicit products verifying Bressoud's conjecture for $k = 1, 2, 3$. We also give the analogous results for $M \neq N$.

Proposition 2. *We have*

(1) $B(M,M,1,1,0)=0$,

(2) $B(M+1,M,2,1,1)=B(M,M,2,1,1)=\prod_{i=1}^{M}(1+q^{2i-1})$,

(3) $B(M,M,2,3/2,1/2)=(1+q^M)\prod_{i=1}^{M-1}(1+q^{2i})$,

(4) $B(M+1,M,2,3/2,1/2)=\prod_{i=1}^{M}(1+q^{2i})$,

(5) $B(M,M,3,2,1)=(1+q^M)\prod_{i=1}^{M-1}(1+q^i+q^{2i})$,

(6) $B(M+1,M,3,2,1)=B(M+2,M,3,2,1)=\prod_{i=1}^{M}(1+q^i+q^{2i})$.

6. The $N\neq M$-case.

In this section we apply the roots of unity technique of §4 to the q-case if $M\neq N$. It develops that we need Lemma 1, which followed from the $q=1$ case, to prove this q-case.

We let

$$B(M,N,k,a,b,x)=\sum_l \begin{bmatrix} M+N \\ M-kl \end{bmatrix}_q q^{(l^2k(a+b)+lk(b-a))/2}cos(lx), \tag{6.1}$$

so that $B(M,N,k,a,b,\pi)=B(M,N,k,a,b)$.

This time the q-binomial theorem implies

$$B(M,N,k,a,k-a,x)=$$

(6.2)

$$Re(\frac{1}{k}\sum_{j=0}^{k-1}\prod_{p=0}^{M-1}(1+q^{p-(2a-1-k)/2}e^{(-x-2\pi j)i/k})\prod_{p=1}^{N}(1+q^{p+(2a-1-k)/2}e^{(x+2\pi j)i/k})).$$

The idea is to suitably specialize the remaining parameters to obtain the square of an absolute value.

Theorem 5. *The Bressoud conjecture holds if $M+N$ is even, $a+b=k$, and $k-2a=N-M\mp 1$.*

Proof. Take $k-2a=N-M-1$, $a+b=k$, in this case (6.2) with $x=\pi$ is

$$\begin{aligned} B=&B(M,N,k,a,b,\pi)=\frac{1}{k}q^{-\binom{(N-M)/2}{2}}\sum_{j=0}^{k-1}2cos((2j+1)\pi/2k)\\ &(cos((2j+1)(N-M-1)\pi/2k)+q^{\frac{M+N}{2}}cos((2j+1)(N-M+1)\pi/2k))\\ &\prod_{p=1}^{(M+N)/2-1}(1+q^{2p}+2q^pcos((2j+1)\pi/k))\end{aligned}$$

After expanding the inner product as a polynomial in $cos((2j+1)\pi/k)$, we have

$$B=\sum_{l=0}^{(M+N)/2-1}a_l(q)L(N-M,k,l)+b_l(q)L(M-N,k,l),$$

for some polynomials $a_l(q)$ and $b_l(q)$ with non-negative coefficients. Lemma 1 then may be applied because $N-M$ is even, $|N-M|\leq k$.

The case $k-2a=N-M+1$ is done similarly. □

7. Conjectures and remarks.

The Borwein conjecture [5] follows from the $k=3$ case of Bressoud's conjecture. For general k, we state the generalized Borwein conjecture. It is easy to prove if $n=\infty$ from the Jacobi Triple product identity.

Conjecture 1. *Let a and k be relatively prime positive integers, $1 \le a \le k/2$, k odd, and put*

$$\prod_{i=0}^{n-1}(1-q^{a+ik})(1-q^{k-a+ik}) = \sum_{j\ge 0} b_j q^j.$$

The sign of b_j is determined by $j \pmod{k}$. If $j \equiv \pm(2l+1)a$ for some l, $0 \le l < k/2$, then $b_j \le 0$, otherwise $b_j \ge 0$.

It appears that the q-analog of Theorem 1 holds for $k \ge 3$.

Conjecture 2. *If M, N, k, ak, and bk are positive integers such that $1 < a+b < 2k-1$, $b-k \le N-M \le k-a$, $3 \le k$, then $B(M,N,k,a,b,x)$ is a polynomial in $1+cos(x)$ and q with non-negative coefficients.*

If x is real, we need only assume $k \ge 2$.

Conjecture 3. *If M, N, k, ak, and bk are positive integers such that $1 < a+b < 2k-1$, $b-k \le N-M \le k-a$, $2 \le k$, then $B(M,N,k,a,b,x)$ is a polynomial in q with non-negative coefficients.*

Conjecture 2 is related to the following generalization of the Borwein conjecture. If

$$\prod_{j=0}^{n-1}(1-q^{1+3j}e^{ix})(1-q^{2+3j}e^{-ix}) = A_n(q^3,x) - qB_n(q^3,x) - q^2C_n(q^3,x)$$

then the real part of the coefficients of the polynomials in q $A_n(q,x)$ is non-negative.

In [9] a different q-analog of Theorem 1 is discussed. The T-Chebyshev polynomial $T_l(z-1)$ is replaced by a q-version $T_l(z-1,q)$. Several q-enumerations are given there, see also [12].

Finally, note that a heuristic for $f(M,N,k,\pi) \ge 0$ is that the largest binomial coefficient should dominate, and it is positive. If we consider instead

$$\tilde{A}(M,N,k) = \sum_l \binom{M+N}{M-kl}^{-1}(-1)^l,$$

the larger of the two tail terms should dominate, since the binomial coefficient is the smallest there. The next Proposition verifies this idea.

Proposition 3. *Let $M = kC_1 + r_1$, $N = kC_2 + r_2$, where $0 \le r_1, r_2 < k$.*

(1) *If $C_1 - C_2$ is even, then*

$$0 < (-1)^{C_1}\tilde{A}(M,N,k) < \binom{M+N}{r_1}^{-1} + \binom{M+N}{r_2}^{-1}$$

(2) *If $C_1 - C_2$ is odd, then for $r_1 = r_2$, $\tilde{A}(M,N,k) = 0$, otherwise for $r_1 < r_2$,*

$$(-1)^{C_1}\tilde{A}(M,N,k) > \binom{M+N}{r_1}^{-1} - \binom{M+N}{r_2}^{-1} > 0.$$

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