ORTHOGONAL POLYNOMIALS AND COMBINATORICS

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Abstract. An introduction is given to the use orthogonal polynomials in distance regular graphs and enumeration. Some examples in each area are given, along with open problems.

1. Introduction. There are many ways that orthogonal polynomials can occur in combinatorics. This paper concentrates on two topics

(1) eigenvalues of distance regular graphs,
(2) generating functions for combinatorial objects.

The introduction to these two topics is well-known to the experts, and is intended for a general audience. Rather than give an exhaustive survey, specific applications which highlight the techniques of each subject are given. For (1), the application is Wilson's proof of the Erdös-Ko-Rado theorem (see §3). For (2), two applications will be given- Foata's proof of the Mehler formula for Hermite polynomials and the Kim-Zeng linearization theorem for Sheffer orthogonal polynomials. Some open problems are given along the way.

Active research topics which are not discussed include group and algebra representations, tableaux and symmetric functions.

The standard references for distance regular graphs and association schemes are [6] and [10], while [26] covers topics (1) and (2). The standard notation for hypergeometric series found in [23] is used here.

2. Association schemes and distance regular graphs. In classical coding theory the Krawtchouk polynomials are important for several problems. This was generalized to polynomials in P-polynomial association schemes by Delsarte [14], and in this section we review this basic setup.

Recall that a graph $G=(V,E)$ consists of a set V of vertices, and a set E of unordered pairs of elements of V, called the edges of G. If the graph G is very regular in a precise sense then associated to G is a finite sequence of orthogonal polynomials.

The adjacency matrix A of the graph G is the $|V|\times|V|$ matrix defined by (the rows and columns are indexed by the vertices of G)

$$A(x,y)=\begin{cases}1 & \text{if } xy \text{ is an edge,}\\ 0 & \text{otherwise.}\end{cases}$$

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We imagine the adjacency matrix as keeping track of which pairs of vertices are distance one from each other, where the distance between two vertices is the length of the shortest path in the graph connecting them. This distance is finite as long as the graph is connected.

The adjacency matrix can be generalized to distance i,

$$A_i(x,y)=\begin{cases}1 & \text{if } d(x,y)=i,\\ 0 & \text{otherwise.}\end{cases}$$

We see that $A_0=I$, $A_1=A$, and if the graph is connected with maximum distance d, then $A_0+A_1+\cdots+A_d=J$, the all ones matrix.

We want a condition on the graph G which mimics the three-term recurrence relation

$$xp_i(x)=\alpha_i p_{i+1}(x)+\beta_i p_(x)+\gamma_i p_{i-1}(x)$$

which is satisfied by any set of orthogonal polynomials. Suppose that $x,y\in V$, with $d(x,y)=i$. If one moves distance one away from y to a vertex z, then we have $d(x,z)=i-1,i$ or $i+1$.

The condition we impose on the connected graph G with maximum distance d is distance regularity, namely the matrices A_i follow the same rule:

$$A_iA_1=c_{i+1}A_{i+1}+a_iA_i+b_{i-1}A_{i-1},\quad 0\le i\le d, \tag{2.1}$$

for some constants c_{i+1}, a_i and b_{i-1}. If such constants exist, then they can be found by explicit counting. For example, if we fix $x,z\in V$, with $d(x,z)=i+1$, c_{i+1} is the number of $y\in V$ such that $d(x,y)=i, d(y,z)=1$. Since c_{i+1} is independent of the choice of x,z with $d(x,z)=i+1$, the graph G must be regular in a very special way, thus the term distance regular.

We now give two examples of distance regular graphs. The first is the N-dimensional cube, which is denoted $H(N,2)$, and is referred to as the binary Hamming scheme. The vertices of $H(N,2)$ are all N-tuples of 0's and 1's, and an edge connects two N-tuples which differ in exactly one position. The distance between two vertices is the number of positions in which they differ. In this case

$$c_{i+1}=i+1,\quad a_i=0,\quad b_{i-1}=N-i+1.$$

For example to compute c_{i+1}, fix $x=(0,0,\cdots,0)$, and $z=(1,\cdots,1,0,\cdots,0)$ where z has $i+1$ 1's so that $d(x,z)=i+1$. If $d(y,z)=1$ and $d(x,y)=i$, then y must be obtained by switching one of the 1's in z to a 0, there are $i+1$ choices for this switch.

The second example, $J(n,k)$, the Johnson scheme, will be considered in detail in the next section. The vertices V of $J(n,k)$ consist of all k-element subsets of the fixed n-element set $\{1,2,\cdots,n\}=[n]$, and two subsets $\alpha,\beta\in V$ are connected by edge if $|\alpha\cap\beta|=k-1$. In this case the distance is given by $d(\alpha,\beta)=k-|\alpha\cap\beta|$, and the constants are

$$c_{i+1}=(i+1)^2,\quad a_i=i(n-k-i)+(k-i)i,\quad b_{i-1}=(k-i+1)(n-i+1).$$

This time to see that $c_{i+1}=(i+1)^2$, fix k-subsets x and z with $|x\cap z|=k-(i+1)$. We must count how many y are distance one from z and distance i from x. To create y, we just delete a point of z from $z-x$, and add a point to z from $x-z$. This can be done in $(i+1)^2$ ways. Note also that $|V|=\binom{n}{k}$.

These two examples have large symmetry groups- permutations of the vertices which preserve the distance in the graph- which ensure distance regularity.

It is clear from (2.1) that $A_i=p_i(A_1)$ is a polynomial of degree i in A_1, where $p_i(x)$ is an orthogonal polynomial, as long as $c_{i+1}\neq 0$ for $i+1\le d$. This is the case if the graph is connected. By considering the eigenvalues of A_1 we will have a real valued polynomial instead of a matrix valued polynomial $p_i(A_1)$. The matrix A_1 is a real symmetric matrix, so A_1 has a complete set of real eigenvalues. In fact there are $d+1$ distinct eigenvalues, $\lambda_0>\lambda_1>\cdots>\lambda_d$, with corresponding eigenspaces $V_0, V_1, \cdots, V_d$. Since A_i is a polynomial in A_1, the eigenvalue of A_i on V_j is $p_i(\lambda_j)$.

Note that the $d+1$ eigenvalues are solutions to the $d+1$ degree polynomial equation (the $i=d$ case of (2.1))

$$\lambda p_d(\lambda)=a_d p_d(\lambda)+b_{d-1}p_{d-1}(\lambda).$$

A discrete orthogonality relation can be found using the eigenspaces. Let E_j be the projection map onto the eigenspace V_j. Then we have

$$A_i=\sum_{j=0}^{d} p_i(\lambda_j)E_j.$$

If this relation is inverted, and the orthogonality of the projection maps E_j is used, the following orthogonality relations are obtained. We let v_i denote the size of a sphere of radius i.

$$\frac{1}{|V|}\sum_{n=0}^{d} p_n(\lambda_i)p_n(\lambda_j)/v_n=\delta_{ij}/dim(V_i), \tag{2.2a}$$

$$\frac{1}{|V|}\sum_{i=0}^{d} p_n(\lambda_i)p_m(\lambda_i)dim(V_i)=\delta_{nm}v_n. \tag{2.2b}$$

It is clear that (2.2b) is a discrete orthogonality relation for the finite set of orthogonal polynomials $\{p_n(x)\}_{n=0}^d$, while (2.2a) may or may not be a polynomial orthogonality relation.

In our two examples,

$$\begin{aligned}
d=&N, \quad \text{and } \lambda_j=N-2j \quad \text{for } H(N,2),\\
d=&min(k,n-k), \quad \text{and } \lambda_j=k(n-k)+j(j-n-1) \quad \text{for } J(n,k),\\
v_i=&\binom{N}{i}, \quad dim(V_i)=\binom{N}{i} \quad \text{for } H(N,2),\\
v_i=&\binom{k}{i}\binom{n-k}{i}, \quad dim(V_i)=\binom{n}{i}-\binom{n}{i-1} \quad \text{for } J(n,k).
\end{aligned}$$

The polynomials are Krawtchouk polynomials ($p=1/2$) and dual Hahn polynomials

$$p_i(\lambda_j)=\binom{N}{i}\,{}_2F_1\left(\begin{matrix}-i, & -j; & 2\\ & -N & \end{matrix}\right) \tag{2.3a}$$

$$p_i(\lambda_j)=\binom{k}{i}\binom{n-k}{i}\,{}_3F_2\left(\begin{matrix}-j, & j-n-1, & -i; & 1\\ & -k, & k-n & \end{matrix}\right). \tag{2.3b}$$

If we put $\hat{p}_i(x) = p_i(x)/v_i$, the orthogonality relation (2.2a) becomes

$$\frac{1}{|V|}\sum_{n=0}^{d} \hat{p}_n(\lambda_i)\hat{p}_n(\lambda_j)v_n = \delta_{ij}/dim(V_i). \tag{2.4}$$

We see that in our two examples, $\hat{p}_i(\lambda_j)$ is a polynomial in i of degree j, so that (2.4) is also a polynomial orthogonality relation. If $\hat{p}_i(\lambda_j) = q_j(\lambda_i^*)$ for some polynomial sequence $q_j(x)$, the distance regular graph is called Q-polynomial.

The classification of distance regular graphs which are Q-polynomial is a very difficult problem. There are several infinite families (see [6], [10]) with arbitrarily large d (including $H(N, 2)$ and $J(n, k)$). These families are closely related to Gelfand pairs (W, W_J) and (G, P_J), for Weyl groups W and maximal parabolic subgroups W_J, and for classical groups G over a finite field and parabolic subgroups P_J. Note that q appears here as a prime power, the order of a finite field. There is a classification theorem for orthogonal polynomials due to Leonard [6, Chap. 3], which says that any set of orthogonal polynomials whose duals (in the sense above) are orthogonal polynomials, must be special or limiting cases of the Askey-Wilson ${}_4\phi_3$ polynomials. Terwilliger [40] has a linear algebraic version of this theorem which hopefully will lead to an algebraic classification of the Q-polynomial distance regular graphs.

A simple application of the graph structure to the polynomials can be given by considering the multiplication

$$A_iA_j = \sum_k c_{ij}^k A_k. \tag{2.5a}$$

We know that c_{ij}^k is the number of y such that $d(x, y) = i$, $d(y, z) = j$, if x and z are fixed with $d(x, z) = k$. If each side is applied to the eigenspace V_s, we find the linearization formula

$$p_i(\lambda_s)p_j(\lambda_s) = \sum_k c_{ij}^k p_k(\lambda_s). \tag{2.5b}$$

We see from (2.5b) that the linearization coefficients are always non-negative and have a combinatorial interpretation. In example one for the Krawtchouk polynomials we have

$$c_{ij}^k = \binom{k}{(k+j-i)/2}\binom{n-k}{n-(k+i+j)/2}.$$

3. The Erdös-Ko-Rado theorem. There are many applications of orthogonal polynomials to coding theory, see for example [15], [26]. In this section a less widely known application to extremal set theory, the Erdös-Ko-Rado theorem, will be given. It uses the eigenvalues of the $J(n, k)$ which are dual Hahn polynomials (2.3b). This proof is due to Wilson [43], with somewhat different details.

We consider the set of all k-element subsets of $[n] = \{1, 2, \cdots, n\}$, this is $J(n, k)$. We assume that $2k \le n$, and look for subsets $\mathcal{F} \subset J(n, k)$ such that for all $A, B \in \mathcal{F}$, $|A \cap B| \ge t$. Such sets $\mathcal{F}$ are called t-intersecting.

The Erdös-Ko-Rado theorem states that

$$|\mathcal{F}| \le \binom{n-t}{k-t} \tag{3.1}$$

as long as $n \geq n_0(k,t)$, for $n_0(k,t)$ which depends only upon k and t. Frankl [22], and then Wilson [43] found an explicit bound $n_0(k,t) = (k-t+1)(t+1)$. It is clear that the set $\mathcal{F}$ is realizable by taking all k-subsets which contain a fixed t-element subset.

If $n < n_0(k,t)$, the bound (3.1) is not correct. For values of n in this range the correct bound was conjectured by Frankl [22] and proven by Ahlswede and Khachatrian [1]. It is based upon the family of subsets

$$\mathcal{F}_i = \{A \in X_k : |A \cap [t+2i]| \geq t+i\}$$

which clearly has the t-intersecting condition.

Theorem A [1]. *If $\mathcal{F}$ is a t-intersecting family, then*

$$|\mathcal{F}| \leq max\{|\mathcal{F}_i| : 0 \leq i \leq (n-t)/2\}.$$

If $n \geq (k-t+1)(t+1)$, the maximum occurs at $i=0$. We shall give Wilson's proof of this case. We use the fact that the dual Hahn polynomials are eigenvalues of the Johnson association scheme $J(n,k)$.

Wilson's method (from Delsarte [14]) is to find an explicit matrix $\mathcal{A}$ in the span of $\{I, A, \cdots, A_d\}$ for $J(n,k)$ such that

(1) $\mathcal{A}_{\alpha\beta} = 0 \quad if \quad |\alpha \cap \beta| \geq t,$

(2) $B = I + \mathcal{A} - J/\binom{n-t}{k-t}$ is a positive semidefinite matrix.

To see that the two conditions above suffice for the proof, let $I_{\mathcal{F}}$ be the characteristic vector of the family $\mathcal{F}$. Then

$$0 \leq (I_{\mathcal{F}}, BI_{\mathcal{F}}) = |\mathcal{F}| - |\mathcal{F}|^2 / \binom{n-t}{k-t}$$

which proves that

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

The inequality $n \geq (k-t+1)(t+1)$ will be needed to prove that the matrix B is positive semidefinite.

To define the element $\mathcal{A}$ and thus B, we take an appropriate linear combination of the basis A_i, $0 \leq i \leq k$. Condition (1) says we should restrict to $k-t+1 \leq i \leq k$. It is somewhat more convenient to use the basis B_j defined by

$$B_j = \sum_{s=j}^{k} \binom{s}{j} A_s. \tag{3.2}$$

Since we know the eigenvalues of each A_s as dual Hahn polynomials, we know the eigenvalues of the B_j's. However even more is true, these eigenvalues factor,

$$B_j(V_e) = (-1)^e \binom{k-e}{j-e} \binom{n-j-e}{k-e} V_e. \tag{3.3}$$

To see (3.3), just use the ${}_3F_2$ form of the $p_j(\lambda_e)$ given in (2.3b). The resulting double sum has two sums, each is which is evaluable from the Chu-Vandermonde ${}_2F_1(1)$ evaluation.

The element $\mathcal{A}$ is defined by

$$\mathcal{A} = \sum_{i=0}^{t-1} (-1)^{t-1-i} \binom{k-1-i}{k-t} \binom{n-k-t+i}{k-t}^{-1} B_{k-i}. \tag{3.4}$$

It is clear that the non-zero matrix elements $\mathcal{A}_{\alpha\beta}$ must have $|\alpha \cap \beta| < t$, so that the first requirement for $\mathcal{A}$ is satisfied.

To prove that $B = I + \mathcal{A} - J/\binom{n-t}{k-t}$ is positive semidefinite, we check that each eigenvalue of B is non-negative. We know the eigenvalues of $\mathcal{A}$, while the eigenvalue of J is $\binom{n}{k}$ on V_0, and is 0 on V_e, $e > 0$. Thus we must show that the eigenvalue θ_e of $\mathcal{A}$ on V_e satisfies

$$\theta_e \geq -1 \quad \textit{for } 1 \leq e \leq k, \tag{3.5a}$$

$$\theta_0 \geq \binom{n}{k} \binom{n-t}{k-t}^{-1} - 1. \tag{3.5b}$$

Proof of (3.5) In fact much more is true, we have equality in (3.5a) for $1 \leq e \leq t$, and equality in (3.5b). We only use the inequality $n \geq (k-t+1)(t+1)$ to establish the inequality in (3.5a) for θ_e when $e > t$.

From (3.3) we have

$$\begin{aligned} \theta_e =& (-1)^{t-1-e} \sum_{i=0}^{t-1} (-1)^i \binom{k-1-i}{k-t} \binom{k-e}{i} \binom{n-k-e+i}{k-e} \binom{n-k-t+i}{k-t}^{-1} \\ =& \frac{(-1)^e}{(t-1)!} \sum_{i=0}^{t-1} (-1)^{t-1-i} \binom{t-1}{i} (k-e-i+1)_{e-1} (n-t-k+i+1)_{t-e}. \end{aligned} \tag{3.6}$$

We can interpret (3.6) as a $t-1$st difference $(\Delta f(x) := f(x+1) - f(x))$

$$\theta_e = \frac{(-1)^e}{(t-1)!} \Delta^{t-1} f(0), \quad f(x) = (k-e-x+1)_{e-1}(n-t-k+x+1)_{t-e}.$$

To prove (3.5a) for $1 \leq e \leq t$, we need only find the leading term of the polynomial $f(x)$, $(-1)^{e-1}x^{t-1}$, to see that $\theta_e = -1$.

For (3.5b), if $e = 0$,

$$f(x) = -x^{t-1} + \text{ lower order terms } + \frac{(n+1-t)_t}{k-x},$$

so that

$$\Delta^{t-1}\left(\frac{1}{k-x}\right) = \frac{(t-1)!}{(k-x-t+1)_t}$$

implies

$$\theta_0 = -1 + \frac{(n+1-t)_t}{(k-t+1)_t} = -1 + \frac{\binom{n}{k}}{\binom{n-t}{k-t}}$$

as required.

We also need the inequalities of $\theta_e \geq -1$ for $e > t$ in (3.5a). For example if we take $e = t+1$, then clearly

$$(-1)^{t+1} f(x) = -x^{t-1} + \text{ lower order terms } + \frac{(-1)^{t+1}(n-2t)_{t-1}}{n-t-k+x},$$

which again implies that θ_{t+1} is a sum of two terms just as θ_0 was

$$\theta_{t+1} = -1 + \frac{(n-2t)_t}{(n-k-t)_t} > -1$$

since $n - 2t > 0$ and $n - k - t > 0$.

If $e = t+2$,

$$\begin{aligned}(-1)^{t+2} f(x) = &- x^{t-1} + \text{ lower order terms} \\ &- \frac{(-1)^{t+2}(n-2t-1)_{t+1}}{n-t-k+x} + \frac{(-1)^{t+2}(n-2t-2)_{t+1}}{n-t-k+x-1},\end{aligned}$$

which implies that θ_{t+2} is a sum of three terms

$$\theta_{t+2} = -1 + \frac{(n-2t-1)_{t+1}}{(n-k-t)_t} - \frac{(n-2t-2)_{t+1}}{(n-k-t-1)_t}.$$

Thus

$$\begin{aligned}\theta_{t+2} &\geq -1 \quad \text{if } n \geq (k-t+1)(t+1), \\ \theta_{t+2} &= -1 \quad \text{if } n = (k-t+1)(t+1).\end{aligned}$$

The remaining inequality $\theta_e > -1$ for $e > t+2$, can be established using a ${}_3F_2(1)$ transformation

$$ {}_3F_2\left(\begin{matrix} a, & b, & -n; & 1 \\ & c, & 1-d-n & \end{matrix}\right) = {}_3F_2\left(\begin{matrix} a, & c-b, & -n; & 1 \\ & c, & a+d & \end{matrix}\right). \tag{3.7}$$

One may write θ_e as a terminating ${}_3F_2$,

$$\begin{aligned}\theta_e = &(-1)^{t-1-e}\binom{k-1}{k-t}\binom{n-k-e}{k-e}\binom{n-k-t}{k-t}^{-1} \\ &\times {}_3F_2\left(\begin{matrix} 1-t, & e-k, & n-k-e+1; & 1 \\ & 1-k, & n-k-t+1 & \end{matrix}\right) \quad \text{if } e \geq t+2.\end{aligned} \tag{3.8}$$

Then applying (3.7) with $a = e-k$, $b = n-k-e+1$, $c = n-k-t+1$, $d = k-t+1$, we find

$$\begin{aligned}\theta_e = &(-1)^{t-1-e}\frac{\binom{n-k-e}{k-e}\binom{e-1}{t-1}}{\binom{n-k-t}{k-t}}\, {}_3F_2\left(\begin{matrix} 1-t, & e-k, & e-t; & 1 \\ & e-t+1, & n-k-t+1 & \end{matrix}\right) \\ = &\frac{(e-1)!(k-t)!}{(t-1)!(e-t)!(k-e)!}\sum_{s=0}^{t-1}\frac{(1-t)_s(e-k)_s}{s!(n-k-e+1)_{e-t+s}}\frac{e-t}{e-t+s}.\end{aligned} \tag{3.9}$$

Since we are assuming $t+2 \le e \le k$, each term in the sum is positive, and each term is a decreasing function of n. Thus $(-1)^{t-1-e}\theta_e > 0$, and $\theta_{t+2} = -1$ for $n = (k-t+1)(t+1)$ implies $0 > \theta_{t+2} > -1$ for $n > (k-t+1)(t+1)$. It is enough to prove that $|\theta_{e+2+j}| < 1$ if $j > 0$. This can be done by induction by comparing the absolute values of the s term in (3.9) for θ_{e+j+2} and θ_{e+j+3} and concluding that the absolute values decrease as a function of j. □

The equalities in (3.5a) and (3.5b) can easily be done without appealing to the finite difference operator Δ. (3.8) is terminating zero-balanced ${}_3F_2$, which can be summed if $1 \le e \le t$, using [23, (1.9.3)]

$$ {}_3F_2\left(\begin{matrix} -m_1-m_2, & b_1+m_1, & b_2+m_2; & 1 \\ & b_1, & b_2 & \end{matrix}\right) = (-1)^{m_1+m_2}\frac{(m_1+m_2)!}{(b_1)_{m_1}(b_2)_{m_2}} $$

where m_1 and m_2 are non-negative integers. The choices $b_1 = 1-k$, $b_2 = n-k-t+1$, $m_1 = e-1$, and $m_2 = t-e$ verify that $\theta_e = -1$ for $1 \le e \le t$ in (4.5a). A ${}_3F_2$ transformation also establishes the case (3.5b).

The Ahlswede-Khachatrian version of the Erdös-Ko-Rado theorem which considers $n < (k-t+1)(t+1)$ is below. The extremal family $\mathcal{F}_r$ that is chosen depends upon which multiple of $(k-t+1)$ bounds n.

Theorem B [1]**.** *If*

$$ (k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right), $$

then $\mathcal{F}_r$ is the largest t-intersecting family, up to permutations.

Note that $r = 0$ is the case that Wilson did. Their proof uses compressing techniques from extremal set theory. An eigenvalue proof along the lines of Wilson's proof is unknown.

A t-intersecting family $\mathcal{F}$ is a subset of $J(n,k)$ such that $d(A,B) \le k-t$ for all $A, B \in \mathcal{F}$. The largest t-intersecting family can be sought for any distance regular graph. Or one could specify a set of distances, and ask for the largest possible subset which avoids these distances. Delsarte [14] did this, and restated it as a linear programming problem using the orthogonal polynomials $p_i(\lambda_j)$. Wilson solves this linear programming problem in the case $n \ge (k-t+1)(t+1)$.

Let M be the allowed set of distances for a subset $\mathcal{F}$ of a distance regular graph. Let a_i, $i \in M$, be the average number of points in $\mathcal{F}$ at distance i,

$$ a_i = \frac{1}{|\mathcal{F}|}|\{(x,y) \in \mathcal{F}\times\mathcal{F} : d(x,y) = i\}|, $$

so $a_0 = 1$, and $\sum_{i\in M} a_i = |\mathcal{F}|$.

Theorem C [14]**.** *For any $\mathcal{F}$,*

$$ \sum_{i\in M} a_i p_i(\lambda_k) \ge 0 \text{ for } k = 1, 2, \cdots, d. \tag{3.10} $$

One then sets up a linear program using the inequalities in (3.10), the objective function is $\sum_{i\in M} a_i$, subject to $a_0 = 1$, $a_i \ge 0, i \in M - \{0\}$. For e-error correcting codes one considers the choice $M = \{0, 2e+1, \cdots, d\}$, there is large literature [15] on this problem in $H(N, q)$ using Krawtchouk polynomials. Many such questions remain open for the other known infinite families of distance regular graphs.

There is a beautiful theorem of Delsarte [14] relating the location of zeros of the kernel polynomials for $p_n(\lambda)$ to the existence of certain subsets of the graph. Also there is very interesting recent work of Curtin and Nomura [11] classifying spin models motivated from knot theory. Several conjectures are stated there concerning the classification of distance regular graphs with specific orthogonal polynomials.

4. Enumeration and classical orthogonal polynomials. The second aspect of orthogonal polynomials in combinatorics to be discussed is enumeration. There are many ways polynomials appear, including rook theory [24] and matching theory [26]. In this section we consider examples related to the exponential formula.

One may consider a set of objects S_n, which has a natural decomposition into smaller disjoint sets $S_n = \cup_{k=0}^{n} S_{nk}$. It is natural to consider the generating function

$$p_n(x) = \sum_{k=0}^{n} |S_{nk}| x^k.$$

One may hope that the analysis related to the polynomials is closely related to the combinatorics of the finite sets S_n.

One example of this setup is to take S_n to be the set of all matchings on an n-element set, or equivalently, the set of all involutions in the symmetric group on n letters. We let S_{nk} be the set of involutions with exactly $n-2k$ fixed points. Here we have

$$p_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \big((2k-1)(2k-3)\cdots 1\big)\; x^{n-2k} = \tilde{H}_n(x),$$

which is a version of a Hermite polynomial.

Foata [18] showed that non-trivial results could result in this way, using combinatorial techniques. One of his most beautiful results, which has been very influential, is his proof [19] of the Mehler formula for Hermite polynomials. It uses the idea of an exponential generating function, which we now explain.

We consider graphs on n vertices whose vertices are labeled with the integers $1, 2, \cdots, n$. Any such graph uniquely decomposes into connected components, whose vertices are also labeled. If $All_n = 2^{\binom{n}{2}}$ is the total number of labeled graphs on n vertices, and $Conn_n$ is the total number of connected labeled graphs on n vertices then we have

$$ALL(t) = \sum_{n=0}^{\infty} All_n \frac{t^n}{n!} = exp(CONN(t)) = exp\Big(\sum_{k=1}^{\infty} Conn_k \frac{t^k}{k!}\Big). \tag{4.1}$$

The basic reason (4.1) holds is the following: if the coefficient of t^n is found on the right side, a sum of multinomial coefficients appears. Each multinomial coefficient counts how many ways there are to shuffle the labels amongst the vertices of the connected components.

Moreover the relation $ALL(t) = exp(CONN(t))$ holds in a weighted version, weights may be attached to the connected components. For example, if we take involutions, the connected components are singleton points and single edges. If we weight each singleton by x and each edge by 1, the exponential generating function of the connected components is $xt + t^2/2$, and we have the generating function for a rescaled version of the Hermite polynomials

$$ALLINV(t) = exp(xt + t^2/2) = \sum_{n=0}^{\infty} \hat{H}_n(x) \frac{t^n}{n!}.$$

We next give Foata's proof [19] of the Mehler formula

$$\sum_{n=0}^{\infty} \hat{H}_n(x)\hat{H}_n(y)\frac{t^n}{n!} = \frac{1}{\sqrt{1-t^2}} exp\left(\frac{2txy + t^2(x^2+y^2)}{2(1-t^2)}\right). \tag{4.2}$$

Proof. We consider the left side as $ALL(t)$, where we have pairs of involutions (σ, τ) each on n letters. We think of the edges of σ as colored blue and those of τ as colored red. Vertices that are blue fixed points have weight x, while red fixed points have weight y. Any edge has weight 1. For example if $n = 11$,

$$\sigma = (18)(2\ 11)(34)(5)(6)(7\ 10)(9),$$
$$\tau = (13)(2)(48)(59)(6\ 10)(7)(11),$$

then $wt(\sigma, \tau) = x^3y^3$.

Suppose that we draw σ and τ on a single diagram. What are the resulting connected components? There are four possibilities:

(1) an even cycle with edges alternating red-blue,
(2) a path of even length, with edges alternating red-blue,
(3) a path of odd length, with edges alternating red-blue, red fixed points at both ends,
(4) a path of odd length, with edges alternating red-blue, blue fixed points at both ends.

In our example these four possibilities all occur:

$$3 \xrightarrow{blue} 4 \xrightarrow{red} 8 \xrightarrow{blue} 1 \xrightarrow{red} 3,$$
$$7 \xrightarrow{blue} 10 \xrightarrow{red} 6,$$
$$2 \xrightarrow{blue} 11,$$
$$5 \xrightarrow{red} 9.$$

If the labeled cycle in (1) has $2n$ vertices, then it has weight t^{2n}. The number of such cycles is $(2n-1)!$, so in this case the exponential generating function is

$$1 + \sum_{n=1}^{\infty} (2n-1)! \frac{t^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{t^{2n}}{2n} = -\frac{1}{2} log(1-t^2).$$

Upon exponentiating we arrive at the factor $1/\sqrt{1-t^2}$ in (4.2).

There are $(2n)!$ labeled paths of length $2n$ of type (2) each with weight xyt^{2n+1}. So the exponential generating function is

$$xy\sum_{n=1}^{\infty}(2n)!\frac{t^{2n+1}}{(2n)!} = xyt/(1-t^2).$$

This is the first term in the exponential of (4.2).

There are $(2n)!/2$ labeled paths of length $2n-1$ of type (3) each with weight x^2t^{2n}. So the exponential generating function is

$$x^2\sum_{n=1}^{\infty}(2n)!\frac{t^{2n}}{2(2n)!} = t^2x^2/2(1-t^2).$$

This is the second term in the exponential of (4.2).

There are $(2n)!/2$ labeled paths of length $2n-1$ of type (4) each with weight y^2t^{2n}. So the exponential generating function is

$$y^2\sum_{n=1}^{\infty}(2n)!\frac{t^{2n}}{2(2n)!} = t^2y^2/2(1-t^2).$$

This is the third and last term in the exponential of (4.2). □

This proof led to a combinatorial study of other classical polynomials: for example Laguerre, Jacobi, Meixner, [7], [12], [35], [36], [41], and to generalized versions of the exponential formula [8]. Foata and Garsia [20] generalized this proof to multilinear generating functions for Hermite polynomials.

The Mehler formula is an example of a bilinear generating function, which naturally gives an integral evaluation. For a general set of orthogonal polynomials $p_n(x)$ with measure $d\mu(x)$, if $h_n = 1/||p_n||^2$, and the bilinear generating function is

$$P(x,y,t) = \sum_{n=0}^{\infty} h_n p_n(x)p_n(y)t^n,$$

then

$$\int_{-\infty}^{\infty} P(x,y,t)P(x,z,w)d\mu(x) = P(y,z,tw). \tag{4.3}$$

This was noted by Bowman [9], who realized that for the q-Hermite polynomials (4.3) becomes the Askey-Wilson integral!

The q-Hermite polynomials may be defined by the generating function

$$\sum_{n=0}^{\infty} H_n(x|q)\frac{t^n}{(q;q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta};q)_\infty}, \quad x=\cos\theta, \tag{q-Hermite GF}$$

and satisfy the orthogonality relation

$$\frac{(q;q)_\infty}{2\pi}\int_0^{\pi} H_n(\cos\theta|q)H_m(\cos\theta|q)(e^{2i\theta}, e^{-2i\theta};q)_\infty d\theta = \delta_{mn}(q;q)_n.$$

The q-Mehler formula is $(x = \cos\theta, y = \cos\phi)$

$$\sum_{n=0}^{\infty} H_n(x|q)H_n(y|q)\frac{t^n}{(q;q)_n} = \frac{(t^2;q)_\infty}{(te^{i(\theta+\phi)}, te^{-i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{i(-\theta+\phi)};q)_\infty}. \tag{4.4}$$

In this case (4.3) becomes, if $z = \cos\gamma$, $a = te^{i\phi}$, $b = te^{-i\phi}$, $c = te^{i\gamma}$, $d = te^{-i\gamma}$

$$\begin{aligned} \frac{(q;q)_\infty}{2\pi}\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta};q)_\infty} &= \\ = \frac{(abcd;q)_\infty}{(ab,ac,ad,bc,bd,cd;q)_\infty}, \end{aligned} \tag{4.5}$$

the Askey-Wilson integral.

One may ask if a q-version of Foata's proof of the Mehler's proves (4.4). One can interpret the q-Hermite polynomials as generating functions of involutions with a q-statistic (see [25] and [32]). A combinatorial version of the q-exponential formula [25] establishes the q-Hermite generating function, although it does not prove the q-Mehler formula (4.4). Hung Ngo [38] found a Foata-style proof, although the q-analogue of the "overlay" map is more complicated. New multilinear versions of the q-Mehler formula (see [31] for what is known) should be found, they presumably lead to a multivariable Askey-Wilson integral, perhaps one due to Gustafson [28]. This may be related to Rogers-Ramanujan identities of higher rank [2]. A combinatorial proof of the Askey-Wilson integral (4.5) exists (see [32]).

A combinatorial proof of an equivalent form of (4.4) by counting subspaces of a finite field of order q exists, we review it here.

Let

$$h_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

be the generating function for the number of k-dimensional subspaces in an n-dimensional vector space over a finite field of order q. The q-Mehler formula (4.4) may be reformulated as

$$\sum_{n=0}^{\infty} h_n(x)h_n(y)\frac{r^n}{(q;q)_n} = \frac{(xyr^2;q)_\infty}{(r,xr,yr,xyr;q)_\infty}. \tag{4.6}$$

Clearly $h_n(x)h_n(y)$ is the generating function for pairs of subspaces, according to their respective dimensions. By choosing first the intersection of these two spaces to be k-dimensional, and then extending the intersection to each space we have

$$h_n(x)h_n(y) = \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_q (xy)^k \sum_{j,l} \begin{bmatrix} n-k \\ j \end{bmatrix}_q x^j \begin{bmatrix} n-k-j \\ l \end{bmatrix}_q q^{lj} y^l. \tag{4.7}$$

Then three applications of the q-binomial theorem show that (4.7) implies (4.6).

Another fundamental generating function whose combinatorics is not understood is (see [30] for references)

$$\sum_{n=0}^{\infty} q^{n^2/4} H_n(x|q)\frac{t^n}{(q;q)_n} = (qt^2;q^2)_\infty \mathcal{E}_q(x;t), \tag{4.8}$$

where $\mathcal{E}_q(x;t)$ is the quadratic q-exponential function,

$$\mathcal{E}_q(x;t) = \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty} \sum_{n=0}^{\infty} \left(\prod_{j=0}^{n-1} (1 + 2iq^{(1-n)/2+j} x - q^{1-n+2j}) \right) q^{n^2/4} \frac{(-it)^n}{(q;q)_n},$$

which satisfies

$$\lim_{q\to 1} \mathcal{E}_q(x; t(1-q)/2) = e^{xt}.$$

Equation (4.8) is another q-version of the Hermite generating function $exp(xt - t^2/2)$. $\mathcal{E}_q(x;t)$ appears in Suslov's addition theorem [39] which generalizes $e^{tx}e^{ty} = e^{t(x+y)}$ and deserves further study (see also [30]).

5. Enumeration and general orthogonal polynomials. A combinatorial model for general orthogonal polynomials was given by Viennot [41]. The key combinatorial structure is a weighted lattice path, and such paths have been extensively studied. In this section we state the initial results and give some examples of the Viennot theory.

We assume that the orthogonal polynomial sequence $p_n(x)$ is monic, so the fundamental three-term recurrence relation takes the form

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad p_{-1}(x) = 0, \quad p_0(x) = 1. \tag{5.1}$$

(5.1) allows for a simple inductive interpretation for $p_n(x)$. We consider all lattice paths starting at the origin $(0,0)$, with three types of steps

(1) northeast $(=(1,1))$ blue edges,
(2) north $(=(1,0))$ red edges,
(3) north-north $(=(2,0))$ black edges.

An example of such a lattice path is

$$(0,0) \xrightarrow{red} (1,0) \xrightarrow{blue} (2,1) \xrightarrow{black} (4,1) \xrightarrow{black} (6,1) \xrightarrow{red} (7,1).$$

Basically we see that there are three ways for a path to terminate at y-coordinate $n+1$, these are the three terms on the right side of (5.1). If we weight a blue northeast edge by x, a red north edge from y-coordinate n to y coordinate $n+1$ by $-b_n$, and a black north-north edge from y-coordinate $n-1$ to y coordinate $n+1$ by $-\lambda_n$, then (5.1) shows that $p_{n+1}(x)$ is the generating function for all paths from $(0,0)$ to the line $y = n+1$. The weight of the lattice path in the example is $(-b_0)(x)(-\lambda_3)(-\lambda_5)(-b_6)$. This tautological combinatorial interpretation for $p_n(x)$ always applies, but it does not necessarily give the "leanest" combinatorial interpretation.

An orthogonality relation is replaced by a moment functional $L(x^n) = \mu_n$. Since a measure may not be uniquely determined, but the moments are determined, this is appropriate. The moments must be given in terms of the recurrence coefficients b_n and λ_n, for example,

$$\mu_0 = 1, \quad \mu_1 = b_0, \quad \mu_2 = b_0^2 + \lambda_1, \quad \mu_3 = b_0^3 + 2b_0\lambda_1 + b_1\lambda_1.$$

In fact as a polynomial in $\vec{b}$ and $\vec{\lambda}$, each μ_n has coefficients which are always positive, and the precise terms which appear can be identified. To do so requires a special type of lattice path in the plane.

A Motzkin path P is a lattice path in the plane starting at $(0,0)$, and ending at $(n,0)$ which stays at or above the x-axis, whose individual steps are either $(1,0)$ (east), $(1,1)$ (northeast), or $(1,-1)$ (southeast). The weight of a Motzkin path is the product of the weights of the individual steps:

$$\begin{aligned} wt((i,k)\to(i+1,k)) &= b_k,\\ wt((i,k)\to(i+1,k+1)) &= 1,\\ wt((i,k)\to(i+1,k-1)) &= \lambda_k. \end{aligned}$$

Theorem D Viennot [41]**.**

$$\mu_n = \sum_{P:(0,0)\xrightarrow{Motzkin}(n,0)} wt(P).$$

As an example, for μ_3 there are four Motzkin paths:

$$\begin{aligned} P_1 &:(0,0)\to(1,0)\to(2,0)\to(3,0), \quad wt(P_1)=b_0^3,\\ P_2 &:(0,0)\to(1,0)\to(2,1)\to(3,0), \quad wt(P_1)=b_0\lambda_1,\\ P_3 &:(0,0)\to(1,1)\to(2,1)\to(3,0), \quad wt(P_1)=b_1\lambda_1,\\ P_4 &:(0,0)\to(1,1)\to(2,0)\to(3,0), \quad wt(P_1)=b_0\lambda_1, \end{aligned}$$

which gives $\mu_3 = b_0^3 + 2b_0\lambda_1 + b_1\lambda_1$.

Theorem D allows one to combinatorially evaluate integrals if one happens to know a representing measure $d\mu(x)$ for the polynomials. For example, the Askey-Wilson integral (4.5) can be evaluated [32] using these lattice paths for the q-Hermite polynomials. The idea is that the combinatorics of the paths replaces the analysis of integration. Viennot's theory was heavily influenced by Flajolet [16],[17].

The linearization coefficients $a(m,n,k)$ are defined by

$$p_n(x)p_m(x) = \sum_{k=|m-n|}^{m+n} a(m,n,k)p_k(x),$$

which is equivalent to

$$a(m,n,k) = \frac{L(p_m p_n p_k)}{L(p_k p_k)}.$$

There has been much combinatorial work on the interpretation of $L(\prod_{i=1}^k p_{n_i}(x))$, which thereby includes $a(m,n,k)$. For example, for the Hermite polynomials, by completing the square in the integral of the product of k generating functions, one finds that

$$\sum_{n_1,\cdots,n_k\ge 0} L\left(\prod_{i=1}^k \hat{H}_{n_i}(x)\right)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_k^{n_k}}{n_k!} = exp(e_2(t_1,\cdots,t_k))$$

where $e_2(t_1,\cdots,t_k)$ is the elementary symmetric function of degree 2. These are connected components for the complete matchings of a graph with k types of vertices, where the edges must connect vertices of different types. We call these matchings "inhomogeneous". Thus $L(\prod_{i=1}^k \hat{H}_{n_i}(x))$ is the number of complete inhomogeneous matchings of the complete graph $K_{n_1+\cdots+n_k}$, where any element of $\{n_1+\cdots+n_{i-1}+1,\cdots,n_1+\cdots+n_i\}$ is a vertex of type i. This is a theorem of Azor, Gillis, and Victor [4], it was proven using the Viennot machinery in [13]. Other versions exist, for Legendre [27], Laguerre [3], [21], Meixner [12], Krawtchouk, Charlier, and Meixner-Pollaczek [44] polynomials (conspicuously missing from this list are the ultraspherical polynomials, and Rahman's result for the Jacobi polynomials), but recently a unified theorem has been given by Kim and Zeng [33].

They consider Sheffer orthogonal polynomials, those that satisfy the recurrence relation

$$p_{n+1}(x) = (x-(ab+u_3n+u_4n))p_n(x) - n(b+n-1)u_1u_2p_{n-1}(x). \tag{5.2}$$

Note that if $a=b=u_1=u_2=u_3=u_4=1$, these are just the Laguerre polynomials $L_n(x)$, whose measure is $e^{-x}dx$ on $[0,\infty)$ and whose nth moment $\mu_n = n!$. For the polynomials in (5.2), μ_n is a polynomial in the parameters a, b, u_1, u_2, u_3, u_4 whose coefficients sum to $n!$. μ_n can be given [33] as a generating function of permutations according to fixed points, cycles, double ascents, double descents, peaks, and valleys,

$$\mu_n = \sum_{\pi\in S_n} u_1^{\#peak(\pi)} u_2^{\#valley(\pi)} u_3^{\#doubleascent(\pi)} u_4^{\#doubledescent(\pi)} \times a^{\#fixpoint(\pi)} b^{\#cycles(\pi)}.$$

The Kim-Zeng theorem combinatorially interprets the integral of a product of these polynomials. They define a permutation analogue of the inhomogeneous matchings which appear in the Hermite result. Let $[i,j]$ denote the closed interval of integers from i to j. Let $N = n_1 + \cdots + n_k$,

$$A_1 = [1, n_1],\ A_i = [n_1+\cdots+n_{i-1}+1, n_1+\cdots+n_i], \quad \text{for } 2\le i\le k.$$

We consider the set D_N of all $N!$ permutations of $A_1\cup A_2\cup\cdots\cup A_k$ such that

$$\pi(A_i)\cap A_i = \varnothing$$

for all i. The set D_N can be considered as a set of generalized derangements.

Theorem E Kim-Zeng [33]. *If $p_n(x)$ are the Sheffer polynomials given by (5.2), then*

$$L\Big(\prod_{i=1}^k p_{n_i}(x)\Big) = \sum_{\pi\in D_N} u_1^{\#peak(\pi)+mat(\pi)} u_2^{\#valley(\pi)+mat(\pi)} \times u_3^{\#doubleascent(\pi)-mat(\pi)} u_4^{\#doubledescent(\pi)-mat(\pi)} b^{\#cycles(\pi)},$$

where $mat(\pi)$ is the number of color matches of π (see [33]*).*

Theorem E specializes to the known results for the classical orthogonal polynomials, and establishes the positivity of the coefficients.

By and large, the exact q-analogues of the classical results for $L(\prod_{i=1}^k p_{n_i}(x))$ are not understood. One would hope for just a weighted version of the Kim-Zeng result, but for the natural choices, this is not the case [33]. Only the q-Hermite polynomials are known to have such results [32].

The orthogonal polynomial framework can suggest new results in enumeration. Here is one example involving the Stirling numbers of the second kind $S(n,k)$, which count the number of set partitions of $[n]$ into k blocks. The q-Stirling numbers of the second kind $S_q(n,k)$ are polynomials in q, with positive integer coefficients which sum to $S(n,k)$. Thus it is natural to consider $S_q(n,k)$ as a generating function for a statistic on set partitions of $[n]$ into k blocks. This was done by Milne [37]. However, since $S_q(n,k)$ appear naturally in the moments of the q-Charlier polynomials

$$\mu_n = \sum_{k=1}^{n} S_q(n,k) a^k,$$

the Viennot theory offers its own statistic, which is not obviously q-Stirling distributed. Considering these two statistics led Wachs and White [42] to new bivariate equidistribution theorems for these two statistics.

There are also many relations to continued fractions [16], [17] and determinants, via the moment generating function [29] and Hankel determinants [34]. Because the entries of the determinants are weighted paths, and such determinants are combinatorially understood from the Gessel-Viennot theory, this has led to widespread work.

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