Classical Orthogonal Polynomials as Moments *

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Abstract

We show that the Meixner, Pollaczek, Meixner-Pollaczek and Al-Salam-Chihara polynomials, in certain normalization, are moments of probability measures. We use this fact to derive bilinear and multilinear generating functions for some of these polynomials. We also comment on the corresponding formulas for the Charlier, Hermite and Laguerre polynomials.

Running Title: Generating Functions

1. Introduction. The umbral calculus of the last century was an attempt to treat polynomials as if they were monomials. For a given sequence of polynomials $\{p_n(x)\}$ this means that one can take an identity involving $\{x^n : n = 0, 1, 2, \cdots\}$ then replace x^n by $p_n(x)$ provided that we develop a calculus to interpret the resulting identity. In the 1970's Rota popularized the umbral calculus by putting it on solid foundations and by showing its significance in combinatorics and special functions, [25], [19].

In this paper we consider linear functionals $\mathcal{L}_a$ whose n th moments are orthogonal polynomials, and which have the integral representation

$$(1.1) \qquad < \mathcal{L}_a | f >= \int_{-\infty}^{\infty} f(x)\, d\mu(x; a) = p_n(a)$$

for some measure $d\mu(x; a)$ that depends upon a parameter a. In particular, we give explicit measures $d\mu(x; a)$ whose moments are various classes of classical orthogonal polynomials. A summary of such polynomials is given in §6. Other authors also used representations of polynomials as moments.

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Rahman and Verma [24] observed that the continuous q-ultraspherical polynomials are multiples of the moments of a probability measure. We generalize their result to the Al-Salam-Chihara polynomials in Theorem 3.1.

The integral representation (1.1) for $\mathcal{L}_a$ can allow for simple evaluations of generating functions. For example, if

$$G(t) = \sum_{n=0}^{\infty} r_n(b) t^n \tag{1.2}$$

is the generating function for any set of polynomials $r_n(b)$, then

$$\int_{-\infty}^{\infty} G(xt) d\mu(x;a) = \sum_{n=0}^{\infty} p_n(a) r_n(b) t^n. \tag{1.3}$$

We will use (1.3) extensively to find bilinear generating functions, when $p_n = r_n$. (This is called the Poisson kernel if p_n is orthonormal.) However (1.3) can used for any set of polynomials $r_n(b)$, once the measure $d\mu(x;a)$ is known. For example, it is easy to give an integral representation for a bilinear generating function for q-ultraspherical and p-ultraspherical polynomials.

In [18] we used the fact that the moments of the measure of the Al-Salam-Carlitz polynomials [1] are the continuous q-Hermite polynomials to derive bilinear and multilinear generating functions for the continuous q-Hermite polynomials. In §4 we similarly use the results of §3 on Al-Salam-Chihara polynomials to give a direct evaluation and an extension of the Poisson kernel of the Al-Salam-Chihara polynomials. We also derive a very general multilinear generating function for the Al-Salam-Chihara polynomials. This extends our earlier results [18] on continuous q-Hermite polynomials because the Al-Salam-Chihara polynomials are a two parameter extension of the continuous q-Hermite polynomials.

In §5 we point out that the continuous q-ultraspherical polynomials are Al-Salam-Chihara polynomials. We then combine this fact with our multilinear generating functions of §4 to establish multilinear generating functions for the continuous q-ultraspherical polynomials. In particular we show how the results of [16] and [6] imply an earlier result of Gasper and Rahman [13]. In §5 we give a new integral representation for the continuous q-ultraspherical polynomials as moments. This integral representation is then used to derive a bilinear generating function for the continuous q-ultraspherical polynomials. As a byproduct we obtain a transformation formula (Theorem 5.5) expressing a sum of two ${}_4\phi_3$'s as a combination of different ${}_4\phi_3$'s. This transformation is of independent interest.

In §2 we consider the functional

$$< \mathcal{M}_{a,b,c} | f > := \frac{(1-c)^{-a-b-1}\Gamma(a+b+2)}{\Gamma(a+1)\,\Gamma(b+1)} \int_c^1 (1-x)^a (x-c)^b f(x) dx. \tag{1.4}$$

It turns out that the moments of $\mathcal{M}_{a,b,c}$ are essentially the Meixner polynomials. We also obtain generating function results for the Meixner-Pollaczek and Pollaczek polynomials in §2.

In §3 we prove (see Theorem 3.1) that the moments of the functional for the big q-Jacobi polynomials are Al-Salam-Chihara polynomials. In this case the functional $\mathcal{L}_{a,b,c}$ is defined by

$$(1.5) \qquad < \mathcal{L}_{a,b,c}|f >= \int_{-\infty}^{\infty} f(x)\, d\mu(x; a, b, c).$$

The measure $\mu(x; a, b, c)$ is

$$(1.6) \qquad \mu(x; a, b, c) = \frac{(aq, bq, cq, abq/c; q)_\infty}{aq\,(q, c/a, aq/c, abq^2; q)_\infty} \frac{(x/a, x/c; q)_\infty}{(x, bx/c; q)_\infty} \times \sum_{n=0}^{\infty} [aq^{n+1}\epsilon_{aq^{n+1}}(x) - cq^{n+1}\epsilon_{cq^{n+1}}(x)],$$

where $\epsilon_u(x)$ is a unit mass at $x = u$. We follow the notation in [12], The measure μ in (1.6) is normalized so that its total mass equals one.

The notion of the q-integral is useful in understanding and motivating identities involving q-series. The q-integral is defined by

$$(1.7) \qquad \int_0^a f(u)\, d_qu := a(1-q) \sum_{m=0}^{\infty} q^m f(aq^m)$$

$$(1.8) \qquad \int_a^b f(u)\, d_qu := \int_0^b f(u)\, d_qu - \int_0^a f(u)\, d_qu.$$

It is clear that integration with respect to μ of (1.6) amounts to q-integration. In fact our representation of the Al-Salam-Chihara polynomials as moments provides a new q-integral representation for the Al-Salam-Chihara polynomials. This q-integral representation has proved to be very useful here and elsewhere [17].

2. Orthogonal Moment Functionals. In this section we give functionals whose moments are the Hermite, Laguerre, and various Meixner families of polynomials. We also use (1.1) to derive new generating functions. We explain how these polynomials are related to the umbral product of Roman and Rota [25].

We now consider functionals whose moments are orthogonal polynomials. A theorem of Boas [9, p. 74] asserts that given any sequence of real numbers $\{\alpha_n\}$ there exists a signed measure α with finite moments such that

$$\alpha_n = \int_0^{\infty} x^n\, d\alpha(x).$$

Thus any sequence of orthogonal polynomials is a sequence of moments for a certain functional.

Let $\mathcal{L}_a$ and $\mathcal{M}_b$ be functionals so that $\{r_n(a)\}$ and $\{s_n(b)\}$,

$$r_n(a) :=< \mathcal{L}_a|x^n >, \qquad s_n(b) :=< \mathcal{M}_b|x^n >, \tag{2.1}$$

are orthogonal polynomials. Roman and Rota [25] defined the product of two linear functionals $\mathcal{L}$ and $\mathcal{M}$ acting on polynomials of binomial type by

$$< \mathcal{L}\mathcal{M}|p_n(x) >:= \sum_{k=0}^{n} \binom{n}{k} < \mathcal{L}|p_k(x) >< \mathcal{M}|p_{n-k}(x) > . \tag{2.2}$$

If $< \mathcal{L}|f >= \int_{-\infty}^{\infty} f(x)d\lambda(x)$ and $< \mathcal{M}|f >= \int_{-\infty}^{\infty} f(y)\, d\mu(y)$ then the product (2.2) becomes

$$< \mathcal{L}\mathcal{M}|f >= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+y)d\lambda(x)d\mu(y). \tag{2.3}$$

The problem is to characterize the pairs $\{r_n(a)\}$ and $\{s_n(b)\}$ so that $\{< \mathcal{L}_a\mathcal{M}_b|x^n >\}$ is a sequence of orthogonal polynomials in a for infinitely many b's or vice versa. Al-Salam and Chihara [2] solved an equivalent problem and their results show that such r_n's and s_n's have to be among the Hermite, Laguerre, Charlier, Meixner, Meixner-Pollaczek or the Al-Salam-Chihara polynomials. For the rest of this section we consider the first five cases. We give the functionals for the Hermite and Laguerre polynomials, show that the Charlier, Meixner and Meixner-Pollaczek polynomials are moments of a gamma distribution on $[a, \infty)$, a beta distribution on $[c, 1]$ and a beta distribution on $[c^{1/2}, c^{-1/2}]$. The Al-Salam-Chihara polynomials are moments of the measure of the big q-Jacobi polynomials, as we shall see in §3.

Hermite and Laguerre Polynomials. It is clear that

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(y-a)^2} y^n\, dy &= \frac{1}{\Gamma(1/2)}\int_{-\infty}^{\infty} e^{-y^2}\,(a+y)^n\, dy \\ &= \sum_{k=0}^{[n/2]} \binom{n}{2k} a^{n-2k}(1/2)_k = \sum_{k=0}^{[n/2]} \frac{n!\, a^{n-2k}}{2^{2k}\, k!\, (n-2k)!}.\end{aligned}$$

Therefore

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(y-a)^2} y^n\, dy = (2i)^{-n} H_n(ia). \tag{2.4}$$

Thus the Hermite polynomials are moments. In fact it is this property that Slepian [26] used to prove the Kibble-Slepian formula. For example the Poisson kernel follows from

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{H_n(ix)H_n(y)}{(2i)^n\, n!} t^n &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} e^{-(u-x)^2}\, \frac{H_n(y)\,(tu)^n}{n!}\, du \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(u-x)^2}\, e^{2uyt+t^2u^2}\, du,\end{aligned}$$

after completing squares in the argument of the exponential.

The Laguerre polynomials $\{n!L_n^\alpha(x)\}$ are moments of an explicit functional

$$(2.5)\qquad n!L_n^\alpha(x) = x^{-\alpha/2}\int_0^\infty e^{x-u}u^{n+\alpha/2}\, J_\alpha(2\sqrt{xu})\, du, \quad n = 0, 1, \cdots; \alpha > -1,$$

[28, (5.4.1)]. The Hille-Hardy formula [28, (5.1.15)]

$$(2.6)\qquad \begin{aligned}&\sum_{n=0}^{\infty}\frac{n!\, t^n}{\Gamma(\alpha+n+1)}L_n^\alpha(x)L_n^\alpha(y)\\ &\quad = (1-t)^{-1}exp\left[-\frac{t(x+y)}{1-t}\right](xyt)^{-\alpha/2}I_\alpha\left(\frac{2\sqrt{xyt}}{1-t}\right),\end{aligned}$$

follows from (2.5) and the generating function [28, (5.1.16)]

$$(2.7)\qquad \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha+n+1)}L_n^\alpha(x) = e^t(xt)^{-\alpha/2}J_\alpha\left(2\sqrt{xt}\right)$$

through the use of Weber's second exponential integral [29, (13.31.1)]

$$(2.8)\qquad \int_0^\infty exp(-p^2t^2)\, J_\nu(at)J_\nu(bt)\, tdt = \frac{1}{2p^2}\, exp\left(-\frac{a^2+b^2}{4p^2}\right) I_\nu\left(\frac{ab}{2p^2}\right).$$

The Hille-Hardy formula is the Poisson kernel for Laguerre polynomials.

The Laguerre polynomials $\{n!\, L_n^\alpha(x)\}$ for $x < 0$ are moments of positive measures for all n. To see this replace x in (2.5) by $-x$ then replace J_α by I_α. The resulting integral converges since [29, (7.23.3)]

$$I_\alpha(x) \approx (2\pi x)^{1/2}\, e^x, \quad x \to +\infty.$$

The Meixner Polynomials. Consider the functional $\mathcal{M}_{A,B,c}$ whose moments are

$$(2.9)\qquad \mu_n = \frac{(1-c)^{-A-B-1}\Gamma(A+B+2)}{\Gamma(A+1)\,\Gamma(B+1)}\int_c^1 (1-x)^A(x-c)^B x^n dx.$$

The Euler integral representation [10, (2.1.10)] implies that

$$(2.10)\qquad \mu_n = c^n{}_2F_1(-n, B+1; A+B+2; 1-1/c).$$

Setting

$$(2.11)\qquad B = -a-1, \quad A = \beta + a - 1,$$

and using the explicit formula for Meixner polynomials [11, (10.24.9)], we have proven the following theorem.

Theorem 2.1 *If Re $a<0$ and Re $a+\beta>0$, the n th moment of the functional $\mathcal{M}_{\beta+a-1,-a-1,c}$ is the Meixner polynomial, $\mu_n = c^n m_n(a;\beta,c)/(\beta)_n$.*

If we put more restrictions on a, c, and β, then the functional in Theorem 2.1 is positive definite. This forces certain determinants to be positive [9, p. 14].

Corollary 2.2 *If $c<1$, $-\beta<a<0$, and $n\ge 0$, then*

$$det\{m_{j+k}(a;\beta,c)c^{k+j}/(\beta)_{k+j} : 0\le j,k\le n\} > 0.$$

The Meixner polynomials have the generating function [11, (10.24.13)]

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}m_n(x;\beta,c) = (1-t)^{-\beta-x}(1-t/c)^x, \quad |t|<min(1,|c|), \tag{2.12}$$

and satisfy the orthogonality relation [11, (10.24.11)]

$$\sum_{k=0}^{\infty}\frac{c^k(\beta)_k}{k!}m_n(k;\beta,c)m_j(k;\beta,c) = n!(\beta)_n c^{-n}(1-c)^{-\beta}\delta_{n,j}, \quad \beta>0,\ 0<c<1. \tag{2.13}$$

We shall assume $0<c<1$.

From (2.12) and Darboux's asymptotic method [28] we see that

$$m_n(x;\beta,c) \approx \frac{c^{-n}\,\Gamma(-x+n)}{(1-c)^{\beta+x}\Gamma(-x)}, \quad x\ne 0,1,\cdots,\ n\to\infty, \tag{2.14}$$

while

$$m_n(x;\beta,c) \approx \frac{\Gamma(\beta+x+n)}{\Gamma(\beta+x)}\,(1-1/c)^x, \quad x=0,1,\cdots,\ n\to\infty. \tag{2.15}$$

In (2.12) we replace x by y and t by tu then apply $\mathcal{L}_{\beta_1+x-1,-x-1,c_1}$ to the variable u. When $x\ne 0,1,\cdots$ this gives

$$\sum_{n=0}^{\infty}\frac{c_1^n\,t^n}{(\beta_1)_n\,n!}m_n(x;\beta_1,c_1)m_n(y;\beta,c) = \frac{(1-c_1)^{1-\beta_1}\,\Gamma(\beta_1)}{\Gamma(-x)\,\Gamma(x+\beta_1)} \times\int_{c_1}^{1}(1-u)^{x+\beta_1-1}(u-c_1)^{-x-1}(1-tu)^{-\beta-y}(1-tu/c)^y\,du. \tag{2.16}$$

Interchanging integration and summation can be justified from (2.14) and (2.15).

After the change of variable $u = c_1 + (1 - c_1)v$ we identify the integral on the right-hand side of (2.16) as an Appell function F_1, [10, (5.7.6)]. The result is [10, (5.8.5)]

$$\begin{aligned}(2.17)\qquad &\sum_{n=0}^{\infty} \frac{c_1^n\, t^n}{(\beta_1)_n\, n!} m_n(x;\beta_1,c_1) m_n(y;\beta,c) \\ &= \frac{(1-tc_1/c)^y}{(1-tc_1)^{\beta+y}}\, F_1\left(-x,-y,\beta+y,\beta_1; \frac{(1-c_1)t}{c-c_1t}, \frac{(1-c_1)t}{1-c_1t}\right).\end{aligned}$$

Although (2.17) was derived under the assumptions $Re\ y < 0$, $Re\ y + \beta_1 > 0$ and $0 < c_1 < 1$ the first two conditions can be relaxed by analytic continuation of the left-hand side of (2.17) taking into account (2.14) and (2.15). The right-hand side of (2.17) can be analytically continued through standard analytic continuation of the F_1 function in [10, 5.8]. In the case $\beta = \beta_1$ the F_1 in (2.17) reduces to a ${}_2F_1$ multiplied by an algebraic function, [10, (5.10.1)]. The result after replacing t by tc is

$$\begin{aligned}(2.18)\qquad &\sum_{n=0}^{\infty} \frac{(cc_1t)^n}{(\beta)_n\, n!} m_n(x;\beta,c_1) m_n(y;\beta,c) \\ &= (1-tc_1)^y (1-tc)^x (1-tcc_1)^{-x-y-\beta}\, {}_2F_1\left(\left.\begin{matrix}-x,-y\\ \beta\end{matrix}\,\right|\, \frac{t(1-c)(1-c_1)}{(1-tc)(1-tc_1)}\right).\end{aligned}$$

Observe that (2.18) is symmetric in the pairs (x, c_1) and (y, c). Also note that the right-hand side of (2.18) is positive when x and y are nonnegative integers and $c, c_1, tc, tc_1 \in (0, 1)$. It is also positive for $x < 0$, $y < 0$ and $c, c_1, tc, tc_1 \in (0, 1)$. In the latter case we require the argument of the ${}_2F_1$ to be in $[0, 1)$. Formula (2.17) is due to Meixner [21].

If in addition to $\beta = \beta_1$ we also require $c = c_1$ then (2.13) shows that (2.17) will be the Poisson kernel for the Meixner polynomials, up to a constant factor.

Theorem 2.3 *The Meixner polynomials satisfy the connection relation*

$$\begin{aligned}(2.19)\qquad &\frac{(1-tc_1/c)^y}{(1-tc_1)^{\beta+y}} \sum_{x=0}^{\infty} \frac{c_1^x(\beta_1)_x}{x!}\, F_1\left(-x,-y,\beta+y,\beta_1; \frac{(1-c_1)t}{c-c_1t}, \frac{(1-c_1)t}{1-c_1t}\right) m_n(x;\beta_1,c_1) \\ &= t^n(1-c_1)^{-\beta_1} m_n(y;\beta,c).\end{aligned}$$

Proof. This follows from (2.13) and the bilinear generating function (2.17).

It is clear that the sum on the left-hand side of (2.19) represents integration with respect to a discrete measure, hence (2.19) can be viewed as an integral equation. In particular the case $\beta = \beta_1$ and $c = c_1$ provides an integral equation whose eigenfunctions are Meixner polynomials. The integral equation (2.19) can be expressed as an integral equation with a symmetric kernel when

$\beta = \beta_1$ and $c = c_1$. The completeness of the Meixner polynomials in the L_2 space weighted by their orthogonality measure shows that the Meixner polynomials are the only eigenfunctions of the integral equation (2.19) and one can identify the eigenvalues from (2.19).

Theorem 2.1 has a curious implication. Assume that a sequence of polynomials $\{p_n(y)\}$ are orthonormal with respect to a functional $\mathcal{L}$ whose moments are given in Theorem 2.1, and let

$$p_0(y) := 1, \quad p_n(y) = \sum_{j=0}^{n} C_{n,j} y^j. \tag{2.20}$$

Then

$$\delta_{n,k} = \mathcal{L}(C_{k,k} y^k p_n(y)) = C_{k,k} \sum_{j=0}^{n} C_{n,j} \mu_{k+j},$$

which may be rewritten as

$$\begin{aligned}\sum_{j=0}^{n} \binom{n}{j} (-1)^j {}_2F_1(j-n, x-n+1; 2-2n-b; d)\, {}_2F_1(-k-j, -x; b; d) \\ = \frac{k! d^{2k} (x-k+1)_k (b+x)_k}{(b)_{2k} (b+k-1)_k}\, \delta_{n,k}, \quad 0 \le k \le n.\end{aligned} \tag{2.21}$$

The Charlier polynomials $\{c_n(x;a)\}$ are a limiting case of the Meixner polynomials, [11, (10.24.16)]

$$\lim_{\beta\to\infty} \beta^{-n} m_n(x; \beta, a/\beta) = c_n(x;a).$$

This means that generating functions for Charlier polynomials will follow as limiting cases. It also follows by direct calculation that $c_n(-\alpha-1;a)$ is a multiple of the n th moment of the weight function $(x-a)^\alpha e^{a-x}$ on $[a,\infty)$. The details are omitted.

The Meixner-Pollaczek Polynomials. They are defined as [11, (10.22.21)]

$$P_n^{\lambda}(x;\phi) := \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1(-n, \lambda+ix; 2\lambda; 1-e^{-2i\phi}). \tag{2.22}$$

They also called Pollaczek polynomials on an infinite interval [11], [28] and the Meixner polynomials of the second kind [9]. We shall follow the terminology in [7].

It is clear from that the Meixner-Pollaczek polynomials can formally be written as Meixner polynomials. In (2.17) set

$$\beta = 2\lambda, \quad \beta_1 = 2\lambda_1, \quad -x = \lambda_1 + i\xi, \quad -y = \lambda + i\eta, \quad c = e^{2i\phi}, \quad c_1 = e^{2i\phi_1}. \tag{2.23}$$

This is equivalent to considering the functional

$$< \mathcal{L}|f >= \frac{(c^{-1/2}-c^{1/2})^{-A-B-1}\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)} \int_{c^{1/2}}^{c^{-1/2}} (c^{-1/2}-x)^A (x-c^{1/2})^B f(x)dx.$$

After replacing t by $te^{i(\phi-\phi_1)}$ in (2.17) we get

$$\sum_{n=0}^{\infty} \frac{n!}{(2\lambda_1)_n} t^n P_n^{\lambda_1}(\xi,\phi_1) P_n^{\lambda}(\eta,\phi) = (1-te^{i(\phi_1-\phi)})^{-\lambda-i\eta}(1-te^{i(\phi_1+\phi)})^{-\lambda+i\eta} \times F_1\left(\lambda_1+i\xi, \lambda+i\eta, \lambda-i\eta, 2\lambda_1, \frac{-2it\sin\phi_1}{e^{i\phi}-e^{i\phi_1}t}, \frac{-2it\sin\phi_1}{e^{-i\phi}-e^{i\phi_1}t}\right). \tag{2.24}$$

Here again the case $\lambda=\lambda_1$ reduces the F_1 to a multiple of a ${}_2F_1$ and we get

$$\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} t^n P_n^{\lambda}(\xi,\phi_1) P_n^{\lambda}(\eta,\phi) = (1-te^{i(\phi_1-\phi)})^{-\lambda-i\eta}(1-te^{i(\phi-\phi_1)})^{-\lambda-i\xi} \times(1-te^{i(\phi_1+\phi)})^{i\xi+i\eta}\, {}_2F_1\left(\begin{matrix}\lambda+i\xi, \lambda+i\eta \\ 2\lambda\end{matrix}\middle| \frac{-4t\sin\phi\sin\phi_1}{(1-te^{i(\phi-\phi_1)})(1-te^{i(\phi_1-\phi)})}\right). \tag{2.25}$$

The orthogonality relation for the Meixner-Pollaczek polynomials is

$$\int_0^{\infty} P_m^{\lambda}(x,\phi)P_n^{\lambda}(x,\phi)\frac{(2\sin\phi)^{2\lambda-1}}{\pi}e^{-(\pi-2\phi)x}|\Gamma(\lambda+ix)|^2 = \frac{(2\lambda)_n}{n!}\delta_{m,n}. \tag{2.26}$$

Therefore (2.25) with $\phi=\phi_1$ is the Poisson kernel for the Meixner-Pollaczek polynomials. The special case $\phi=\phi_1$ of (2.24) is [23, (5.2)]. Rahman also noted (2.25).

The Pollaczek Polynomials. These polynomials are defined by [11, (10.21.10)]

$$P_n^{\lambda}(\cos\theta,a,b) := \frac{(2\lambda)_n}{n!}\, {}_2F_1(-n,\lambda+ih(\theta);2\lambda;1-e^{-2i\theta}), \tag{2.27}$$

where

$$h(\theta) := \frac{a\cos\theta+b}{\sin\theta} = \frac{ax+b}{\sqrt{1-x^2}}, \quad x=\cos\theta. \tag{2.28}$$

They satisfy the orthogonality relation

$$\int_0^{\pi} P_m^{\lambda}(\cos\theta,a,b)P_n^{\lambda}(\cos\theta,a,b)e^{(2\theta-\pi)h(\theta)}(\sin\theta)^{2\lambda}|\Gamma(\lambda+ih(\theta))|^2 = 2^{1-2\lambda}\pi\frac{\Gamma(2\lambda+n)}{n!(\lambda+a+n)}\delta_{m,n}. \tag{2.29}$$

Note that (2.22) and (2.27) indicate that x and ϕ in (2.22) are now replaced by $h(\theta)$ and θ in (2.27). Thus (2.24) and (2.25) can be translated to generating functions for the Pollaczek polynomials. In order to find the Poisson kernel for the Pollaczek polynomials we have to incorporate the term $a+n+\lambda$ which amounts to taking a linear combinations of the left-hand side of (2.25) and its derivative with respect to t.

3. Al-Salam-Chihara Polynomials. The big q-Jacobi polynomials are [12, (7.3.10), p. 167]

$$P_n(x;a,b,c;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}\,\middle|\, q, q\right). \tag{3.1}$$

Using (1.5) their orthogonality relation is [12, (7.3.12)],

$$< \mathcal{L}_{a,b,c}|P_nP_m >= \frac{(q, bq, abq/c;q)_n(-ac)^{-n}q^{n(n-1)/2}}{(abq, aq, cq;q)_n(1-abq^{2n+1})}\delta_{m,n}.$$

In this case we have

$$\begin{aligned} &\mu_n \frac{(q, c/a, aq/c, abq^2;q)_\infty}{(aq, bq, cq, abq/c;q)_\infty} q^{-n} \\ &= \frac{(q, aq/c;q)_\infty\, a^n}{(aq, abq/c;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} aq, abq/c \\ aq/c \end{matrix}\,\middle|\, q, q^{n+1}\right) \\ &\quad - \frac{c^{n+1}\,(cq/a, q;q)_\infty}{a(cq, bq;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} cq, bq \\ cq/a \end{matrix}\,\middle|\, q, q^{n+1}\right). \end{aligned} \tag{3.2}$$

To simplify the right-hand side of (3.2) we appeal to the three term transformation formula in [12, p. 64, (4.3.5)], namely

$$\begin{aligned} &{}_2\phi_1\left(\begin{matrix} A, B \\ C \end{matrix}\,\middle|\, q, Z\right) \\ &\quad + \frac{(B, q/C, C/A, AZ/q, q^2/AZ;q)_\infty}{(C/q, Bq/C, q/A, AZ/C, Cq/AZ;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} Aq/C, Bq/C \\ q^2/C \end{matrix}\,\middle|\, q, Z\right) \\ &= \frac{(ABZ/C, q/C;q)_\infty}{(AZ/C, q/A;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} C/A, Cq/ABZ \\ Cq/AZ \end{matrix}\,\middle|\, q, Bq/C\right). \end{aligned} \tag{3.3}$$

In (3.3) we choose A, B, C and Z as

$$A = aq, \quad B = abq/c, \quad C = aq/c, \quad Z = q^{n+1}. \tag{3.4}$$

The result is the three term relation

$$\begin{aligned} &{}_2\phi_1\left(\begin{matrix} aq, abq/c \\ \\ aq/c \end{matrix}\,\middle|\, q, q^{n+1}\right) + \frac{(abq/c, c/a, 1/c, aq^{n+1}, q^{-n}/a;q)_\infty}{(a/c, bq, 1/a, cq^{n+1}, q^{-n}/c;q)_\infty} \\ &\times {}_2\phi_1\left(\begin{matrix} cq, bq \\ \\ qc/a \end{matrix}\,\middle|\, q, q^{n+1}\right) = \frac{(abq^{n+2}, c/a;q)_\infty}{(cq^{n+1}, 1/a;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} 1/c, q^{-n-1}/ba \\ \\ q^{-n}/c \end{matrix}\,\middle|\, q, qb\right). \end{aligned} \tag{3.5}$$

This leads to the following more compact form of (3.5)

$$\begin{aligned} &{}_2\phi_1\left(\begin{matrix} aq, abq/c \\ \\ aq/c \end{matrix}\,\middle|\, q, q^{n+1}\right) \frac{a^n\,(aq/c;q)_\infty}{(aq, ba/cq;q)_\infty} \\ &+\frac{c^n\,(1-c/a)\,(cq/a;q)_\infty}{(1-a/c)\,(bq, cq;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} cq, bq \\ \\ qc/a \end{matrix}\,\middle|\, q, q^{n+1}\right) \\ &= \frac{a^n\,(aq/c, c/a, abq^{n+2};q)_\infty}{(abq/c, aq, 1/a, cq^{n+1};q)_\infty}\, {}_2\phi_1\left(\begin{matrix} 1/c,\, q^{-n-1}/\,ba \\ \\ q^{-n}/\,c \end{matrix}\,\middle|\, q, qb\right). \end{aligned} \tag{3.6}$$

Therefore (3.2) reduces to

$$\mu_n = \frac{(aq)^n(bq, cq, abq^{n+2};q)_\infty}{(1/a, cq^{n+1}, abq^2;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} 1/c,\, q^{-n-1}/\,ab \\ \\ q^{-n}/\,c \end{matrix}\,\middle|\, q, qb\right).$$

The q-analogue of the Euler transformation is (III.3) in [12, p. 241] is

$${}_2\phi_1\left(\begin{matrix} 1/c,\, q^{-n-1}/\,ab \\ \\ q^{-n}/\,c \end{matrix}\,\middle|\, q, qb\right) = \frac{(1/a;q)_\infty}{(bq;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, abq/c \\ \\ q^{-n}/\,c \end{matrix}\,\middle|\, q, 1/a\right). \tag{3.7}$$

This further simplifies μ_n to the form

$$(3.8)\qquad \mu_n = a^n q^n \frac{(cq;q)_n}{(abq^2;q)_n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, abq/c \\ \\ q^{-n}/c \end{matrix}\,\middle|\, q, 1/a\right).$$

The Al-Salam-Chihara polynomials, [5], [7], [8], are

$$\begin{aligned} p_n(x, t_1, t_2) &= {}_3\phi_2\left(\begin{matrix} q^{-n}, t_1e^{i\theta}, t_1e^{-i\theta} \\ \\ t_1t_2, 0 \end{matrix}\,\middle|\, q, q\right) \\ &= \frac{\left(t_1e^{-i\theta};q\right)_n t_1^n e^{in\theta}}{(t_1t_2;q)_n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, t_2e^{i\theta} \\ \\ q^{1-n}e^{i\theta}\big/ t_1 \end{matrix}\,\middle|\, q, \frac{qe^{-i\theta}}{t_1}\right). \end{aligned}$$

The second equality follows from a ${}_3\phi_2$ transformation and the q-analogue of the Pfaff-Kummer transformation [8]. In order to relate the μ's to the Al-Salam-Chihara polynomials we make the parameter identification

$$(3.9)\qquad a = t_1\, e^{i\theta}\big/ q, \quad c = t_1\, e^{-i\theta}\big/ q, \quad b = t_2\, e^{-i\theta}\big/ q$$

Hence $\mu_n = p_n(\cos\theta; t_1, t_2)$.

Theorem 3.1 *The moments of the probability measure $\mu(x; a, b, c)$ are the Al-Salam-Chihara polynomials $\{p_n(\cos\theta; t_1, t_2)\}$ where a, b, c and t_1, t_2, θ are related through (3.9).*

It is important to observe that Theorem 3.1 is equivalent to the q-integral representation (see [12, p. 19])

$$(3.10)\quad p_n(\cos\theta; t_1, t_2) = \frac{(t_1e^{i\theta}, t_1e^{-i\theta}, t_2e^{i\theta}, t_2e^{-i\theta};q)_\infty}{(1-q)t_1e^{i\theta}(q, e^{-2i\theta}, qe^{2i\theta};q)_\infty} \int_{t_1e^{-i\theta}}^{t_1e^{i\theta}} \frac{(qxe^{i\theta}/t_1, qxe^{-i\theta}/t_1 : q)_\infty}{(x, t_2x/t_1;q)_\infty}\, x^n\, d_qx.$$

Let us now integrate certain functions against the measure $\mu(x; a, b, c)$ with respect to which the big q-Jacobi polynomials are orthogonal. The simplest function is the q-analogue of the binomial theorem

$$(3.11)\quad f(x) = \frac{(\lambda x;q)_\infty}{(\mu x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(\lambda/\mu;q)_n}{(q;q)_n} (\mu x)^n.$$

On one hand we have

$$\mathcal{L}_{a,b,c} f = \sum_{n=0}^{\infty} \frac{(\lambda/\mu; q)_n}{(q;q)_n} \mu^n p_n(\cos\theta; t_1, t_2).$$

On the other hand we can explicitly evaluate $\mathcal{L}_{a,b,c} f$. This implies

$$\begin{aligned}
(3.12) \qquad & \sum_{n=0}^{\infty} \frac{(\lambda/\mu; q)_n}{(q;q)_n} p_n(\cos\theta; t_1, t_2)\, \mu^n \\
&= \int_{-\infty}^{\infty} \frac{(\lambda x; q)_\infty}{(\mu x; q)_\infty}\, d\mu(x; a, b, c) \\
&= \frac{(aq, bq, cq, abq/c; q)_\infty}{(c/a, abq/c, abq^2; q)_\infty} \left\{ \frac{(\lambda aq, abq/c; q)_\infty}{(\mu aq, aq, abq/c; q)_\infty} {}_3\phi_2\left(\begin{matrix} a\mu q, aq, abq/c \\ \lambda aq, aq/c \end{matrix} \,\middle|\, q, q \right) \right. \\
& \left. - \frac{c\,(\lambda cq, cq/a; q)_\infty}{a(\mu cq, cq, bq; q)_\infty} {}_3\phi_2\left(\begin{matrix} q\mu c, cq, bq \\ \lambda cq, cq/a \end{matrix} \,\middle|\, q, q \right) \right\}.
\end{aligned}$$

We next combine the above ${}_3\phi_2$'s into a single ${}_3\phi_2$ by applying the transformation [12, (4.3.1), p. 63]

$$\begin{aligned}
(3.13) \qquad & {}_3\phi_2\left(\begin{matrix} A, B, C \\ D, E \end{matrix} \,\middle|\, q, \frac{DE}{ABC} \right) = \frac{(E/B, E/C; q)_\infty}{(E, E/BC; q)_\infty} {}_3\phi_2\left(\begin{matrix} D/A, B, C \\ D, BCq/E \end{matrix} \,\middle|\, q, q \right) \\
& + \frac{(D/A, B, C, DE/BC; q)_\infty}{(D, E, BC/E, DE/ABC; q)_\infty} {}_3\phi_2\left(\begin{matrix} E/B, E/C, DE/ABC \\ DE/BC, aE/BC \end{matrix} \,\middle|\, q, q \right).
\end{aligned}$$

With the parameter identification in (3.13) as

$$(3.14) \qquad D/A = abq/c, \quad B = aq, \quad C = a\mu q, \quad BC/E = a/c,$$

so that

$$(3.15) \qquad A = \lambda c/b, \quad B = aq, \quad C = a\mu q, \quad D = \lambda aq, \quad E = a\mu q^2 c,$$

we obtain

$$(3.16) \qquad \frac{(\lambda aq, aq/c; q)_\infty}{(aq, \mu aq, abq/c; q)_\infty} {}_3\phi_2\left(\begin{matrix} abq/c, aq, aq\mu \\ \lambda aq, aq/c \end{matrix} \,\middle|\, q, q \right)$$

$$
+\frac{(\lambda qc, c/a;q)_\infty}{(1-a/c)(cq,\mu qc, bq;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}\mu qc, cq, bq\\ \\ \lambda qc, cq/a\end{matrix}\,\middle|\, q,q\right)
$$

$$
=\frac{(c/a,\mu caq^2,\lambda aq, qa/c;q)_\infty}{(\mu qc, cq, aq,\mu aq, abq/c;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\lambda c/b, aq,\mu aq\\ \\ \lambda aq, a\mu cq^2\end{matrix}\,\middle|\, q,qb\right).
$$

Thus we have established the generating function

$$
(3.17)\qquad \sum_{n=0}^{\infty}\frac{(\lambda/\mu;q)_n}{(q;q)_n}\,\mu^n\,p_n\left(\cos\theta;t_1,t_2\right)=\frac{(bq,\mu acq^2,\lambda aq;q)_\infty}{(abq^2,\mu qc,\mu aq;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\lambda c/b, aq, a\mu q\\ \\ \lambda aq, a\mu cq^2\end{matrix}\,\middle|\, q,bq\right)
$$

$$
=\frac{(t_2e^{-i\theta},\mu t_1^2,\lambda t_1e^{i\theta};q)_\infty}{(t_1t_2,\mu t_1e^{-i\theta},\mu t_1e^{i\theta};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\lambda t_1/t_2, t_1e^{i\theta},\mu t_1e^{i\theta}\\ \\ \lambda t_1e^{i\theta},\mu t_1^2\end{matrix}\,\middle|\, q,t_2e^{-i\theta}\right).
$$

Formula (3.17) is in Suslov's unpublished notes [27] and we acknowledge his priority. The special case $\lambda = tt_2$ and $\mu = t/t_1$ leads to the known generating function [2], [5], [8]

$$
(3.18)\qquad \sum_{n=0}^{\infty}\frac{(t_1t_2;q)_n t^n}{(q;q)_n t_1^n}\,p_n\left(\cos\theta;t_1,t_2\right)=\frac{(tt_1,tt_2;q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty},
$$

via the q-analogue of Gauss's theorem, [12, (II.8)].

It is worth mentioning that the special case $\lambda = \mu q = tq$ of (3.17) is

$$
(3.19)\qquad \sum_{n=0}^{\infty}p_n\left(\cos\theta;t_1,t_2\right)t^n \;=\; \frac{(t_2e^{-i\theta},tt_1^2;q)_\infty}{(1-tt_1e^{i\theta})(t_1t_2,tt_1e^{-i\theta};q)_\infty}
\times{}_3\phi_2\left(\begin{matrix}qtt_1/t_2, t_1e^{i\theta}, tt_1e^{i\theta}\\ \\ qtt_1e^{i\theta}, tt_1^2\end{matrix}\,\middle|\, q,t_2e^{-i\theta}\right).
$$

Since p_n is the n th moment of $\mu(x;a,b,c)$ and μ has a compact support, then

$$
(3.20)\qquad \sum_{n=0}^{\infty}z^{-n-1}\mu_n=\int_{-\infty}^{\infty}\frac{d\mu(u;a,b,c)}{z-u}\qquad z\notin supp\{\mu\}.
$$

This gives a direct evaluation of the Stieltjes function (the right-hand side of (3.20)) for the big q-Jacobi polynomials. The result is

$$
(3.21)\qquad \int_{-\infty}^{\infty}\frac{d\mu(u;a,b,c)}{z-u} \;=\; \frac{(t_2e^{-i\theta},t_1^2/z;q)_\infty}{(z-t_1e^{i\theta})(t_1t_2,t_1e^{-i\theta}/z;q)_\infty}
$$

$$\times {}_3\phi_2\left(\begin{matrix} qt_1/t_2z, t_1e^{i\theta}, t_1e^{i\theta}/z \\ \\ qt_1e^{i\theta}/z, t_1^2/z \end{matrix}\,\middle|\, q, t_2e^{-i\theta}\right).$$

The relationship (3.21) is in [14] but the present proof is much more elementary.

The Al-Salam-Carlitz polynomials $\{V_n^{\alpha}(x;q)\}$ have the generating function

$$\frac{(xw;q)_\infty}{(w,\alpha w;q)_\infty} = \sum_{n=0}^{\infty}\frac{(-w)^n}{(q;q)_n}q^{n(n-1)/2}V_n^{\alpha}(x;q). \tag{3.22}$$

Clearly we can identify the right-hand sides of (3.22) and (3.18) by choosing $t_2 = 0$ and adjusting the remaining parameters. The V_n^{α}'s are also moments and their multilinear generating functions will follow as special cases of this work.

4. Poisson Kernel and Multilinear Generating Functions. In this section we use the results of §3 to find the Poisson kernel and multilinear generating functions for the Al-Salam-Chihara polynomials.

We evaluate the action of $\mathcal{L}_{a,b,c}$ on $x^k(\lambda x;q)_\infty/(\mu x;q)_\infty$ in two different ways and generalize (3.17). The calculations are similar and the result is

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(\lambda/\mu;q)_n}{(q;q)_n}\,p_{n+k}\left(\cos\theta;t_1,t_2\right)\mu^n \\ &= \frac{(aq,bq,cq,abq/c;q)_\infty}{(c/a,abq/c,abq^2;q)_\infty}\left\{\frac{a^kq^k\left(\lambda aq,abq/c;q\right)_\infty}{(\mu aq,aq,abq/c;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} a\mu q, aq, abq/c \\ \\ \lambda aq, aq/c\end{matrix}\,\middle|\, q, q^{k+1}\right)\right. \\ &\left.-\frac{c^{k+1}q^k\left(\lambda cq,cq/a;q\right)_\infty}{a(\mu cq,cq,bq;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} q\mu c, cq, bq \\ \\ \lambda cq, cq/a\end{matrix}\,\middle|\, q, q^{k+1}\right)\right\}.\end{aligned} \tag{4.1}$$

Recall that the relation between a, b, c and $t_1, t_2, e^{i\theta}$ is as in (3.9). We cannot simplify the above combination because we do not have an suitable analogue of (3.16).

We now come to the Poisson kernel of the Al-Salam-Chihara polynomials. Set

$$\alpha = s_1e^{i\phi}/q, \quad \beta = s_2e^{-i\phi}/q, \quad \gamma = s_1e^{-i\phi}/q. \tag{4.2}$$

Theorem 4.1 *The Al-Salam-Chihara polynomials have the bilinear generating function*

$$
\begin{aligned}
(4.3)\qquad & \sum_{n=0}^{\infty} \frac{(t_1t_2;q)_n}{(q;q)_n t_1^n} t^n p_n(\cos\theta;t_1,t_2)p_n(\cos\phi;s_1,s_2) \\
&= \frac{(s_1e^{-i\phi}, s_2e^{-i\phi}, tt_1s_1e^{i\phi}, tt_2s_1e^{i\phi};q)_\infty}{(s_1s_2, e^{-2i\phi}, ts_1e^{i(\theta+\phi)}, ts_1e^{i(\phi-\theta)};q)_\infty} \\
&\times {}_4\phi_3\left(\left.\begin{matrix} s_1e^{i\phi}, s_2e^{i\phi}, ts_1e^{i(\theta+\phi)}, ts_1e^{i(\phi-\theta)} \\ qe^{2i\phi}, tt_1s_1e^{i\phi}, tt_2s_1e^{i\phi}\end{matrix}\right| q,q\right) \\
&+ \frac{(s_1e^{i\phi}, s_2e^{i\phi}, tt_1s_1e^{-i\phi}, tt_2s_1e^{-i\phi};q)_\infty}{(s_1s_2, e^{2i\phi}, ts_1e^{i(\theta-\phi)}, ts_1e^{-i(\theta+\phi)};q)_\infty} \\
&\times {}_4\phi_3\left(\left.\begin{matrix} s_1e^{-i\phi}, s_2e^{-i\phi}, ts_1e^{-i(\theta+\phi)}, ts_1e^{i(\theta-\phi)} \\ qe^{-2i\phi}, tt_1s_1e^{-i\phi}, tt_2s_1e^{-i\phi}\end{matrix}\right| q,q\right).
\end{aligned}
$$

Proof. In (3.18) replace t by tx and apply $\mathcal{L}_{\alpha,\beta,\gamma}$ to both sides of the resulting equality. The result is

$$
\begin{aligned}
(4.4)\qquad & \sum_{n=0}^{\infty} \frac{(t_1t_2;q)_n}{(q;q)_n t_1^n} t^n p_n(\cos\theta;t_1,t_2)p_n(\cos\phi;s_1,s_2) \\
&= \frac{(s_1e^{-i\phi}, s_2e^{-i\phi}, tt_1s_1e^{i\phi}, tt_2s_1e^{i\phi};q)_\infty}{(s_1s_2, e^{-2i\phi}, ts_1e^{i(\theta+\phi)}, ts_1e^{i(\phi-\theta)};q)_\infty} \\
&\times {}_4\phi_3\left(\left.\begin{matrix} s_1e^{i\phi}, s_2e^{i\phi}, ts_1e^{i(\theta+\phi)}, ts_1e^{i(\phi-\theta)} \\ qe^{2i\phi}, tt_1s_1e^{i\phi}, tt_2s_1e^{i\phi}\end{matrix}\right| q,q\right) \\
&- \frac{e^{-2i\phi}(s_1e^{i\phi}, s_2e^{i\phi}, tt_1s_1e^{-i\phi}, tt_2s_1e^{-i\phi};q)_\infty}{(1-e^{-2i\phi})(s_1s_2, qe^{2i\phi}, ts_1e^{i(\theta-\phi)}, ts_1e^{-i(\theta+\phi)};q)_\infty} \\
&\times {}_4\phi_3\left(\left.\begin{matrix} s_1e^{-i\phi}, s_2e^{-i\phi}, ts_1e^{-i(\theta+\phi)}, ts_1e^{i(\theta-\phi)} \\ qe^{-2i\phi}, tt_1s_1e^{-i\phi}, tt_2s_1e^{-i\phi}\end{matrix}\right| q,q\right).
\end{aligned}
$$

After some simplification this implies (4.3) and the proof is complete.

In this generality the ${}_4\phi_3$'s in (4.3) can not be combined into a single basic hypergeometric function because they are not necessarily balanced. Recall that a ${}_{r+1}\phi_r$ is balanced if the product of the numerator parameters times q equals the product of the denominator parameters. On the other hand it is easy to see that the ${}_4\phi_3$'s are balanced if and only if $s_1s_2 = t_1t_2$.

Theorem 4.2 *If $s_1s_2 = t_1t_2$ then*

$$\sum_{n=0}^{\infty} \frac{(t_1t_2;q)_n}{(q;q)_n t_1^n} t^n p_n(\cos\theta; t_1, t_2)\, p_n(\cos\phi; s_1, s_2) \tag{4.5}$$

$$= \frac{(tt_1s_1e^{i\phi}, tt_2s_1e^{i\phi}, ts_1s_2e^{i\theta}, ts_1^2e^{i\theta}; q)_\infty}{(tt_1t_2s_1e^{i(\theta+\phi)}, ts_1e^{i(\theta-\phi)}, ts_1e^{i(\theta+\phi)}, ts_1e^{i(\phi-\theta)}; q)_\infty}$$

$$\times {}_8\phi_7\left(\begin{matrix} s_1^2ts_2e^{i(\theta+\phi)}/q, s_1\sqrt{qts_2}e^{i(\theta+\phi)/2}, -s_1\sqrt{qts_2}e^{i(\theta+\phi)/2}, t_2e^{i\theta}, t_1e^{i\theta}, \\ s_1\sqrt{ts_2/q}e^{i(\theta+\phi)/2}, -s_1\sqrt{ts_2/q}e^{i(\theta+\phi)/2}, tt_1s_1e^{i\phi}, tt_2s_1e^{i\phi}, \end{matrix}\right.$$

$$\left.\begin{matrix} s_1e^{i\phi}, s_2e^{i\phi}, ts_1e^{i(\theta+\phi)} \\ ts_1s_2e^{i\theta}, s_1^2te^{i\theta}, s_1s_2 \end{matrix}\;\middle|\; q, ts_1e^{-i(\theta+\phi)}\right).$$

Proof. Formula (2.10.10) in [12, p. 42] is

$${}_8\phi_7\left(\begin{matrix} A, q\sqrt{A}, -q\sqrt{A}, B, C, D, E, F \\ \sqrt{A}, -\sqrt{A}, Aq/B, Aq/C, Aq/D, Aq/E, Aq/F \end{matrix}\;\middle|\; q, \frac{A^2q^2}{BCDEF}\right) \tag{4.6}$$

$$= \frac{(Aq, Aq/DE, Aq/DF, Aq/EF; q)_\infty}{(Aq/D, Aq/E, Aq/F, Aq/DEF; q; q)_\infty}\, {}_4\phi_3\left(\begin{matrix} Aq/BC, D, E, F \\ Aq/B, Aq/C, DEF/A \end{matrix}\;\middle|\; q, q\right)$$

$$+ \frac{(Aq, Aq/BC, D, E, F, A^2q^2/BDEF, A^2q^2/CDEF; q)_\infty}{(Aq/B, Aq/C, Aq/D, Aq/E, Aq/F, A^2q^2/BCDEF, DEF/Aq; q)_\infty}$$

$$\times {}_4\phi_3\left(\begin{matrix} Aq/DE, Aq/Df, Aq/EF, A^2q^2/BCDEF \\ A^2q^2/BDEF, A^2q^2/CDEF, Aq^2/DEF \end{matrix}\;\middle|\; q, q\right).$$

Choose

$$A = s_1^2ts_2e^{i(\theta+\phi)}/q, \quad B = t_2e^{i\theta}, \quad C = t_1e^{i\theta}, \quad D = s_1e^{i\phi}, \quad E = s_2e^{i\phi}, \quad F = ts_1e^{i(\theta+\phi)}$$

in (4.6) to replace the combination of ${}_4\phi_3$'s in (4.3) by an ${}_8\phi_7$ taking into account that $t_1t_2 = s_1s_2$. The result is (4.5) and the proof is complete.

Theorem 4.2, in its present form, was proved independently in [6] and [16]. When $t_1 = s_1$ and $t_2 = s_2$ the generating function (4.5) becomes the Poisson kernel for the Al-Salam-Chihara polynomials since their orthogonality relation is

$$\int_0^\pi p_m(\cos\theta; t_1, t_2) p_n(\cos\theta; t_1, t_2)\, \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty\, d\theta}{(t_1e^{i\theta}, t_1e^{-i\theta}, t_2e^{i\theta}, t_2e^{-i\theta}; q)_\infty} \tag{4.7}$$

$$= \frac{2\pi (q;q)_n \, t_1^{2n}}{(q, t_1t_2;q)_\infty \, (t_1t_2;q)_n} \, \delta_{m,n}.$$

One can generalize Theorem 4.1 by multiplying (3.18) by x^k, replace x by xt then act with $\mathcal{L}_{a,b,c}$. This leads to our next result.

Theorem 4.3 *We have*

$$\sum_{n=0}^{\infty} \frac{(t_1t_2;q)_n}{(q;q)_n t_1^n} t^n p_n(\cos\theta; t_1, t_2) p_{n+k}(\cos\phi; s_1, s_2) \tag{4.8}$$

$$= (s_1e^{i\phi})^k \frac{(s_1e^{-i\phi}, s_2e^{-i\phi}, tt_1s_1e^{i\phi}, tt_2s_1e^{i\phi};q)_\infty}{(s_1s_2, e^{-2i\phi}, ts_1e^{i(\theta+\phi)}, ts_1e^{i(\phi-\theta)};q)_\infty}$$

$$\times {}_4\phi_3\left(\begin{matrix} s_1e^{i\phi}, s_2e^{i\phi}, ts_1e^{i(\theta+\phi)}, ts_1e^{i(\phi-\theta)} \\ qe^{2i\phi}, tt_1s_1e^{i\phi}, tt_2s_1e^{i\phi} \end{matrix} \middle| q, q^{k+1} \right)$$

$$+ (s_1e^{-i\phi})^k \frac{(s_1e^{i\phi}, s_2e^{i\phi}, tt_1s_1e^{-i\phi}, tt_2s_1e^{-i\phi};q)_\infty}{(s_1s_2, e^{2i\phi}, ts_1e^{i(\theta-\phi)}, ts_1e^{-i(\theta+\phi)};q)_\infty}$$

$$\times {}_4\phi_3\left(\begin{matrix} s_1e^{-i\phi}, s_2e^{-i\phi}, ts_1e^{-i(\theta+\phi)}, ts_1e^{i(\theta-\phi)} \\ qe^{-2i\phi}, tt_1s_1e^{-i\phi}, tt_2s_1e^{-i\phi} \end{matrix} \middle| q, q^{k+1} \right).$$

Note that (4.5) and (4.8) will look more symmetric if we replace t by t/s_1.

Our next result is a multilinear generating function for the Al-Salam-Chihara polynomials and was motivated by our earlier result in [18] for continuous q-Hermite polynomials.

Theorem 4.4 *If $aq := t_1e^{i\theta}$, $bq := t_2e^{-i\theta}$, $cq := t_1e^{-i\theta}$ then the Al-Salam-Chihara polynomials have the multilinear generating function*

$$\sum_{n_1,\cdots,n_m=0}^{\infty} \prod_{j=1}^{m} \left\{ \lambda_j^{n_j} \frac{(a_{2j-1}a_{2j};q)_{n_j}}{(q;q)_{n_j}(a_{2j-1})^{n_j}} \, p_{n_j}(\cos\theta_j, a_{2j-1}, a_{2j}) \right\} p_{n_1+\cdots+n_m}(\cos\theta, t_1, t_2) \tag{4.9}$$

$$= \frac{(bq, cq;q)_\infty}{(c/a, abq^2;q)_\infty} \prod_{j=1}^{m} \left\{ \frac{(aq\lambda_j a_{2j-1}, aq\lambda_j a_{2j};q)_\infty}{(aq\lambda_j e^{i\theta_j}, aq\lambda_j e^{-i\theta_j};q)_\infty} \right\}$$

$$\times {}_{2m+2}\phi_{2m+1}\left(\begin{matrix} aq, abq/c, aq\lambda_1e^{i\theta_1}, aq\lambda_1e^{-i\theta_1}, \cdots, aq\lambda_me^{i\theta_m}, aq\lambda_me^{-i\theta_m} \\ aq/c, aq\lambda_1a_1, aq\lambda_1a_2, \cdots, aq\lambda_ma_{2m-1}, aq\lambda_ma_{2m} \end{matrix} \middle| q, q \right)$$

$$+ \frac{(aq, abq/c;q)_\infty}{(a/c, abq^2;q)_\infty} \prod_{j=1}^{m} \left\{ \frac{(cq\lambda_j a_{2j-1}, cq\lambda_j a_{2j};q)_\infty}{(cq\lambda_j e^{i\theta_j}, cq\lambda_j e^{-i\theta_j};q)_\infty} \right\}$$

$$\times {}_{2m+2}\phi_{2m+1}\left(\begin{matrix} bq, cq, cq\lambda_1e^{i\theta_1}, cq\lambda_1e^{-i\theta_1}, \cdots, cq\lambda_me^{i\theta_m}, cq\lambda_me^{-i\theta_m} \\ cq/a, cq\lambda_1a_1, cq\lambda_1a_2, \cdots, cq\lambda_ma_{2m-1}, cq\lambda_ma_{2m} \end{matrix} \middle| q, q \right).$$

Proof. Rewrite (3.18) in the form

$$(4.10)\qquad \frac{(a_{2j-1}\lambda_j x, a_{2j}\lambda_j x; q)_\infty}{(\lambda_j x e^{i\theta_j}, \lambda_j x e^{-i\theta_j}; q)_\infty} = \sum_{n_j=0}^{\infty} \frac{(a_{2j-1}\, a_{2j}; q)_{n_j}}{(q;q)_{n_j}(a_{2j-1})^{n_j}}\, p_{n_j}(\cos\theta_j, a_{2j-1}, a_{2j})\lambda_j^{n_j} x^{n_j}.$$

Multiply (4.10) for $j = 1, 2, \cdots, m$ then apply $\mathcal{L}_{a,b,c}$. The result is (4.9).

Note that we may replace x in (4.10) by x^{c_j} then apply $\mathcal{L}_{a,b,c}$. The result will replace

$$p_{n_1+\cdots+n_m}(\cos\theta, t_1, t_2)$$

by

$$p_{c_1 n_1+\cdots+c_m n_m}(\cos\theta, t_1, t_2).$$

The ϕ functions will get more involved and become q-analogues of the Wright functions but we do not really gain anything with this degree of generality. Another variation is to multiply by x^k before acting with $\mathcal{L}_{a,b,c}$. Here again there is nothing conceptually new.

Finally we mention a similar case when the number of a's is odd. In this case we multiply (4.10) for $j = 1, 2, \cdots, m$, multiply the result by

$$\frac{(a_{2m+1}\lambda_{m+1} x; q)_\infty}{(\lambda_{m+1} x; q)_\infty} = \sum_{n_{m+1}=0}^{\infty} \frac{(a_{2m+1}; q)_{n_{m+1}}}{(q;q)_{n_{m+1}}} \lambda_{m+1}^{n_{m+1}} x^{n_{m+1}}$$

then apply $\mathcal{L}_{a,b,c}$.

5. Continuous q-Ultraspherical Polynomials. The continuous q-ultraspherical polynomials $\{C_n(x;\beta|q)\}$ have the generating function [12, (7.4.1)]

$$(5.1)\qquad \sum_{n=0}^{\infty} C_n(\cos\theta;\beta|q)t^n = \frac{(t\beta e^{i\theta}, t\beta e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty}.$$

Their orthogonality relation is [12, (7.4.15)]

$$(5.2)\qquad \int_0^\pi C_m(\cos\theta;\beta|q)C_n(\cos\theta;\beta|q)\frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty}\, d\theta = \frac{2\pi(\beta, \beta q; q)_\infty}{(q, \beta^2; q)_\infty}\frac{(\beta^2;q)_n(1-\beta)}{(q;q)_n(1-\beta q^n)}\delta_{m,n}.$$

Comparing (3.18) and (5.1) we find

$$(5.3)\qquad C_n(\cos\theta;\beta|q) = \frac{(\beta^2;q)_n}{(q;q)_n}\, \beta^{-n} e^{-in\theta}\, p_n(\cos\theta; \beta e^{i\theta}, \beta e^{-i\theta}).$$

This means that we can translate every generating function in §3 and §4 to a generating function for the C_n's. For example (3.19) becomes

$$\text{(5.4)} \qquad \sum_{n=0}^{\infty} \frac{(q;q)_n}{(\beta^2;q)_n} C_n(\cos\theta;\beta|q)t^n = \frac{(\beta e^{-2i\theta}, \beta t e^{i\theta};q)_\infty}{(1-te^{i\theta})(\beta^2, te^{-i\theta};q)_\infty} \times {}_3\phi_2\left(\begin{matrix} qte^{i\theta}/\beta, \beta e^{2i\theta}, te^{i\theta} \\ qte^{i\theta}, \beta t e^{i\theta}\end{matrix} \middle| q, \beta e^{-2i\theta}\right).$$

Another example is (4.3) with

$$t_1 = \beta e^{i\theta}, \quad t_2 = \beta e^{-i\theta}, \quad s_1 = \beta_1 e^{i\phi}, \quad s_2 = \beta_1 e^{-i\phi},$$

which yields

$$\begin{aligned} \text{(5.5)} \qquad & \sum_{n=0}^{\infty} \frac{(q;q)_n}{(\beta_1^2;q)_n} C_n(\cos\theta;\beta|q) C_n(\cos\phi;\beta_1|q)t^n \\ &= \frac{(\beta_1, \beta_1 e^{-2i\phi}, t\beta e^{i(\theta+\phi)}, t\beta e^{i(\phi-\theta)};q)_\infty}{(\beta_1^2, e^{-2i\phi}, te^{i(\theta+\phi)}, te^{i(\phi-\theta)};q)_\infty} \\ &\times {}_4\phi_3\left(\begin{matrix} \beta_1 e^{2i\phi}, \beta_1, te^{i(\theta+\phi)}, te^{i(\phi-\theta)} \\ qe^{2i\phi}, \beta t e^{i(\theta+\phi)}, \beta t e^{i(\phi-\theta)}\end{matrix} \middle| q, q\right) \\ &+ \frac{(\beta_1, \beta_1 e^{2i\phi}, t\beta e^{i(\theta-\phi)}, t\beta e^{-i(\theta+\phi)};q)_\infty}{(\beta_1^2, e^{2i\phi}, te^{i(\theta-\phi)}, te^{-i(\theta+\phi)};q)_\infty} \\ &\times {}_4\phi_3\left(\begin{matrix} \beta_1 e^{-2i\phi}, \beta_1, te^{i(\theta-\phi)}, te^{-i(\theta+\phi)} \\ qe^{-2i\phi}, \beta t e^{i(\theta-\phi)}, \beta t e^{-i(\theta+\phi)}\end{matrix} \middle| q, q\right). \end{aligned}$$

When $\beta_1 = \beta$ the ${}_4\phi_3$'s in (5.5) combine into an ${}_8\phi_7$ and the result is

$$\begin{aligned} \text{(5.6)} \qquad & \sum_{n=0}^{\infty} \frac{(q;q)_n}{(\beta^2;q)_n} C_n(\cos\theta;\beta|q) C_n(\cos\phi;\beta|q)t^n \\ &= \frac{(t\beta e^{i(\theta+\phi)}, t\beta e^{i(\phi-\theta)}, t\beta e^{i(\theta-\phi)}, t\beta e^{i(\theta+\phi)};q)_\infty}{(t\beta^2 e^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{i(\theta+\phi)}, te^{i(\phi-\theta)};q)_\infty} \\ &\times {}_8\phi_7\left(\begin{matrix} \beta^2 te^{i(\theta+\phi)}/q, \beta\sqrt{qt}e^{i(\theta+\phi)/2}, -\beta\sqrt{qt}e^{i(\theta+\phi)/2}, \beta, \beta e^{2i\theta}, \beta e^{2i\phi}, \beta, te^{i(\theta+\phi)} \\ \beta\sqrt{t/q}e^{i(\theta+\phi)/2}, -\beta\sqrt{t/q}e^{i(\theta+\phi)/2}, \beta t e^{i(\theta+\phi)}, \beta t e^{i(\phi-\theta)}, \beta t e^{i(\theta-\phi)}, \beta t e^{i(\theta+\phi)}, \beta^2 \end{matrix} \middle| q, te^{-i(\theta+\phi)}\right). \end{aligned}$$

The kernel (5.6) was evaluated by Gasper and Rahman in [13]. Their form of the right-hand side is different but one can transform our answer to theirs by using the two term transformation (2.10.1) in [12] connecting two very well-poised ${}_8\phi_7$'s with the choices

$$a = \beta^2 t e^{i(\theta+\phi)}, \quad b = e = \beta, \quad c = \beta e^{2i\theta}, \quad d = \beta e^{2i\phi}, \quad f = t e^{i(\theta+\phi)}.$$

Note that (5.5) is a special case of (4.5) because (4.5) has three free parameters among t_1, t_2, s_1, s_2 but (5.5) has only one free parameter, the parameter β.

We now introduce another measure whose moments are the continuous q-ultraspherical polynomials.

Theorem 5.1 *Let*

$$\begin{aligned}(5.7)\qquad \nu(x;\beta,u) \;=\;& \frac{(\beta, \beta u^{-2}; q)_\infty}{(q, u^{-2}; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/\beta, qu^2/\beta; q)_n}{(q, qu^2; q)_n}\, \beta^{2n} \epsilon_{q^n u} \\ &+ \frac{(\beta, \beta u^2; q)_\infty}{(q, u^2; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/\beta, qu^{-2}/\beta; q)_n}{(q, qu^{-2}; q)_n}\, \beta^{2n} \epsilon_{q^n/u}.\end{aligned}$$

Then

$$(5.8)\qquad \int_{-\infty}^{\infty} y^n d\nu(y;\beta,u) = \frac{u^n (\beta u^{-2}; q)_n}{(q;q)_n}\; {}_2\phi_1\left(\left. \begin{matrix} q^{-n}, \beta u^2 \\ q^{1-n} u^2/\beta \end{matrix} \right| q, q/\beta \right).$$

Proof. It is clear that the left-hand side of (5.8) is

$$\begin{aligned}(5.9)\qquad & u^n \frac{(\beta, \beta u^{-2}; q)_\infty}{(q, u^{-2}; q)_\infty}\; {}_2\phi_1\left(\left. \begin{matrix} q/\beta, qu^2/\beta \\ qu^2 \end{matrix} \right| q, \beta^2 q^n \right) \\ &+ u^{-n} \frac{(\beta, \beta u^2; q)_\infty}{(q, u^2; q)_\infty}\; {}_2\phi_1\left(\left. \begin{matrix} q/\beta, qu^{-2}/\beta \\ qu^{-2} \end{matrix} \right| q, \beta^2 q^n \right).\end{aligned}$$

In (3.3) we make the choices

$$(5.10)\qquad A = q/\beta, \quad B = qu^2/\beta, \quad C = qu^2, \quad Z = q^n \beta^2,$$

and the expression in (5.9) simplifies to the right-hand side of (5.8).

Corollary 5.2 *We have*

$$(5.11)\qquad \int_{-\infty}^{\infty} y^n \, d\nu(y;\beta,e^{i\theta}) = C_n(\cos\theta;\beta|q).$$

Proof. Denote the right-hand side of (5.8) by $r_n(\theta)$. Then

$$\sum_{n=0}^{\infty} r_n(\theta)\, t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\beta u^{-2};q)_n}{(q;q)_n} u^n t^n \frac{(q^{-n},\beta u^2;q)_{n-k}}{(q^{1-n}u^2/\beta,q;q)_{n-k}} \left(\frac{q}{\beta}\right)^{n-k}.$$

$$\begin{aligned}\sum_{n=0}^{\infty} r_n(\theta)\, t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\beta u^2;q)_{n-k}u^{k-n}}{(q;q)_{n-k}} \frac{(\beta u^{-2};q)_k}{(q;q)_k}\, t^n u^k \\ &= \left[\sum_{k=0}^{\infty} \frac{(\beta u^{-2};q)_k}{(q;q)_k} u^k t^k\right] \left[\sum_{n=0}^{\infty} \frac{(\beta u^2;q)_n u^{k-n}}{(q;q)_n}\, t^n u^{-n}\right].\end{aligned}$$

Using the q-binomial theorem (3.10) we get

$$\sum_{n=0}^{\infty} r_n(\theta)\, t^n = \frac{(\beta t u^{-1},\beta t u;q)_\infty}{(tu,tu^{-1};q)_\infty},$$

and the corollary follows from (5.1).

We note that (7.4.2) and (7.4.4) in [12] show that the expression (5.9) equals $C_n(\cos\theta;\beta|q)$ which also equals the right-hand side of (5.8) with $u = e^{i\theta}$. The proof given here makes this work as self-contained as possible.

Let

$$\mathcal{L}_{\beta,\theta}(f) = \int_{-\infty}^{\infty} f(y)d\nu(y;\beta,e^{i\theta}). \tag{5.12}$$

Our next result is a bilinear generating function for the continuous q-ultraspherical polynomials. It follows from (5.1) when we replace β, θ and t by β_1, ϕ and yt, respectively, then apply $\mathcal{L}_{\beta,\theta}$.

Theorem 5.3 *The continuous q-ultraspherical polynomials satisfy*

$$\begin{aligned}&\sum_{n=0}^{\infty} C_n(\cos\theta;\beta|q)C_n(\cos\phi;\beta_1|q)t^n \\ &= \frac{(\beta,\beta e^{-2i\theta},\beta_1 t e^{i(\theta+\phi)},\beta_1 t e^{i(\theta-\phi)};q)_\infty}{(q,e^{-2i\theta},te^{i(\theta+\phi)},te^{i(\theta-\phi)};q)_\infty} \\ &\quad\times {}_4\phi_3\left(\left.\begin{matrix} q/\beta, qe^{2i\theta}/\beta, te^{i(\theta+\phi)}, te^{i(\theta-\phi)} \\ qe^{2i\theta}, \beta_1 te^{i(\theta+\phi)}, \beta_1 te^{i(\theta-\phi)}\end{matrix}\,\right| q,\beta^2\right) \\ &+\frac{(\beta,\beta e^{2i\theta},\beta_1 t e^{i(\phi-\theta)},\beta_1 t e^{-i(\theta+\phi)};q)_\infty}{(q,e^{2i\theta},te^{i(\phi-\theta)},te^{-i(\theta+\phi)};q)_\infty} \\ &\quad\times {}_4\phi_3\left(\left.\begin{matrix} q/\beta, qe^{-2i\theta}/\beta, te^{i(\phi-\theta)}, te^{-i(\theta+\phi)} \\ qe^{-2i\theta}, \beta_1 te^{i(\phi-\theta)}, \beta_1 te^{-i(\theta+\phi)}\end{matrix}\,\right| q,\beta^2\right).\end{aligned} \tag{5.13}$$

Theorem 5.4 *The continuous q-ultraspherical polynomials have the bilinear generating function*

$$\sum_{n=0}^{\infty} C_{n+k}(\cos\theta;\beta|q)C_n(\cos\phi;\beta_1|q)t^n \tag{5.14}$$
$$= e^{ik\theta}\,\frac{(\beta,\beta e^{-2i\theta},\beta_1 te^{i(\theta+\phi)},\beta_1 te^{i(\theta-\phi)};q)_\infty}{(q,e^{-2i\theta},te^{i(\theta+\phi)},te^{i(\theta-\phi)};q)_\infty}$$
$$\times {}_4\phi_3\left(\left.\begin{matrix} q/\beta, qe^{2i\theta}/\beta, te^{i(\theta+\phi)}, te^{i(\theta-\phi)} \\ qe^{2i\theta}, \beta_1 te^{i(\theta+\phi)}, \beta_1 te^{i(\theta-\phi)}\end{matrix}\right| q,\beta^2q^k\right)$$
$$+e^{-ik\theta}\,\frac{(\beta,\beta e^{2i\theta},\beta_1 te^{i(\phi-\theta)},\beta_1 te^{-i(\theta+\phi)};q)_\infty}{(q,e^{2i\theta},te^{i(\phi-\theta)},te^{-i(\theta+\phi)};q)_\infty}$$
$$\times {}_4\phi_3\left(\left.\begin{matrix} q/\beta, qe^{-2i\theta}/\beta, te^{i(\phi-\theta)}, te^{-i(\theta+\phi)} \\ qe^{-2i\theta}, \beta_1 te^{i(\phi-\theta)}, \beta_1 te^{-i(\theta+\phi)}\end{matrix}\right| q,\beta^2q^k\right).$$

Proof. Multiply (5.1) by y^k then apply the procedure that led to (5.13).

It is clear that the left-hand side of (5.13) is symmetric if the ordered pairs (θ,β) and (ϕ,β_1) are interchanged. Imposing this symmetry on the right-hand side of (5.13) leads to the surprising transformation formula which is of independent interest. We shall state this as a theorem.

Theorem 5.5 *The following combination of ${}_4\phi_3$'s is invariant under the exchange* $(\theta,\beta)\to(\phi,\beta_1)$

$$\frac{(\beta,\beta e^{-2i\theta},\beta_1 te^{i(\theta+\phi)},\ b_1 te^{i(\theta-\phi)};q)_\infty}{(q,e^{-2i\theta},te^{i(\theta+\phi)},te^{i(\theta-\phi)};q)_\infty} \tag{5.15}$$
$$\times {}_4\phi_3\left(\left.\begin{matrix} q/\beta, qe^{2i\theta}/\beta, te^{i(\theta+\phi)}, te^{i(\theta-\phi)} \\ qe^{2i\theta}, \beta_1 te^{i(\theta+\phi)}, \beta_1 te^{i(\theta-\phi)}\end{matrix}\right| q,\beta^2\right)$$
$$+\frac{(\beta,\beta e^{2i\theta},\beta_1 te^{i(\phi-\theta)},\beta_1 te^{-i(\theta+\phi)};q)_\infty}{(q,e^{2i\theta},te^{i(\phi-\theta)},te^{-i(\theta+\phi)};q)_\infty}$$
$$\times {}_4\phi_3\left(\left.\begin{matrix} q/\beta, qe^{-2i\theta}/\beta, te^{i(\phi-\theta)}, te^{-i(\theta+\phi)} \\ qe^{-2i\theta}, \beta_1 te^{i(\phi-\theta)}, \beta_1 te^{-i(\theta+\phi)}\end{matrix}\right| q,\beta^2\right).$$

We conclude this section by stating a multilinear generating function for the continuous q-ultraspherical polynomials.

Theorem 5.6 *We have*

$$\sum_{n_1,\cdots,n_m=0}^{\infty}\prod_{j=1}^{m}\left\{t_j^{n_j}C_{n_j}(\cos\theta_j;\beta_j|q)\right\}C_{n_1+\cdots+n_m}(\cos\theta;\beta|q) \tag{5.16}$$
$$=\frac{(\beta,\beta e^{-2i\theta};q)_\infty}{(q,e^{-2i\theta};q)_\infty}\prod_{j=1}^{m}\left\{\frac{(t_j\beta_j e^{i(\theta+\theta_j)},t_j\beta_j e^{i(\theta-\theta_j)};q)_\infty}{(t_je^{i(\theta+\theta_j)},t_je^{i(\theta-\theta_j)};q)_\infty}\right\}$$

$$\times {}_{2m+2}\phi_{2m+1}\left(\begin{matrix} q/\beta, qe^{2i\theta}/\beta, t_1e^{i(\theta+\theta_1)}, t_1e^{i(\theta-\theta_1)}, \cdots, t_me^{i(\theta+\theta_m)}, t_me^{i(\theta-\theta_m)} \\ \\ qe^{2i\theta}, t_1\beta_1e^{i(\theta+\theta_1)}, t_1\beta_1e^{i(\theta-\theta_1)}, \cdots, t_m\beta_me^{i(\theta+\theta_m)}, t_m\beta_me^{i(\theta-\theta_m)} \end{matrix}\middle| q, \beta^2\right)$$

$$+\frac{(\beta, \beta e^{2i\theta}; q)_\infty}{(q, e^{2i\theta}; q)_\infty}\prod_{j=1}^m\left\{\frac{(t_j\beta_je^{i(\theta_j-\theta)}, t_j\beta_je^{-i(\theta+\theta_j)}; q)_\infty}{(t_je^{i(\theta_j-\theta)}, t_je^{-i(\theta+\theta_j)}; q)_\infty}\right\}$$

$$\times {}_{2m+2}\phi_{2m+1}\left(\begin{matrix} q/\beta, qe^{-2i\theta}/\beta, t_1e^{i(\theta_1-\theta)}, t_1e^{-i(\theta+\theta_1)}, \cdots, t_me^{i(\theta_m-\theta)}, t_me^{-i(\theta+\theta_m)} \\ \\ qe^{-2i\theta}, t_1\beta_1e^{i(\theta_1-\theta)}, t_1\beta_1e^{-i(\theta+\theta_1)}, \cdots, t_m\beta_me^{i(\theta_m-\theta)}, t_m\beta_me^{-i(\theta+\theta_m)} \end{matrix}\middle| q, \beta^2\right).$$

Proof. In (5.1) replace θ and t by θ_j and t_j; respectively, $j = 1, \cdots m$, multiply the results and apply $\mathcal{L}_{\beta,\theta}$. We obtain (5.16).

6. Remarks and Summary. The q-analogue of (2.2) is

$$(6.1) \qquad <\mathcal{LM}|p_n>_q = \sum_{k=0}^n \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} <\mathcal{L}|p_k>_q <\mathcal{M}|p_{n-k}>_q$$

where $\{p_n(x)\}$ is any Eulerian family of polynomials, [3], [15]. When $\mathcal{L}$ and $\mathcal{M}$ have the representations

$$(6.2) \qquad \mathcal{L}f(x) = \int_{-\infty}^{\infty} f(x)\, d\lambda(x), \quad \mathcal{M}f(x) \int_{-\infty}^{\infty} f(y)\, d\mu(y)$$

then

$$(6.3) \qquad <\mathcal{LM}|f>_q = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(xy) d\lambda(x) d\mu(y).$$

The product functional (6.3) was used in the previous sections when we repeatedly applied certain functionals with different parameters.

It is clear that the products of functionals in (2.2) and (6.3) correspond to convolutions of measures and as such they are very natural. It is also clear that the products (2.2) and (6.3) correspond to additive and multiplicative algebraic structures. It was observed in [22] that in many orthogonal polynomial systems what is designated as a variable may not be the "natural" variable. So we denote the natural variable by s and the polynomials variable by x and write $x = x(s)$. A mapping $s \to x(s)$ from the complex plane into itself defines a grid or a lattice on a subset S of the complex s plane on which the mapping is one-to-one. The functional products (2.2) and (6.3) correspond to the linear grid $x(s) = s$ and the q-linear grid $x(s) = q^s$, respectively. These are the simplest grids. The grid used is also related to a comultiplication of a bialgebra. The linear grid corresponds to $x \to x \otimes 1 + 1 \otimes x$ while the q-linear grid corresponds to $x \to x \otimes x$. It will be useful to find the comultiplication associated with the quadratic grid $x(s) = s^2 + cs$ and the q-quadratic grid $x(s) = (q^s + q^{-s})/2$. The former will give an umbral calculus for the Wilson polynomials and Wilson divided difference operators while the latter will give an umbral calculus for the Askey-Wilson polynomials and the corresponding divided difference operators.

It is worth noting that Slepian [26] essentially used the Rodrigues formula for Hermite polynomials and the fact that the Fourier transform of a Gaussian is another Gaussian to derive the multilinear generating function known as the Kibble-Slepian formula, [20], [26]. In the case of Laguerre polynomials formula (2.5) also follows from

$$(6.4) \qquad x^\alpha e^{-x} = \int_0^\infty e^{-u}(xu)^{\alpha/2} J_\alpha(2\sqrt{xu})\, du, \quad \alpha > -1,$$

by successive differentiations, [29, (3.2.5)] and the Rodrigues formula for the Laguerre polynomials.

We conclude with the list of classical orthogonal polynomials given in this paper which are moments. For each polynomial we give the (non-normalized) measure $d\mu(x)$, the monic orthogonal polynomials $p_n(x)$ for the measure, and the coefficients α_n and β_n in the three term recurrence relation

$$(6.5) \qquad p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x),$$

for $p_n(x)$, [9, p. 215].

Hermite polynomials: $\mu_n = H_n(ia)/i^n$, $d\mu(x;a) = e^{-(x/2+a)^2}dx$ on $(-\infty, \infty)$,

$$p_n(x) = H_n(x/2 + a), \alpha_n = -2a, \beta_n = 2n.$$

Laguerre polynomials: $\mu_n = n!L_n^\alpha(-b)$, $d\mu(x; b, \alpha) = b^{-\alpha/2}e^{-b-x}x^{\alpha/2}I_\alpha(2\sqrt{xb})dx$ on $(0, \infty)$. Here $p_n(x)$, α_n, and β_n are unknown.

Laguerre polynomials: $\mu_n = n!L_n^\alpha(ib)/(\alpha + 1)_n$,

$$\alpha_n = 1 + \frac{ib(2 - \alpha)}{(\alpha + 2n - 2)(\alpha + 2n)},$$

and

$$\beta_n = \frac{nb^2(\alpha + n - 2)}{(\alpha + 2n - 3)_3(\alpha + 2n - 2)}.$$

$p_n(x)$ and $d\mu(x)$ are given in [30].

Meixner polynomials: $\mu_n = c^n m_n(a,\beta,c)/(\beta)_n$, $d\mu(x) = (1-x)^{\beta+a-1}(x-c)^{-a-1}dx$ on $[c,1]$,

$$p_n(x) = n!(1-c)^n P_n^{(\beta+a-1,-a-1)}((2x-(1+c))/(1-c))/(\beta+n-1)_n, (Jacobipolynomials)$$

$$\alpha_n = c + \frac{(c-1)((\beta-2)a - 2n\beta - 2n(n-1))}{(\beta+2n-2)(\beta+2n)},$$

and

$$\beta_n = \frac{(c-1)^2 n(\beta+n-2)(n-1-a)(\beta+a+n-1)}{(\beta+2n-3)_3(\beta+2n-2)}.$$

Al-Salam-Chihara polynomials: $\mu_n = p_n(cos(\theta); t_1, t_2)$, $d\mu(x)$ is given by (1.6) where (3.9) holds, $p_n(x) = P_n(x; a, b, c; q)(aq, cq; q)_n/(abq^{n+1}; q)_n$ (big q-Jacobi polynomials), for α_n and β_n see [12, Ex. 7.10, p. 186].

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