A CONVOLUTION FORMULA FOR THE TUTTE POLYNOMIAL

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Let M be a finite matroid with rank function r. We will write $A \subseteq M$ when we mean that A is a subset of the ground set of M, and write $M|_A$ and M/A for the matroids obtained by restricting M to A, and contracting M on A respectively. Let M^* denote the dual matroid to M. (See [1] for definitions). The main theorem is

Theorem 1. *The Tutte polynomial $T_M(x,y)$ satisfies*

$$T_M(x,y) = \sum_{A \subseteq M} T_{M|_A}(0,y) T_{M/A}(x,0). \tag{1}$$

First we define a convolution product and note a useful lemma.

Let $\mathbb{M}$ be the set of all isomorphism classes of finite matroids, and let K be a commutative ring with 1. For any functions $f, g : \mathbb{M} \to K$, define $f \circ g : \mathbb{M} \to K$ by

$$(f \circ g)(M) = \sum_{A \subseteq M} f(M|_A) g(M/A). \tag{2}$$

The convolution $\circ$ is associative, with identity element δ,

$$\delta(M) = \begin{cases} 1 \text{ if } M = \varnothing, \\ 0 \text{ otherwise.} \end{cases}$$

Following Crapo [2], let $\zeta(x,y)(M) = x^{r(M)} y^{r(M^*)}$, where $K = \mathbb{Z}[x,y]$.

Lemma 1. $\zeta(x,y)^{-1} = \zeta(-x,-y)$.

Proof. Note that

$$\begin{aligned}(\zeta(x,y) \circ \zeta(-x,-y))(M) &= \sum_{A \subseteq M} x^{r(M|_A)} y^{r((M|_A)^*)} (-x)^{r(M/A)} (-y)^{r((M/A)^*)} \\ &= x^{r(M)} y^{r(M^*)} \sum_{A \subseteq M} (-1)^{|M|-|A|} \\ &= \delta(M).\end{aligned}$$

Proof of Theorem 1. The Tutte polynomial may be defined by [1,2]

$$T_M(x+1, y+1) = (\zeta(1,y) \circ \zeta(x,1))(M), \tag{3}$$

so also

$$T_M(x+1,0) = (\zeta(1,-1)\circ\zeta(x,1))(M),$$
$$T_M(0,y+1) = (\zeta(1,y)\circ\zeta(-1,1))(M).$$

Therefore

$$\begin{aligned}\sum_{A\subseteq M} T_{M|A}(0,y+1)T_{M/A}(x+1,0) &= (\zeta(1,y)\circ\zeta(-1,1))\circ(\zeta(1,-1)\circ\zeta(x,1))(M)\\ &= \zeta(1,y)\circ(\zeta(-1,1)\circ\zeta(1,-1))\circ\zeta(x,1)(M)\\ &= \zeta(1,y)\circ\zeta(x,1)(M)\\ &= T_M(x+1,y+1),\end{aligned}$$

where the third equality is by Lemma 1. □

Remark 1.
Note that Theorem 1 can be rewritten as

$$T_M(x,y) = \sum_{\text{isthmus-free flats } V} T_V(0,y)T_{M/V}(x,0). \tag{4}$$

This is because when $A\subseteq M$ is not a flat, M/A contains a loop e and

$$T_{M/A}(x,0) = \left[y\,T_{(M/A)-e}(x,y)\right]_{y=0} = 0.$$

Similarly if A contains an isthmus e, then

$$T_{M|A}(0,y) = \left[x\,T_{(M|_A)/e}(x,y)\right]_{x=0} = 0.$$

Remark 2.
Theorem 1 can also be proven using Tutte's original definition of the Tutte polynomial involving *basis activities* [1,2]: for any ordering of the ground set of M,

$$T_M(x,y) := \sum_{\text{bases } B \text{ of } M} x^{|IA_M(B)|}y^{|EA_M(B)|}$$

where here $IA_M(B)$ (resp. $EA_M(B)$) denotes the set of internally (resp. externally) active elements of M with respect to the base B. Theorem 1 in [3] asserts that any base B can be uniquely decomposed $B = B_1\cup B_2$ with $B_1\cap B_2 = \varnothing$ and

$$IA_V(B_1) = EA_{M/V}(B_2) = \varnothing$$

where V is the flat $\overline{B_1}$ spanned by B_1. It turns out that in this decomposition one furthermore has

$$IA_M(B) = IA_{M/V}(B_2), \qquad EA_M(B) = EA_V(B_1). \tag{5}$$

We omit the details of this verification, which are straightforward. Given this, one then has

$$\begin{aligned}
T_M(x,y) &= \sum_{\text{bases } B \text{ of } M} x^{|IA_M(B)|} y^{|EA_M(B)|} \\
&= \sum_{\text{flats } V \text{ of } M} \sum_{\substack{\text{bases } B_1 \text{ of } V \\ \text{with } IA_V(B_1)=\varnothing}} \sum_{\substack{\text{bases } B_2 \text{ of } M/V \\ \text{with } EA_{M/V}(B_2)=\varnothing}} x^{|IA_{M/V}(B_2)|} y^{|EA_V(B_1)|} \\
&= \sum_{\text{flats } V \text{ of } M} \left(\sum_{\substack{\text{bases } B_1 \text{ of } V \\ \text{with } IA_V(B_1)=\varnothing}} y^{|EA_V(B_1)|} \right) \left(\sum_{\substack{\text{bases } B_2 \text{ of } M/V \\ \text{with } EA_{M/V}(B_2)=\varnothing}} x^{|IA_{M/V}(B_2)|} \right) \\
&= \sum_{\text{flats } V \text{ of } M} T_V(0,y)\, T_{M/V}(x,0).
\end{aligned}$$

Remark 3.

The version (4) of Theorem 1 may also be proven by deletion-contraction, as we now explain. Recall [1] that the Tutte polynomial is characterized by the following three properties.

(i) $T_M(x,y) = x$ if M consists of a single isthmus, and $T_M(x,y) = y$ if M consists of single loop.

(ii) $T_{M_1 \oplus M_2}(x,y) = T_{M_1}(x,y) \cdot T_{M_2}(x,y)$.

(iii) $T_M(x,y) = T_{M-e}(x,y) + T_{M/e}(x,y)$ if e is neither an isthmus nor a loop of M.

Let $T'_M(x,y)$ be the right side of (4), and we must show that it also satisfies (i),(ii),(iii). Properties (i),(ii) are straightforward and omitted. To show (iii), fix an element e which is neither an isthmus nor a loop of M, and then use property (iii) for $T_M(x,0), T_M(0,y)$ to write

$$\begin{aligned}
T'_M(x,y) &= \sum_{\text{isthmus-free flats } V} T_V(0,y) T_{M/V}(x,0) \\
&= \sum_{\substack{\text{i.f. flats } V \\ e \in V}} T_V(0,y) T_{M/V}(x,0) + \sum_{\substack{\text{i.f. flats } V \\ e \notin V}} T_V(0,y) T_{M/V}(x,0) \\
&= \sum_{\substack{\text{i.f. flats } V \\ e \in V}} T_{V-e}(0,y) T_{M/V}(x,0) + \sum_{\substack{\text{i.f. flats } V \\ e \in V}} T_{V/e}(0,y) T_{M/V}(x,0) \\
&+ \sum_{\substack{\text{i.f. flats } V \\ e \notin V}} T_V(0,y) T_{M/V-e}(x,0) + \sum_{\substack{\text{i.f. flats } V \\ e \notin V}} T_V(0,y) T_{(M/V)/e}(x,0) \\
&= \sum_{\substack{V, V-e \text{ both i.f.} \\ e \in V}} T_{V-e}(0,y) T_{M/V}(x,0) + \sum_{\substack{\text{i.f. flats } V \\ e \in V}} T_{V/e}(0,y) T_{M/V}(x,0) \\
&+ \sum_{\substack{\text{i.f. flats } V \\ e \notin V}} T_V(0,y) T_{M/V-e}(x,0) + \sum_{\substack{\text{i.f. flats } V \\ V \cup \{e\} \text{ a flat}}} T_V(0,y) T_{(M/V)/e}(x,0) \qquad (6)
\end{aligned}$$

where the last equality comes from the fact that $T_{V-e}(0,y) = 0$ unless $V - e$ is isthmus-free, and dually $T_{(M/V)/e}(x,0) = 0$ unless $V \cup \{e\}$ is a flat of M.

On the other hand, we wish to show that the above sum is the same as

$$
\begin{aligned}
&T'_{M-e}(x,y) + T'_{M/e}(x,y) \\
&= \sum_{\substack{\text{i.f. flats } W\\ \text{of } M-e}} T_{(M-e)|_W}(0,y)T_{(M-e)/W}(x,0) + \sum_{\substack{\text{i.f. flats } W\\ \text{of } M/e}} T_{(M/e)|_W}(0,y)T_{(M/e)/W}(x,0) \\
&= \sum_{\substack{\text{i.f. flats } W \text{ of } M-e,\\ W \text{ not a flat of } M}} T_{(M-e)|_W}(0,y)T_{(M-e)/W}(x,0) + \sum_{\substack{\text{i.f. flats } W \text{ of } M/e,\\ W \text{ not a flat of } M}} T_{(M/e)|_W}(0,y)T_{(M/e)/W}(x,0) \\
(7)\quad &+ \sum_{\substack{\text{i.f. flats } W \text{ of } M-e,\\ W \text{ a flat of } M}} T_{(M-e)|_W}(0,y)T_{(M-e)/W}(x,0) + \sum_{\substack{\text{i.f. flats } W \text{ of } M/e,\\ W \text{ a flat of } M}} T_{(M/e)|_W}(0,y)T_{(M/e)/W}(x,0)
\end{aligned}
$$

The terms W in the sums on the right-hand side of equation (7) biject with the terms V in the sums on the right-hand side of equation (6) as follows: in the first sum $W = V - e$, in the second sum $W = V/e$, in the third sum $W = V$ and in the fourth sum $W = V$. We leave it to the reader to check that this gives a bijection of the terms which shows the equality of the right-hand sides in (6) and (7). The only tricky point here is in the fourth sum, where one must note that not only are W, V equal as subsets of the ground sets of $M/e, M$ respectively, but also the flats W, V of $M/e, M$ are isomorphic as matroids, due to the fact that e is an isthmus of $V \cup \{e\}$.

Remark 4.
Lemma 1 can be used to prove other convolution identities. For example, if we define

$$\rho(x,y,z,w)(M) := (\zeta(z,y) \circ \zeta(x,w))(M)$$

then equation (3) implies

$$
\begin{aligned}
T_M(x,y) &= \rho(x-1, y-1, 1, 1)(M) \\
T_M(0,y) &= \rho(-1, y-1, 1, 1)(M) \\
T_M(x,0) &= \rho(x-1, -1, 1, 1)(M)
\end{aligned}
$$

and Theorem 1 is the specialization $z = w = 1$ of the more general identity

$$
\begin{aligned}
\rho(x-1, y-1, z, w) &= \zeta(z, y-1) \circ \zeta(x-1, w) \\
&= \zeta(z, y-1) \circ \zeta(-1, 1) \circ \zeta(1, -1) \circ \zeta(x-1, w) \\
&= \rho(-1, y-1, z, 1) \circ \rho(x-1, -1, 1, w)
\end{aligned}
$$

As another example, of the use of Lemma 1, one can start with equation (2) and multiply both sides by $\zeta(-1, -y)$. Using the notation $T(x,y)(M) := T_M(x,y)$, we obtain

$$\zeta(-1, -y) \circ T(x+1, y+1) = \zeta(x, 1)$$

which gives an apparently new recursion for the Tutte polynomial

$$T_M(x,y) = (x-1)^{r(M)} - \sum_{\varnothing \neq A \subseteq M} (-1)^{r(M|_A)} (1-y)^{r(M|^*_A)} T_{M/A}(x,y).$$

Remark 5.

The convolution product defined by equation (2) suggests a certain coalgebra (actually a Hopf algebra) naturally associated with matroids. Let A be a free K-module with basis $\mathbb{M}$ equal to the isomorphism classes of finite matroids $[M]$. The coproduct $\Delta : A \to A \otimes A$ is defined K-linearly by

$$\Delta([M]) = \sum_{A \subseteq M} [M|_A] \otimes [M/A],$$

and the product $\mu : A \otimes A \to A$ is defined K-linearly by

$$\mu([M] \otimes [M']) = [M \oplus M'].$$

Define a bigrading on A by setting the bidegree of $[M]$ to be $(r(M), r(M^*))$. One can check that this makes A a co-associative, commutative, bigraded, connected, Hopf algebra over K, whose unit $\eta : K \to A$ is $\eta(1) = [\varnothing]$, and whose co-unit $\epsilon : A \to K$ is $\epsilon([M]) = \delta_{M,\varnothing}$. If $\phi : A \to A$ is the involution $\phi([M]) = [M^*]$ extended K-linearly to all of A, then one can check that the identity $M^*|_{M-A} \cong (M/A)^*$ leads to the equation

$$\Delta \circ \phi = (\phi \otimes \phi) \circ \Delta^{op}.$$

Therefore $\phi^* : A^* \to A^*$ is an algebra anti-automorphism. Note that ϕ also exchanges the bigrading in the sense that if a has bidegree (s, t) then $\phi(a)$ has bidegree (t, s).

Motivated by this, let A be any co-associative, bigraded, connected coalgebra over K with coproduct Δ and co-unit η, having a distinguished K-basis of bihomogeneous elements $\mathbb{M}$. Let $\circ$ denote the product dual to Δ in the dual algebra A^*, and $\phi : A \to A$ be any involution which exchanges the bigrading and such that $\phi^* : A^* \to A^*$ is an anti-automorphism. Define $\zeta \in A^*$ by $\zeta(x, y)(M) = x^s y^t$ for all $M \in \mathbb{M}$ having bidegree (s, t). We can then define a *Tutte functional* $T(x, y) \in A^*$ by $T(x, y) = \zeta(1, y - 1) \circ \zeta(x - 1, 1)$. One can then check that the familiar Tutte polynomial identity [1]

$$T_{M^*}(x, y) = T_M(y, x)$$

has the counterpart

$$\phi^*(T(x, y)) = T(y, x)$$

which follows formally from the assumed properties of ϕ.

Furthermore, the proof of Lemma 1 actually shows the following in this context:

$$\text{If } \zeta(1, 1)^{-1} = \zeta(-1, -1) \text{ then } \zeta(x, y)^{-1} = \zeta(-x, -y)^{-1}.$$

Consequently, if we impose the extra condition on A that $\zeta(1, 1)^{-1} = \zeta(-1, -1)$, then the counterpart to Theorem 1

$$T(x, y) = T(0, y) \circ T(x, 0)$$

ensues as a formal consequence.

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References

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