Homework #4

NOTE: I have deleted a few problems from those on the syllabus.
Cameron 1. p. 45 This problem is to show that

$$n! \leq e\sqrt{n+1}(\frac{n}{e})^n$$

2. p. 45 This problem is

$$\binom{2n}{n} \sim 2^{2n}/\sqrt{\pi n}.$$

3. p. 69 The problem on Fibonacci numbers and seating on a line.

4. p. 71, the Catalan number problem on elections.

5. p. 71, the exponential generating function of $exp(t + t^2/2)$.

Balakrishnan 3.96, 3.97 (find $f(n)$ two ways, by solving the recurrence relation, and by using an exponential generating function), 3.98

Supplement 1. Let a_n be the number of regions into which the plane is divided by n lines in general position, for example $a_1 = 2$, and $a_2 = 4$. Find a_3 and a_4, find a recurrence relation for a_n, and then solve your recurrence.

2. Suppose that $2n$ people are seated around a large circular table. Let b_n be the number of ways that n pairs of handshakes can be made (each person shakes with one other person) without crossings ($b_2 = 2$). Find a recurrence relation for b_n, and solve it.

3. Suppose that all diagonals of an $(n+2)$-gon are drawn, and that no three diagonals intersect. Let a_n be the number of regions created. Show that

$$a_n = a_{n-1} + \binom{n+1}{3} + n, \quad a_1 = 1, \quad a_2 = 4$$

and solve for a_n.

4. Recall that we defined the x-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = \sum_{\lambda \text{ inside an } k\times(n-k) \text{ box}} x^{||\lambda||}.$$

For example

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_x = 1 + x + 2x^2 + x^3 + x^4.$$

Show that the x-binomial coefficient satisfies the recurrence relation

$$\text{(1)} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_x = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_x + x^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_x$$

Prove that a solution to (1) is

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = \frac{n!_x}{k!_x(n-k)!_x}$$

where

$$n!_x = [n]_x[n-1]_x \cdots [2]_x[1]_x, \quad [j]_x = \frac{1-x^j}{1-x}.$$

Verify that this formula works for $n = 4$ and $k = 2$.