Homework #5

Cameron

1. p. 84, the opinion poll question

2. p. 84, the explicit form of the Stirling numbers of the second kind

3. p. 84, the Stirling numbers of the first kind question

Balakrishnan 2.160, 2.163, 2.166, 2.169, 2.173, 2.174

Supplement

1. Use the explicit formula for

$$\sum_{k=0}^{n} |s(n,k)| x^k$$

to find the average number of cycles that a permutation of $\{1, 2, \cdots, n\}$ has. What is the behavior as $n \to \infty$?

2. Show that the number of permutations of $\{1, 2, \cdots, n\}$ in which i is never immediately followed by $i+1$, $1 \le i \le n-1$, is $D_n + D_{n-1}$.

3. By considering the number of ways to distribute m distinguishable balls into n distinguishable boxes, each box non-empty, $m < n$, prove that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m = 0.$$

4. Fix positive integers m and n with $m \le n/2$. Use rook polynomials to find the number of permutations of $\{1, 2, \cdots, n\}$ in which
(a) i does not occur in position i, $1 \le i \le m$,
(b) i does not occur in positions $2i-1$, $2i$, $1 \le i \le m$.

5. Let $\phi(n)$ be the number of positive integers $\le n$ which are relatively prime to n. For example, $\phi(7) = 6, \phi(6) = 2$. Suppose that n has distinct prime factors of $p_1, p_2, \cdots p_m$. Use inclusion-exclusion to show that

$$\phi(n) = n - \frac{n}{p_1} - \frac{n}{p_2} - \cdots \frac{n}{p_m} + \frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \cdots + (-1)^m \frac{n}{p_1 p_2 \cdots p_m}.$$

Conclude that

$$\phi(n) = n(1 - 1/p_1)(1 - 1/p_2) \cdots (1 - 1/p_m).$$