Homework #7

Supplement

(B) 1. Let a_n be the number of permutations π of n such that $\pi^3 = id$. Find

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Find an explicit formula for a_n and a recurrence relation for a_n.

(A) 2. Let $\overline{fp}$ be the average number of fixed points of a permutation of length n. Find $\overline{fp}$, and the standard deviation $\sigma = \sqrt{\overline{fp^2} - (\overline{fp})^2}$.

(A) 3. Fix a positive integer k. If n is large, what is a good approximation for the probability that a permutation of length n has exactly k fixed points?

(B) 4. How many lattice paths start at $(0,0)$, end at $(14,2)$, and stay at or above the x-axis?

(C) 5. How many connected labeled graphs are there on 5 vertices?

(C) 6. How many permutations of [9] have a longest increasing subsequence of length 5?

(C) 7. In the greedy LEX match of 4-subsets of [9] to 5-subsets of [9], which set is matched to $\{2,3,5,8\}$? to $\{1,2,5,8,9\}$?

(A) 8. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, define

$$inv(\pi) = |\{(i,j) : 1 \le i < j \le n, \pi_i > \pi_j\}|.$$

For example $inv(7236154) = 6+1+1+3+0+1 = 12$. Find a simple form of the generating function

$$G_n(x) = \sum_{\pi \in S_n} x^{inv(\pi)}.$$

What is the average and maximum number of inversions a permutation may have?

(A) 9. Let $a_{n,k}$ be the number of lattice paths from $(0,0)$ to $(2n,0)$ which stay at or above the x-axis, and intersect the x-axis at exactly k points besides $(0,0)$ and $(2n,0)$. Let

$$F_n(x) = \sum_{k=0}^{\infty} a_{n,k} x^k.$$

(a) Find $F_n(1)$.

(b) Find $a_{n,0}$ and $G(t) = \sum_{n=0}^{\infty} a_{n,0} t^n$.

(c) Show that

$$\sum_{n=0}^{\infty} F_n(x) t^n = \frac{1}{1 - xG(t)},$$

and find an explicit formula for $a_{n,k}$. (This may be a bit hard.)